L2. A bead of mass $m = 1$ kg slides on a frictionless wire. The wire is held rigidly in the shape of a circular hoop of radius $R = 1$ m, as shown in the figure. Initially the bead is at rest at the top of the hoop. The bead will begin to slide through an angle θ when slightly perturbed from this unstable position. The moment of inertia of the bead may be neglected.

(a) Show that, for small angles, θ evolves according to the equation of motion

$$\ddot{\theta} - \lambda^2\theta \approx 0, \text{ where } \lambda = g/R.$$

(b) Suppose that the bead is balanced at the top of the hoop, $\theta(0) = 0$, and given a very light push such that initial angular velocity is $\dot{\theta}(0) = 10^{-17} s^{-1}$, is on the order of the minimum allowed by the uncertainty principle. Solve the equation of motion and determine how long (in seconds) it will take the bead to slide through an angle of 30 degrees.

(c) Using the fact that energy is a conserved quantity, show that the time Δt it takes a bead to slide between any two angles θ_1 and θ_2 is given *exactly* by

$$\Delta t = \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\frac{2E}{mR^2} - \frac{2g}{R}\cos\theta}}.$$

where E is the total energy and the potential energy has been taken to be zero at θ= 90 degrees.

(d) After reaching the angle of 30 degrees, approximately how much longer (in seconds) will it take the bead to slide the rest of the way to the bottom of the hoop?

Useful integral:

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{1-\cos\theta}} = \sqrt{2}\ln\left[\frac{\sin(\theta_1/2)}{\sin(\theta_2/2)}\left(\frac{1-\cos(\theta_2/2)}{1-\cos(\theta_1/2)}\right)\right].$$