PHYC 467: Methods of Theoretical Physics II

Spring 2016

Final Exam

Date and Time: 05/11/2016, 10:00-13:00

Instructions:

- This is an open-book, open-note exam. All reference material allowed.
- The exam consists of three problems, which are equally weighted.

1- The two-dimensional spinors ψ_L and ψ_R belong to the $(1/2, 0)$ and $(0, 1/2)$ representations of the $SO(3,1)$ group respectively. Show that $\psi_{\rm R}^\dagger\psi_{\rm L}, \psi_{\rm L}^\dagger\psi_{\rm R}$ bilinears behave scalars and $\psi_{\rm L}^\dagger\sigma^\mu\psi_{\rm L}, \psi_{\rm R}^\dagger\bar{\sigma}^\mu\psi_{\rm R}$ behave are four-vectors under the group transformations. Here $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, where σ^i are Pauli matrices.
[Hint: You may use elements of the $SO(3,1)$ group in the $(1/2, 0)$ and $(0, 1/2)$ representations, $\Lambda_{\rm L}$ and $\lambda_{\rm R}$ respectively, to find how the above bilinears are transformed under (infinitesimal) rotations and boosts.]

2- A quantum system consists of N identical bosons each of which can be at either of the two levels 1 and 2 with energies E_1 and E_2 respectively ($E_1 < E_2$). Determine all possible energy eigenvalues of this system. Obtain the probability function $P(E)$ for the system to have energy E at temperature T, and claculate the mean and variance of its energy as a function of T. Find the energy mean and variance in the high temperature $kT \gg E_2 - E_1$ and low temperature $kT \ll E_2 - E_1$ limits, and comment.
[Hint: You may use the Maxwell-Boltzmann distribution $P(E) \propto \exp(-E/kT)$ for the probability to occupy a state with energy E at temperature T.]

3- Positive sample values x_i ($i = 1, ..., N$) are drawn independently from the distribution $P(x|\lambda) = \lambda \exp(-\lambda x)$. Find the maximum likelihood estimate of the parameter λ, denoted by $\hat{\lambda}_{\rm ML}$, and calculate its mean $E[\hat{\lambda}_{\rm ML}]$ and variance $V[\hat{\lambda}_{\rm ML}]$. Show that the maximum likelihood estimator is neither unbiased nor the most efficient.
[Hint: You may compare $V[\hat{\lambda}_{\rm ML}]$ with the lower bound in Fisher's inequaltiy.]