PHYC 467: Methods of Theoretical Physics II

Spring 2016

Midterm Exam

- This is a take home exam. Distributed on Friday April 8, due on Monday April 11 by 8:00 pm.
- All reference material allowed, but no DISCUSSING PROBLEMS.
- Any questions are to be directed to the instructor.
- Two problems, equally weighted.

1- Consider the three-dimensional sphere S^2 of radius a. The zenith and azimuthal angles, ϕ and θ respectively, can be used to define a coordinates system on this sphere. Write down the distance between two nearby points (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ and find the metric tensor g_{ij} for S^2.

(a) Calculate the Christoffel symbols of the second kind $\Gamma^i{}_{jk}$.

(b) Use the result in part (a) to find the components of the Riemann tensor:

$$R^l{}_{ijk} = \frac{\partial \Gamma^l{}_{ik}}{\partial u^j} - \frac{\partial \Gamma^l{}_{ij}}{\partial u^k} + \Gamma^m{}_{ik}\Gamma^l{}_{mj} - \Gamma^m{}_{ij}\Gamma^l{}_{mk} \,.$$

(c) Use the result in part (b) to find the Ricci tensor $R_{ij} \equiv R^k{}_{ikj}$, and then calculate the Ricci scalar $R \equiv R^i{}_i$.

(d) Comment on the value and sign of R.

2- Show that generators of the $SO(N)$ group are traceless antisymmetric matrices. Considering the N-dimensional representation of $SO(N)$, show that this group has $N(N-1)/2$ generators.

(a) Specializing to $SO(4)$, show that the 6 matrices T_{ij} defined as follows ($1 \leq i, j \leq 4$ and $i < j$) form a set of generators for $SO(4)$:

$$(T_{ij})_{mn} = -i(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \, .$$

(b) Defining $L_1 = T_{23}, \; L_2 = T_{31}, \; L_3 = T_{12}, \; M_1 = T_{14}, \; M_2 = T_{24}, \; M_3 = T_{34}$, show that the following commutation relations are satisfied:

$$[L_i, L_j] = i\epsilon_{ijk} L_k \quad , \quad [M_i, M_j] = i\epsilon_{ijk} L_k \quad , \quad [M_i, L_j] = i\epsilon_{ijk} M_k \, .$$

(c) Now define new generators $J_i = (L_i + M_i)/2$ and $K_i = (L_i - M_i)/2$. Show that these generators obey the following commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad , \quad [K_i, K_j] = i\epsilon_{ijk} K_k \quad , \quad [J_i, K_j] = 0 \, .$$

(d) Use the result in part (c) and show that the $SO(4)$ algebra is equivalent to $SU(2) \bigotimes SU(2)$. Discuss the rank of $SO(4)$ and the number of its Casimir operators.

(e) Show explicitly that $C_1 = \sum_i (L_i^2 + M_i^2)$ and $C_2 = \sum_i L_i M_i$ commute with all the $SO(4)$ generators, and hence are Casimir operators. What is the relation between C_1, C_2 and the Casimir operators of the two $SU(2)$'s in part (d)?

(f) As discussed in the class, the dynamical symmetry group of the Hydrogen atom is $SO(4)$ where L_i and M_i generators correspond to the components of the orbital angular momentum and the Runge-Lenz vector respectively. Use the fact that $\vec{L} \cdot \vec{M} = 0$ in this case, and the result in part (e), to find the dimensionality of irreducible representations of $SO(4)$ that describe the energy eigenstates of the Hydrogen atom. Verify that this agrees with what you know about the degeneracy in a given energy level of the Hydrogen atom.