PHYC 467: Methods of Theoretical Physics II

Spring 2016

Homework Assignment #2

(Due February 16, 2016)

1- Show that if v_i are covariant components of a vector, then $v_{i,j} - v_{j,i}$ is a second-order tensor under general coordinate transformations. We now contract this tensor with the Levi-Civita symbol ϵ^{ijk} to find $u^i = -\epsilon^{ijk}(v_{j,k} - v_{k,j})/2\sqrt{g}$. How do u^i behave under a general coordinate transformation? Does this object look familiar? Comment.

2- Show that Christoffel symbols (of the second kind) $\Gamma^i{}_{kj}$ satisfy the following relation:

$$\Gamma^i{}_{ki} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial u^k} \,,$$

where g is the determinat of the metric tensor. Use this and show that the divergence of a vecotr field $\vec{v}$ can be expressed as follows:

$$v^i{}_{;i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^j} (\sqrt{g} v^j) \,.$$

Specializing to spherical coordinates, verify that this expression gives the familiar result for the divergence of a vecotr in spherical coordinates.
Hint: You may use the fact that $g = \sum_j g_{ij} \Delta^{ij}$ (where Δ^{ij} denotes the cofactor of the element g_{ij} and summation is performed over j only), and the relation between g^{ij} and Δ^{ij}, to first show that:

$$\frac{\partial g}{\partial u^k} = g g^{ij} \frac{\partial g_{ij}}{\partial u^k} \,.$$

3- Defining the second covarinat derivative of a vecotr v_i as $v_{i;jk} \equiv (v_{i;j})_{;k}$, show that:

$$v_{i;jk} - v_{i;kj} \equiv R^l{}_{ijk} v_l \,,$$

where

$$R^l{}_{ijk} = \frac{\partial \Gamma^l{}_{ik}}{\partial u^j} - \frac{\partial \Gamma^l{}_{ij}}{\partial u^k} + \Gamma^m{}_{ik} \Gamma^l{}_{mj} - \Gamma^m{}_{ij} \Gamma^l{}_{mk} \,,$$

are the components of the Riemann curvature tensor. Verify that $R^l{}_{ijk}$ is indeed a tensor under general coordinate transformations. Use this to show that in three-dimensional Euclidean space all components of the Riemann tensor identically vanish for any coordinate system. Hint: You may use the definition of the Chrsitoffel symbols $\Gamma^k{}_{ij} \equiv \vec{e}^k \cdot \partial\vec{e}_i/\partial u^j$ to find how they are transformed under a change of coordinates:

$$\Gamma'^k{}_{ij} = \frac{\partial u'^k}{\partial u^l}\frac{\partial^2 u^l}{\partial u'^j \partial u'^i} + \frac{\partial u'^k}{\partial u^n}\frac{\partial u^l}{\partial u'^i}\frac{\partial u^m}{\partial u'^j}\Gamma^n{}_{lm}\,.$$

4- One may define Christoffel symbols of the first kind by $\Gamma_{ijk} = g_{il}\Gamma^l{}_{jk}$. Verify that:

$$\Gamma_{ijk} + \Gamma_{jik} = \frac{\partial g_{ij}}{\partial u^k}\,,$$

and use this to show that the covariant derivative of the metric tensor identiucally vanishes, i.e., $g_{ij;k} = 0$.