PHYC 467: Methods of Theoretical Physics II

Spring 2016

Homework Assignment #4

(Due March 29, 2016)

1- Generators L_i of a Lie group can be normalized such that $\mathrm{Tr}(L_i L_j) = \delta_{ij}/2$. Show that the resulting structure constants are purely imaginary and totally antisymmetric in this case. Verify that the matrices $\sigma_i/2$ are normalized generators of $SU(2)$ (σ_i are the Pauli matrices).

2- The adjoint representation of an algebra is defined as the representation in which the matrix elements of generators are given by $(L_i)_{jk} = c_{ijk}$, where c_{ikj} are the structure constants of the algebra.

(a) Assuming normalzied L_i's show that $\det(L_i) = \det(-L_i)$ in the adjoint representation.

(b) Explicitly write down the $SU(2)$ generators in its adjoint representation. Use the result in part (a) to show that $\det(L_i) = 0$ in the representation of $SU(2)$ and verify this.

3- Let $a_1^\dagger$, a_1 and $a_2^\dagger$, a_2 denote two sets of creation and annihilation operators that satisfy the following commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij} \quad , \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\,.$$

(a) Show that the three operators

$$T_1 = a_1^\dagger a_2 + a_2^\dagger a_1 \quad , \quad T_2 = -i(a_1^\dagger a_2 - a_2^\dagger a_1) \quad , \quad T_3 = a_1^\dagger a_1 - a_2^\dagger a_2\,,$$

generate the $SU(2)$ algebra.

(b) Show that the number operator defined as $N = a_1^\dagger a_1 + a_2^\dagger a_2$ is a Casimir operator of this $SU(2)$ algebra.

4- The translation-rotation group in three dimensions (called the Euclidean group E_3) has 6 generators $P_{1,2,3}$ (for translations) and $J_{1,2,3}$ (for rotations) obeying the following commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad , \quad [P_i, P_j] = 0 \quad , \quad [P_i, J_j] = i\epsilon_{ijk} P_k \, .$$

(a) What is the rank of this group?

(b) Show that $\vec{P} \cdot \vec{P}$ and $\vec{J} \cdot \vec{P}$ are Casimir operators of E_3.

5- Consider an n-dimensional unitary representation of a Lie group that is spanned by an orthonormal basis $|i\rangle$ $(1 \leq i \leq n)$.

(a) Show that the basis $\langle i|$ also spans a representation of the group. What do the generators look like in this representation? Conclude that the two representations are complex conjugate of each other.

(b) A representation is called real if generators in that representation and its complex conjugate are related by a similarity transformation. Show that the Pauli matrices σ_i obey $\sigma_2 \sigma_i \sigma_2 = -\sigma_i^*$, and use this to prove that the fundamental representation of $SU(2)$ is real.