Physics 511: Electrodynamics

Spring 2018

Final Exam

May 9, 2018

Instructions:

- Do any 3 of the 4 problems. All problems carry equal weight.
- This is a closed-book exam.
- You may use personal notes including useful formulas and relations.

1- Consider a hollow sphere of radius R that carries a surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$, where σ_0 is constant and θ is the zenith angle in the spherical coordinate system.

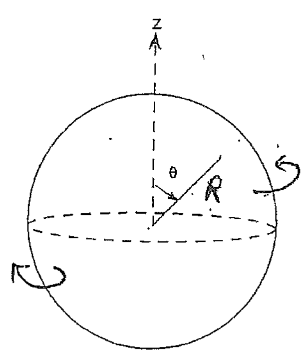
(a) Evaluate the total charge, and the electric dipole and quadrupole moments for this charge distribution.

(b) The northern and southern hemispheres now start oscillations about the z axis at frequency ω in opposite directions described by:

$$\phi(t) = +\phi_0 \sin\omega t \qquad 0 \leq \theta < \frac{\pi}{2}$$
$$\phi(t) = -\phi_0 \sin\omega t \qquad \frac{\pi}{2} < \theta \leq \pi,$$

where ϕ is the azimuth angle in the spherical coordinate system. Determine the lowest non-vanishing multipole moment that contributes to radiation fields.

(c) Find the angular distribution of radiation in the long wavelength limit.

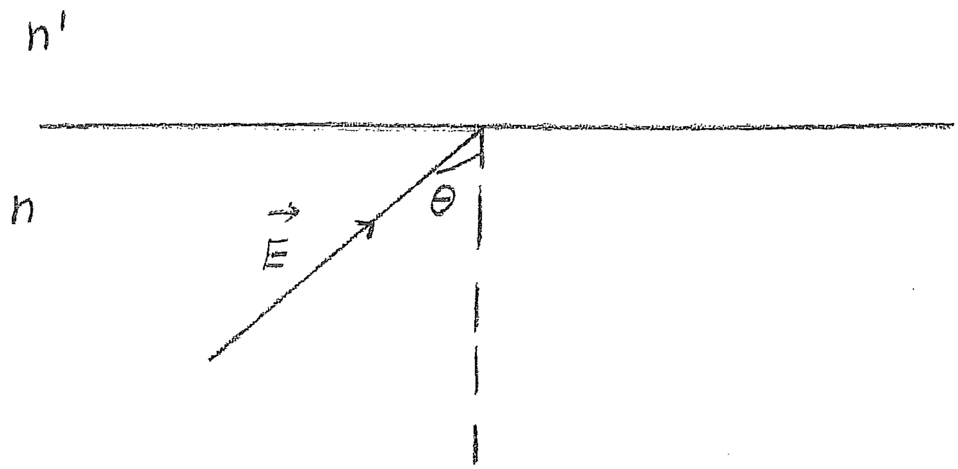


2- A plane wave of frequency ω is incident at an angle θ at the planar boundary between two non-magnetic lossless media with refraction indices n and n', where $n < n'$, as shown in the figure. The wave has positive circular polarization $\vec{E} = E\hat{e}_+$ (using coomplex notation).

(a) Write the electric field vector of the incident wave in a basis consisitng of unit vectors $\hat{s}$ and $\hat{p}$ corresponding to s and p polarizations rescpetively.

(b) Write the electric field vector of the reflected wave, choosing a suitable basis, and describe its polarization. How does the sense of rotation of the electric field vector varies with the incident angle θ?

(c) Find all incident angles for which the reflected wave is linearly or circularly polarized.



3- Two semi-infinite slabs of lossless dielectric material with refractive index n are separated by a narrow horizontal air gap of width d. A plane wave of frequency ω with s polarization is incident on the gap at an angle θ as shown in the figure.

(a) What is the minimum incident angle θ_C for which the wave is totally reflected by the bottom slab-air interface?

(b) For the rest of the problem assume that $\theta > \theta_C$. Construct the transfer matrix for propagation of the incident wave across the bottom interface, through the air gap, and across the top interface. Show that the power reflection coefficient for this system may be expressed as follows:

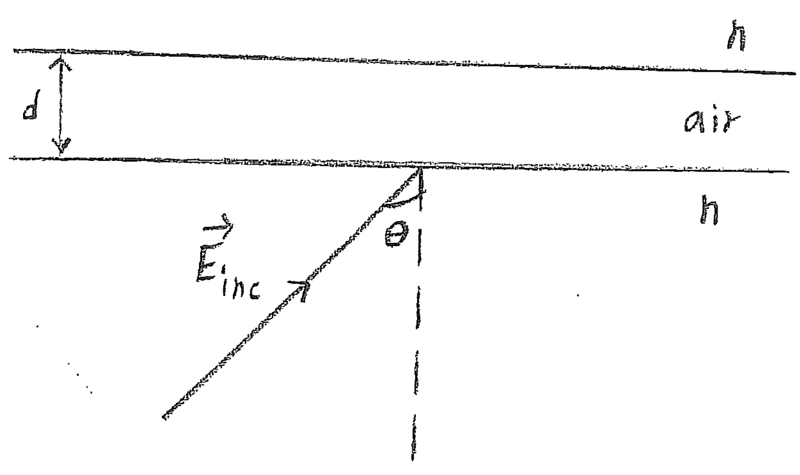
$$R = \frac{\cosh(2\omega\alpha d/c) - 1}{\cosh(2\omega\alpha d/c) - \mathrm{Re}(r_{21}^2)},$$

where

$$\alpha + \sqrt{\frac{\sin^2\theta}{\sin^2\theta_C} - 1} \quad , \quad r_{21} = -\frac{n\,\cos\theta - i\alpha}{n\,\cos\theta + i\alpha}.$$

(c) Show that for $d \gg c/\omega\alpha$, the net power transmission coefficient may be approximated as:

$$T = \frac{4n^2\alpha^2\,\cos^2\theta}{(n^2\,\cos^2\theta + \alpha^2)^2}\,\exp(-2\omega\alpha d/c)\,.$$



4- The power radiated per unit solid angle by a charge q in linear relativistic motion with velocity $\vec{v}$ is given by:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c}\frac{|\dot{\vec{\beta}} \times \hat{n}|^2}{(1-\vec{\beta}\cdot\hat{n})^5},$$

where $\vec{\beta} = \vec{v}/c$ and $\hat{n}$ is the unit observation vector.

(a) Show that the total radiated power is given by:

$$P(t) = \frac{q^2\dot{\beta}^2}{6\pi\epsilon_0 c}\gamma^6.$$

You may use the following integral identity:

$$\int_{-1}^{1}\frac{(1-x^2)}{(1-\beta x)^5}dx = \frac{4}{3(1-\beta^2)^3}.$$

(b) By noting that $P(t) = -\dot{\gamma}mc^2$, show that:

$$\frac{d\gamma}{dt} = -\frac{\beta^2}{\tau},$$

where $\tau^{-1} = 6\pi\epsilon_0 mc^3/q^2$ is a characteristic rate for radiative power loss from the charge.

(c) Integrate the equation in part (b) to obtain the following implicit solution for γ as a function of time:

$$\gamma - \gamma_0 + \frac{1}{2}\ln\frac{(\gamma-1)(\gamma_0+1)}{(\gamma+1)(\gamma_0-1)} = -\frac{t}{\tau},$$

where γ_0 is the value of γ at $t = 0$.