Physics 511: Electrodynamics

Spring 2019

Final Exam

May 8, 2019

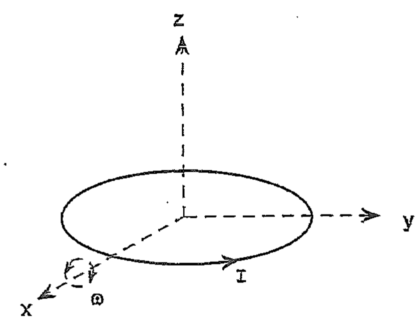
Instructions:

- Do any 1 of problems 1 and 2, and any 1 of problems 3 and 4. All problems carry the same weight.
- This is an open-book open-note exam.

1- A circular loop of radius R carrying a current I is centered at the origin in the xy plane. At time $t = 0$, the loop plane is set into a small-amplitude oscillation at frequency ω about the x axis so that the angle that the plane normal makes with the z axis at time t has the following form:

$$\theta(t) = \theta_0 \cos \omega t,$$

where $theta_0 << 1$. Such oscillations will result in radiation at frequency ω. In the following, do not make any assumptions about the size of the loop in relation to the wavelength of emission.



(a) Using the following coordinate transformation uUnder rotation by angle θ about the x axis:

$$x' = x \quad , \quad y' = y\cos\theta + z\sin\theta \quad , \quad z' = -y\sin\theta + z\cos\theta \, ,$$

show that a current element $I \, d\vec{x}'$ may be expressed as:

$$I \; d\vec{x}' = I \; d\vec{x} - \hat{x} \times I\theta_0 \; d\vec{x} \; \cos\omega t \, ,$$

where $\hat{x}$, as usual, is the unit vector along the x axis.

(b) Derive the following formula for the vector potential in the radiation zone in the complex notation:

$$\vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I\theta_0 \; \hat{x} \times \oint d\vec{l}' \exp(-ik\hat{n} \cdot \vec{x}') \, ,$$

where the vector line element around the loop, $d\vec{x}$, has been replaced by its more conventional $d\vec{l}'$ notation.

(c) Writing $d\vec{l}'$ in polar coordinates, show that the radiation-zone vector potential may be expressed as:

$$\vec{A}(\vec{x}) = -\hat{z} \, \frac{\mu_0}{4\pi} I\theta_0 R \, \frac{e^{ikr}}{r} \int_0^{2\pi} d\phi' \cos\phi' \exp[-ikR(n_x \cos\phi' + n_y \sin\phi')] \, ,$$

where n_x, n_y, n_z are the Cartesian components of the unit observation vector $\hat{n}$. Show that the oscillating current loop does not radiate along any direction in the yz plane, but it radiates along a direction in the xz plane.

2- A plane wave with circular polarization $\hat{e}_+ = (\hat{x} + i\hat{y})/\sqrt{2}$ is incident on a perfectly conducting sphere of radius a. In the following, assume the long wavelenght limit.

(a) Show that the differential scattering cross section, when summed over outgoing polarizations, is:

$$\frac{d\sigma}{d\Omega}(\theta) = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right] ,$$

where θ is the scattering angle.

(b) Find the maximum and minimum values of the differential cross section and the corresponding scattering angles.

(c) Calculate the total scattering cross section.

Hint: In part (a), you may use the result for arbitrary initial polarization $\hat{e}_0$:

$$\frac{d\sigma}{d\Omega}(\hat{e}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\hat{e}_0 \cdot \hat{n}|^2 - \frac{1}{4}|\hat{n} \cdot (\hat{n}_0 \times \hat{e}_0)|^2 - \hat{n}_0 \cdot \hat{n}\right] ,$$

where $\hat{n}_0$ and $\hat{n}$ are the directions of the incident and scattered radiations.

3- The power radiated per unit solid angle by a charge q in linear relativistic motion with velocity $\vec{v}$ is given by:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c} \frac{|\dot{\vec{\beta}} \times \hat{n}|^2}{(1 - \vec{\beta}\cdot\hat{n})^5},$$

where $\vec{\beta} = \vec{v}/c$ and $\hat{n}$ is the unit observation vector.

(a) Show that the total radiated power is given by:

$$P(t) = \frac{q^2\dot{\beta}^2}{6\pi\epsilon_0 c}\gamma^6.$$

You may use the following integral identity:

$$\int_{-1}^{1} \frac{(1-x^2)}{(1-\beta x)^5}dx = \frac{4}{3(1-\beta^2)^3}.$$

(b) By noting that $P(t) = -\dot{\gamma}mc^2$, show that:

$$\frac{d\gamma}{dt} = -\frac{\beta^2}{\tau},$$

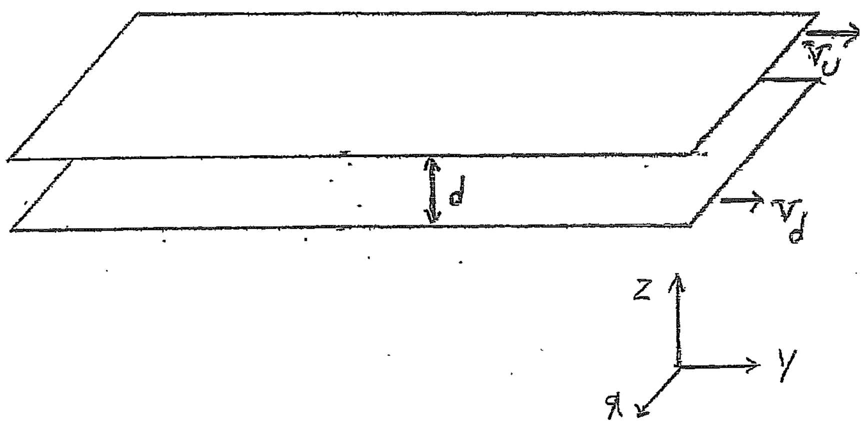
where $\tau^{-1} = 6\pi\epsilon_0 mc^3/q^2$ is a characteristic rate for radiative power loss from the charge.

(c) Integrate the equation in part (b) to obtain the following implicit solution for γ as a function of time:

$$\gamma - \gamma_0 + \frac{1}{2}\ln\frac{(\gamma-1)(\gamma_0+1)}{(\gamma+1)(\gamma_0-1)} = -\frac{t}{\tau},$$

where γ_0 is the value of γ at $t = 0$.

4- Two infinite planar sheet that are parallelt to the xy plane are seperated by a distance d as shown in the figure. The upper sheet has surface charge density σ_0 in its rest frame and is moving with speed v_u in the $+y$ direction. The lower sheet has surface charge density $2\sigma_0$ in its rest frame and is moving with speed v_d in the $+y$ direction.



(a) Find the electric and magnetis fields at all points in the frame shown in the figure. (Hint: You may find it easier to first find the fields associated with each sheet in its rest frame then use the transformaiton laws for the $\vec{E}$ and $\vec{B}$ fields under a boost.)

(b) Derive a relation between v_u and v_d such that all fields vanish in the region between the two sheets.

(c) Discuss the results of part (a) in the limit that $v_u,\ v_d \to 0$.