Physics 581 - Quantum Optics II
Lecture 4: Quasiprobability Functions of the Field

Motivation

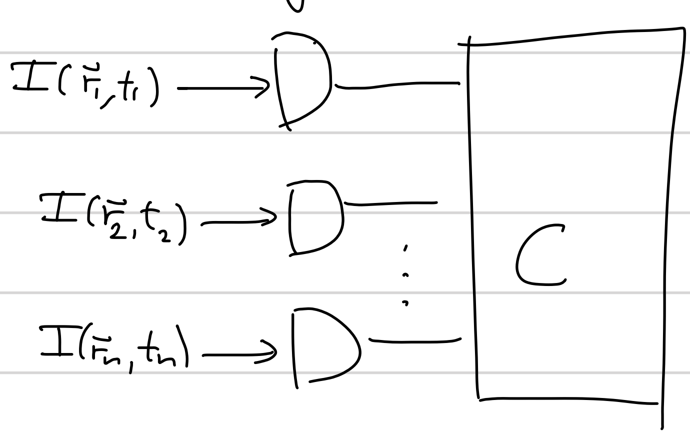
We have defined "classical light" as that whose state can be expressed as a statistical mixture of coherent state

$$\hat{\rho} = \int d^2\{\alpha_k\}\; P(\{\alpha_k\})\, |\{\alpha_k\}\rangle\langle\{\alpha_k\}|$$

$P(\{\alpha_k\})$ = Glauber-Sudharshan "P-representation." Classical $\Leftrightarrow P(\{\alpha_k\}) \geq 0$

This definition of classicality is defined by the condition that a photon counting experiment is describable by the semiclassical theory (i.e. classical wave inducing photo-electric emission).

E.g. Coincidence counting from n-detectors



$$C = \langle :\hat{I}(\vec{r}_1,t_1)\,\hat{I}(\vec{r}_2,t_2)\cdots\hat{I}(\vec{r}_n,t_n):\rangle \qquad \hat{I}(\vec{r},t) = \hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r},t)$$

normal order

$$= \sum(\;)\,\langle \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_n}\,\hat{a}_1\hat{a}_2\cdots\hat{a}_n\rangle$$

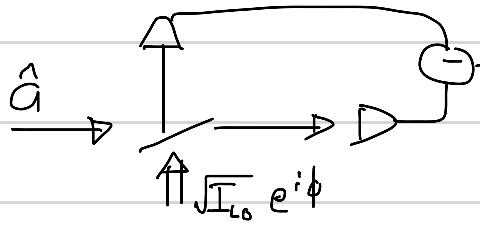
$$\hat{\rho}_{classical} \Rightarrow C = \overline{|E_1|^2\,|E_2|^2\cdots|E_n|^2}$$

Classical statical average

$$= \sum(\;)\int d^2\{\alpha_k\}\; P(\{\alpha_k\})\, |\alpha_1|^2|\alpha_2|^2\cdots|\alpha_n|^2$$

Any state of the field arising from a classical (maybe noise) current source is described by a statistical mixture of coherent states, so in this sense the light is classical.

We can also, however, consider other detection methods, and ask whether there is a classical, statistical description of the process. Consider, e.g., homodyne detection:



$$\longrightarrow \hat{I}_{out} \propto \hat{X}_\phi = \frac{e^{-i\phi}\hat{a} + e^{i\phi}\hat{a}^\dagger}{\sqrt{2}}$$

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)^2 \right\rangle = \frac{1}{2}\langle \underbrace{\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger}_{\text{symmetric order}} \rangle$$

A squeezed state is non-classical in the sense that it cannot be represented as a statistical mixture of coherent states (and thus does not arise as the radiation of a classical noise source), but note

$$\langle \hat{X}^2 \rangle = \int dX \, \frac{e^{-\frac{X^2}{2\Delta X^2}}}{\sqrt{2\pi \Delta X^2}} \, X^2 = \text{classical statistical}$$

average, even when $\Delta X^2 < \frac{1}{2}$ (vacuum limit). So in some sense, the squeezed vacuum can be understood in a classical statistical noise theory.

That is, there exists another way of representing the state in terms of coherent states, such that

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a}+\hat{a}^\dagger}{2}\right)^2 \right\rangle = \int d^2\alpha \, W(\alpha) \left(\frac{\alpha + \alpha^*}{\sqrt{2}}\right)^2$$

$W(\alpha) \equiv$ "Wigner function"

Quantum Mechanics in Phase Space

The fact that there are many different ways to represent a state in terms coherent states follows from the fact that they are "phase space eigenstates."

$$(\hat{X}+i\hat{P})\,|\alpha\rangle = (X+iP)\,|\alpha\rangle \qquad X=\sqrt{2}\,Re(\alpha) \quad P=\sqrt{2}\,Im(\alpha)$$

$|\alpha\rangle$ is not an eigenstate of a Hermitian operator, and thus, they are not orthogonal, $|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2} \Rightarrow \{|\alpha\rangle\}$ is an <u>over-complete</u> basis: $\int d^2\alpha\,|\alpha\rangle\langle\alpha| = \mathbb{1}$, but smaller sets will do.

The question of how to represent quantum states in phase space has a long history. From a fundamental perspective, it allows us to compare classical and quantum behavior. Classically, we can formulate Hamiltonian dynamics via Liouville theory.

Classical State: Probability distribution in phase space $P(\vec{q},\vec{p},t)$

Liouville equation for dynamics:

$$\frac{dP}{dt} = \underbrace{\{H(\vec{q},\vec{p}),\, P(\vec{q},\vec{p},t)\}}_{\text{Poisson bracket}} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial P}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial P}{\partial q_i}\right)$$

$$\overline{f(\vec{q},\vec{p})}\Big|_t = \int d^n\vec{q}\, d^n\vec{p}\; P(\vec{q},\vec{p},t)\, f(\vec{q},\vec{p})$$

expected value of some observable function of the canonical coordinates.

In quantum mechanics we cannot generally represent the state as a classical probability on phase space, because $\hat{X}$ and $\hat{P}$ don't commute. However, use the basis of phase space eigenstates to define quasi probability distributions such that

$$\underbrace{\langle f(\hat{X},\hat{P})\rangle_\sigma}_{\substack{\text{Quantum observable in phase}\\ \text{space with ordering } \sigma}} = \int dX\,dP\, \underbrace{W_{-\sigma}(X,P)}_{\substack{\text{Quasi-probability distribution}\\ \text{for ordering } -\sigma \text{ (sign to be explained)}}}\, f_\sigma(X,P)$$

The issue of operating ordering arises because $\hat{X}$ and $\hat{P}$ don't commute

Consider $f(\hat{X},\hat{P}) = (\hat{X}+\hat{P})^2 = \hat{X}^2 + \hat{P}^2 + \hat{X}\hat{P} + \hat{P}\hat{X}$: Symmetric

$= \hat{X}^2 + \hat{P}^2 + 2\hat{X}\hat{P} + i$: X then P

$= \hat{X}^2 + \hat{P}^2 + 2\hat{P}\hat{X} - i$ P then X

Substituting $\hat{X} \Rightarrow X$, $\hat{P} \Rightarrow P$ gives different functions $f_\sigma(X,P)$.

Often it is convenient to work with complex amplitudes α, α^* (quantum analogues: $\hat{a}, \hat{a}^\dagger$)

Seek $W_\sigma(\alpha)$ such that $\langle f(\hat{a},\hat{a}^\dagger)\rangle_\sigma = \int d^2\alpha\, W_{-\sigma}(\alpha)\, f_\sigma(\alpha,\alpha^*)$

The function $f_\sigma(\alpha,\alpha^*)$ is defined from power series expansions

Normal order: $f(\hat{a},\hat{a}^\dagger) = \sum C^{+1}_{nm} (\hat{a}^\dagger)^n (\hat{a})^m$

Anti-normal order: $= \sum C^{-1}_{nm} (\hat{a})^m (\hat{a}^\dagger)^n$

Symmetric Order $= \sum C^{0}_{nm} \{(\hat{a}^\dagger)^n (\hat{a})^m\}_{sym}$

$\{(\hat{a}^\dagger)^n(\hat{a})^m\}_{sym} = \binom{n+m}{m}^{-1} \sum$ Permutations of $(\hat{a}^{\dagger n}\hat{a}^m)$

For example: $\{\hat{a}^\dagger\hat{a}^2\}_{sym} = \hat{a}^\dagger\hat{a}^2 + \hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^2\hat{a}^\dagger$

Then $f_\sigma(\alpha, \alpha^*) = \sum_{n,m}' c^\sigma_{nm} (\alpha^*)^n \alpha^m$

<u>Example:</u> $f(\hat{a}^\dagger, \hat{a}) = \hat{n}^2 = \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}$

$$= \underbrace{(\hat{a}^\dagger)^2 (\hat{a})^2 + \hat{a}^\dagger \hat{a}}_{c^{+1}_{22} = 1, \; c^{+1}_{11} = 1} = \underbrace{(\hat{a})^2 (\hat{a}^\dagger)^2 - 3\hat{a}\hat{a}^\dagger + 1}_{c^{-1}_{22} = 1, \; c^{-1}_{11} = -3, \; c^{-1}_{00} = 1}$$

$$\Rightarrow f_{+1}(\alpha, \alpha^*) = |\alpha|^4 + |\alpha|^2, \quad f_{-1}(\alpha, \alpha^*) = |\alpha|^4 - 3|\alpha|^2 + 1$$

The quasiprobability distribution is the representation of the density operator with respect to a given operator ordering:

$$\hat{\rho} = \sum c^S_{nm} \{\hat{a}^{\dagger n} \hat{a}^m\}_{sym} = \sum' c^N_{nm} \hat{a}^{\dagger n} \hat{a}^m = \sum c^A_{nm} \hat{a}^m \hat{a}^{\dagger n}$$

$$\boxed{\begin{aligned} W(\alpha) &\equiv \tfrac{1}{\pi} \sum c^S_{nm} \alpha^{*n} \alpha^m \equiv \text{Wigner Function} \\ Q(\alpha) &= \tfrac{1}{\pi} \sum c^N_{nm} \alpha^{*n} \alpha^m \equiv \text{Husimi Function} \\ P(\alpha) &\equiv \tfrac{1}{\pi} \sum c^A_{nm} \alpha^{*n} \alpha^m \equiv \text{Glauber-Sudharshan P-function} \end{aligned}}$$

inserted for normalization

<u>Weyl Group and Operator Ordering</u>

Expressing observables as an order power series is tedious! We can achieve the operator ordering more compactly using the properties of the desplacement operator (Cahill & Glauber, Phy Rev <u>177</u>, 1857 (1969))

$$\hat{D}(\beta) = e^{\beta \hat{a}^\dagger - \beta^* \hat{a}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\beta \hat{a}^\dagger - \beta^* \hat{a})^n$$

$$= \mathbb{1} + (\beta \hat{a}^\dagger - \beta^* \hat{a}) + \tfrac{1}{2} \left(\beta^2 \hat{a}^{\dagger 2} - |\beta|^2 (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}) + \beta^* \hat{a}^2\right) + \cdots$$

$$\Rightarrow \boxed{\hat{D}(\beta) = \underbrace{\sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n! \, m!} \{(\hat{a}^\dagger)^n (\hat{a})^m\}_{sym}}_{\text{Symmetrically ordered power series in } \hat{a} \text{ and } \hat{a}^\dagger}}$$

We can use Baker-Campbell-Hausdorff $e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]} \leftarrow$ when c-number
to express $\hat{D}(\beta)$ in normal and antinormal order:

$$\hat{D}(\beta) = e^{-|\beta|^2/2} e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}} = e^{-|\beta|^2/2} :\hat{D}(\beta,\beta^*): \leftarrow \text{normal order}$$

$$\Rightarrow \boxed{:\hat{D}(\beta): = \underbrace{\sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n!\, m!} (\hat{a}^\dagger)^n (\hat{a})^m}_{\text{normally ordered power series in } \hat{a} \text{ and } \hat{a}^\dagger} = e^{+|\beta|^2/2} \hat{D}(\beta)}$$

$$\hat{D}(\beta) = e^{+|\beta|^2/2} e^{-\beta^* \hat{a}} e^{+\beta \hat{a}^\dagger} = e^{+|\beta|^2/2} \vdots \hat{D}(\beta,\beta^*) \vdots \leftarrow \text{anti-normal order}$$

$$\Rightarrow \boxed{\vdots \hat{D}(\alpha) \vdots = \underbrace{\sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n!\, m!} \hat{a}^m (\hat{a}^\dagger)^n}_{\text{anti-normally ordered power series in } \hat{a} \text{ and } \hat{a}^\dagger} = e^{-\frac{|\beta|^2}{2}} \hat{D}(\beta)}$$

This construction allows us to define a "continuum" between normally ordered and antinormally ordered power series

$$\hat{D}_\sigma(\beta) \equiv e^{\sigma|\beta|^2/2} \hat{D}(\beta)$$

$$\hat{D}_0(\beta) = e^{\beta\hat{a}^\dagger - \beta^*\hat{a}} \text{ (symmetric order)}; \quad \hat{D}_{+1}(\beta) = e^{\beta\hat{a}^\dagger} e^{-\beta^*\hat{a}} \text{ (normal order)}$$
$$\hat{D}_{-1}(\beta) = e^{-\beta^*\hat{a}} e^{\beta\hat{a}^\dagger} \text{ (antinormal order)}$$

Given the operator-ordered products generated by the displacement operator, we can express any operator as an operator-ordered power series by expressing them in terms the displacement operators. For that purpose, we turn to the "Weyl algebra."

Operator Bases

We know that plane waves $\frac{e^{-iX_0P}}{\sqrt{2\pi}}, \frac{e^{+iP_0X}}{\sqrt{2\pi}}$ and delta function $\delta(x-x_0)$, $\delta(p-p_0)$ form complete sets of functions for expanding functions in position and momentum space. These are the position and momentum representations of position and momentum eigenvectors.

$$\langle x|x_0\rangle = \delta(x-x_0), \quad \langle x|p_0\rangle = \frac{e^{ixP_0}}{\sqrt{2\pi}}, \quad \langle p|x_0\rangle = \frac{e^{-ipX_0}}{\sqrt{2\pi}}, \quad \langle p|p_0\rangle = \delta(p-p_0)$$

The sets $\{|x\rangle\langle x|\}$ $\{|p\rangle\langle p|\}$ form complete sets:

$$\int dx\, |x\rangle\langle x| = \int dp\, |p\rangle\langle p| = \hat{\mathbb{1}}$$

$$\Rightarrow |\psi\rangle = \int dp_0\, |p_0\rangle\langle p_0|\psi\rangle = \int dp_0\, |p_0\rangle \tilde{\psi}(p_0)$$

$$\psi(x) = \langle x|\psi\rangle = \int \frac{dp_0}{\sqrt{2\pi}}\, e^{iP_0X}\, \tilde{\psi}(p_0)$$

The Displacement Operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-i(x\hat{P} - p\hat{X})}$ is like a "plane wave for operators" in phase space. They ~~too~~ form a "basis for operators"

We can define an inner product between two operator via the trace

$$(\hat{A}|\hat{B}) \equiv \mathrm{Tr}(\hat{A}^\dagger\hat{B}) : \quad \text{Projection of } \hat{A} \text{ onto } \hat{B}$$

This follows simply for matrices; if we stack all the columns into one giant vector, this is the usual inner product for a complex vector space.

One can show (see homework) that $\hat{D}(\alpha)$ form and orthonormal complete set:

$$(\hat{D}(\alpha)|\hat{D}(\beta)) = \mathrm{Tr}(\hat{D}^\dagger(\alpha)|\hat{D}(\beta)) = \pi\delta^{(2)}(\alpha-\beta) = 2\pi\,\delta(x-x')\,\delta(p-p')$$

$$\hat{A} = \int \frac{d^2\beta}{\pi}\, \hat{D}(\beta)\, (\hat{D}(\beta)|\hat{A}) \qquad \beta = \frac{x'+ip'}{\sqrt{2}}, \quad \alpha = \frac{x+ip}{\sqrt{2}}$$

Note, this is actually an over complete set. Formally we call such bases "frames" so $\{\hat{D}(\alpha)\}$ is an "operator frame"

In partular $\hat{\rho} = \int \frac{d^2\beta}{\pi}\, \hat{D}(\beta)\, (\hat{D}(\beta)|\hat{\rho})$

We can now substitute the operator-order product expansion by $\hat{D}(\beta)$ is this representation.

$$\hat{\rho} = \sum_{n,m} \underbrace{\int \frac{d^2\beta}{\pi} \frac{\beta^n (-\beta^*)^m}{n!\, m!} (\hat{D}(\beta) | \hat{\rho})}_{C^S_{n,m}} \{\hat{a}^{\dagger n} (\hat{a})^m\}_{sym}$$

The symmetric quasi-probability representation of $\hat{\rho}$, or Wigner function

$$W(\alpha) = \frac{1}{\pi} \sum_{n,m} C^S_{n,m} \alpha^{*n} \alpha^m = \int \frac{d^2\beta}{\pi^2} \sum_{n,m} \frac{(\alpha^*\beta)^n}{n!} \frac{(\alpha\beta^*)^m}{m!} Tr(\hat{D}^\dagger(\beta)\hat{\rho})$$

$$\boxed{\begin{aligned} \Rightarrow W(\alpha) &= \int \frac{d^2\beta}{\pi^2} e^{\alpha^*\beta - \alpha\beta^*} Tr(\hat{\rho}\, \hat{D}^\dagger(\beta)) = \int \frac{d^2\beta}{\pi^2} e^{\alpha\beta^* - \alpha^*\beta} Tr(\hat{\rho}\, \hat{D}(\beta)) \\ &= \int \frac{d^2\beta}{\pi^2} Tr(\hat{\rho}\, e^{\beta(\hat{a}^\dagger - \alpha^*) - \beta^*(\hat{a} - \alpha)}) = \frac{1}{\pi} Tr(\hat{\rho}\, \hat{T}(\alpha)) \\ \hat{T}(\alpha) &= \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \beta\alpha^*} \hat{D}(\beta) = \int \frac{d^2\beta}{\pi} e^{\beta(\hat{a}^\dagger - \alpha^*) - \beta(\hat{a} - \alpha)} = \pi \{\delta(\hat{a}^\dagger - \alpha^*)\, \delta(\hat{a} - \alpha)\}_{sym} \end{aligned}}$$

$$\boxed{\begin{aligned} W(x,p) &= \int \frac{dx'dp'}{(2\pi)^2} Tr(\hat{\rho}\, e^{-ip'(\hat{x}-x) + ix'(\hat{p}-p)}) = \frac{1}{2\pi} Tr(\hat{\rho}\, \hat{T}(x,p)) \\ \hat{T}(x,p) &= \int \frac{dx'dp'}{2\pi} e^{-ix'p + p'x} \hat{D}(x,p) = \int \frac{dx'dp'}{2\pi} e^{-ip'(\hat{x}-x) + ix'(\hat{p}-p)} = 2\pi \{\delta(\hat{x}-x)\, \delta(\hat{p}-p)\}_{sym} \end{aligned}}$$

The operator $\hat{T}(\alpha)$ is the Fourier transform of the Displacement operator

$$\boxed{\text{As } \hat{D}(\alpha) \text{ is a plane wave} \Leftrightarrow \hat{T}(\alpha) \text{ is } \cancel{\text{to}} \text{ delta function}}$$

The expression for the Wigner function makes intuitive sense.

$$W(\alpha) = \frac{1}{\pi} Tr(\hat{\rho}\, \hat{T}(\alpha)) = Tr(\hat{\rho}\, \{\delta(\hat{a}-\alpha)\, \delta(\hat{a}^\dagger - \alpha^*)\}_{sym})$$

We seek to replace $\hat{a} \Rightarrow \alpha$, and $\hat{a}^\dagger \Rightarrow \alpha^*$ in the density operator in a symmetric way. The operator $\hat{T}(\alpha)$ gives us a way to achieve this.

We can take the inverse Fourier transform to express the density operator in terms of the Wigner function:

$$\boxed{\begin{aligned}\hat{\rho} &= \int d^2\alpha\, W(\alpha)\, \hat{T}(\alpha) = \int d^2\alpha\, W(\alpha) \{\pi\, \delta(\alpha - \hat{a})\, \delta(\alpha^* - \hat{a}^\dagger)\}_{sym} \\ &= \int dX dP\, W(X,P) \{2\pi\, \delta(X - \hat{X})\, \delta(P - \hat{P})\}_{sym}\end{aligned}}$$

This relation between the distribution function on phase space and the quantum operator is known as Weyl quantization after the mathematical physicist Herman Weyl who first wrote it down, and the transform $\hat{\rho} \Leftrightarrow W(\alpha)$ is known as the Wigner-Weyl transform.

We can generalize the Wigner-Weyl transform for any operator, not just the density operator. Formally, the operators $\hat{T}(\alpha)$ form an (over) complete set of operators (an operator frame). They are Hermitian and orthogonal

$$\hat{T}^\dagger(\alpha) = \hat{T}(\alpha), \qquad (\hat{T}(\alpha) | \hat{T}(\alpha')) = \mathrm{Tr}(\hat{T}(\alpha)\hat{T}(\alpha')) = \pi\, \delta^{(2)}(\alpha - \alpha')$$

$$\hat{A} = \int \frac{d^2\alpha}{\pi}\, \hat{T}(\alpha)\, \underbrace{\mathrm{Tr}(\hat{A}\,\hat{T}(\alpha))}_{A_{sym}(\alpha,\alpha^*)} \Longleftrightarrow A_{sym}(\alpha,\alpha^*) = \mathrm{Tr}(\hat{A}\,\hat{T}(\alpha))$$

$A_{sym}(\alpha,\alpha^*)$ is the symmetric representation of $\hat{A}$ on phase space

Thus, in the same way that we can use the position or momentum representations to calculate probability of measurement outcomes and expectation values, we can use the phase space representation (here the symmetric Wigner representation.

$$\boxed{\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \int d^2\alpha\, A_{sym}(\alpha,\alpha^*)\, W(\alpha)}$$

General operator ordered phase space distributions

Let us define: $\hat{D}_\sigma(\beta) \equiv e^{\sigma\frac{|\beta|^2}{2}} \hat{D}(\beta) = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n!\, m!} \{\hat{a}^{\dagger n} \hat{a}^m\}_\sigma$

$\sigma = +1$: Normal order, $\sigma = 0$: Symmetric order, $\sigma = -1$: Anti-normal order

From this we can define the operator order

$$\hat{T}_\sigma(\alpha) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \beta^*\alpha} \hat{D}_\sigma(\beta) = \pi \{\delta(\hat{a}^\dagger - \alpha^*)\, \delta(\hat{a} - \alpha)\}_\sigma$$

$$\hat{D}_\sigma(\beta) = \int \frac{d^2\alpha}{\pi} e^{\beta\alpha^* - \alpha^*\beta} \hat{T}_\sigma(\alpha) = \{e^{\beta\hat{a}^\dagger - \beta^*\hat{a}}\}_\sigma$$

We can use this to find the general operator ordered phase-space representation of $\hat{\rho}$ (or any operator) using our expansion in terms of the $\hat{D}(\alpha)$

$$\hat{\rho} = \int \frac{d^2\beta}{\pi} \underbrace{\hat{D}(\beta)\, Tr(\hat{\rho}\, \hat{D}^\dagger(\beta))}_{e^{+\sigma\frac{|\beta|^2}{2}} e^{-\sigma\frac{|\beta|^2}{2}}} = \int \frac{d^2\beta}{\pi} \hat{D}_{+\sigma}(\beta)\, Tr(\hat{\rho}\, \hat{D}^\dagger_{-\sigma}(\beta))$$

Now, $\hat{D}_\sigma(\beta) = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n!\, m!} \{(\hat{a}^\dagger)^n (\hat{a})^m\}_\sigma \leftarrow$ σ-ordered product

$$\Rightarrow \hat{\rho} = \sum_{n,m} c^\sigma_{nm} \{(\hat{a}^\dagger)^n (\hat{a})^m\}_\sigma, \qquad c^\sigma_{n,m} = \int \frac{d^2\beta}{\pi} \frac{\beta^n (-\beta^*)^m}{n!\, m!} Tr(\hat{\rho}\, \hat{D}^\dagger_{-\sigma}(\beta))$$

$$\Rightarrow W_\sigma(\alpha) = \frac{1}{\pi} \sum_{n,m} c^\sigma_{nm} \alpha^{*n} \alpha^m = \int \frac{d^2\beta}{\pi^2} e^{\alpha^*\beta - \beta^*\alpha} Tr(\hat{\rho}\, \hat{D}^\dagger_{-\sigma}(\beta))$$

$$= \int \frac{d^2\beta}{\pi^2} e^{\beta^*\alpha - \alpha^*\beta} Tr(\hat{\rho}\, \hat{D}_{-\sigma}(\beta))$$

$$\Rightarrow \boxed{\begin{aligned} &W_\sigma(\alpha) = \frac{1}{\pi} Tr(\hat{\rho}\, \hat{T}_{-\sigma}(\alpha)), \quad \sigma\text{-ordered representation} \\ &\hat{T}_{-\sigma}(\alpha) \equiv \int \frac{d^2\beta}{\pi} e^{\beta^*\alpha - \alpha^*\beta} \hat{D}_{-\sigma}(\beta) = \pi\{\delta(\hat{a}^\dagger - \alpha^*)\, \delta(\hat{a} - \alpha)\}_{-\sigma} \end{aligned}}$$

<u>Note</u>: To obtain the σ-order representation of $\hat{\rho}$ we project on the $\underline{\underline{-\sigma}}$ representation of $\hat{T}_{-\sigma}(\alpha)$. This is the duality of sym/antisym representations

The operators $\hat{T}_\sigma(\alpha)$ allow us to transform and operator $\hat{A}$ into a phase-space representation of a desire operator ordering. This is now easily seen in hindsight.

$$\hat{A} = \sum C^{+1}_{nm} (\hat{a}^\dagger)^n (\hat{a})^m \underset{\substack{\hat{a}\to\alpha \\ \hat{a}^\dagger \to \alpha^*}}{\Longrightarrow} A_{+1}(\alpha) = \sum_{n,m} C^{+1}_{n,m} (\alpha^*)^n (\alpha^m)$$

We achieve this a delta function $A_{+1}(\alpha) = \mathrm{Tr}(\hat{A}\, \hat{T}_{-1}(\alpha)) = \pi \mathrm{Tr}(\hat{A}\, \delta(\hat{a}-\alpha)\, \delta(\hat{a}^\dagger - \alpha^*))$

$$\Rightarrow A_{+1}(\alpha) = \sum_{n,m} C^{+1}_{n,m}\, \pi \mathrm{Tr}(\hat{a}^{\dagger n} \hat{a}^m \delta(\hat{a}-\alpha)\, \delta(\hat{a}^\dagger - \alpha^*)) =$$

$$= \sum_{n,m} C^{+1}_{nm} \pi \mathrm{Tr}(\delta(\hat{a}^\dagger - \alpha^*)\, \hat{a}^{\dagger n} \hat{a}^m \delta(\hat{a}-\alpha)) = \sum_{n,m} C^{+1}_{nm} \pi \int \frac{d^2\alpha'}{\pi} \langle \alpha' | \delta(\hat{a}^\dagger - \alpha)\, \hat{a}^{\dagger n} \hat{a}^m \delta(\hat{a}-\alpha) |\alpha\rangle$$

$$= \sum_{n,m} C^{+1}_{nm} \int d\alpha'\, d\alpha'^* (\alpha'^*)^n (\alpha')^m \, \delta(\alpha^* - \alpha'^*)\, \delta(\alpha - \alpha') = \sum_{nm} C^{+1}_{nm} \alpha^{*n} \alpha^m = A_{+1}(\alpha) \checkmark$$

Similarly $A_{-1}(\alpha) = \mathrm{Tr}(\hat{A}\, \hat{T}_{+1}(\alpha)) = \pi \mathrm{Tr}(\hat{A}\, \delta(\hat{a}^\dagger - \alpha^*)\, \delta(\hat{a}-\alpha)) = \sum_{n,m} C^{-1}_{nm} \pi \mathrm{Tr}(\hat{a}^{\dagger m} \delta(\hat{a}^\dagger - \alpha^*) \delta(\hat{a}-\alpha) \hat{a}^n)$

The generalization thus follows for symmetric ordering or any value of σ.

Formally $\hat{T}_\sigma(\alpha)$ forms an operator frame, and $\hat{T}_{-\sigma}(\alpha)$ is the dual frame

$$(\hat{T}_{-\sigma}(\alpha) | \hat{T}_\sigma(\alpha')) = \mathrm{Tr}(\hat{T}^\dagger_{-\sigma}(\alpha)\, \hat{T}(\alpha')) = \mathrm{Tr}(\hat{T}_{-\sigma}(\alpha)\, \hat{T}_\sigma(\alpha')) = \pi\, \delta^{(2)}(\alpha - \alpha')$$

$\Rightarrow$ An operator $\hat{A} = \int \frac{d^2\alpha}{\pi} \mathrm{Tr}(\hat{A}\, \hat{T}_{-\sigma}(\alpha))\, \hat{T}_\sigma(\alpha)$

In particular $\boxed{\hat{\rho} = \int d^2\alpha\, W_\sigma(\alpha)\, \hat{T}_\sigma(\alpha) = \int d^2\alpha\, \pi W_\sigma(\alpha) \{\delta(\hat{a}^\dagger - \alpha^*)\, \delta(\hat{a}-\alpha)\}_\sigma}$

One can show $\hat{T}_{-1}(\alpha) = |\alpha\rangle\langle\alpha|$ (see home work)

$\Rightarrow$ $W_{+1}(\alpha) = \frac{1}{\pi} \mathrm{Tr}(\hat{\rho}\, \hat{T}_{-1}(\alpha)) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle \equiv Q(\alpha)$ Husmi function

Normally order representation of $\hat{\rho}$

Also note $\hat{\rho} = \int d^2\alpha\, W_{-1}(\alpha)\, \hat{T}_{-1}(\alpha) = \int d^2\alpha\, W_{-1}(\alpha)\, |\alpha\rangle\langle\alpha|$

$\Rightarrow$ $W_{-1}(\alpha) = P(\alpha)$ = Glauber - Sudarshan P-representation

We can now calculate the expectation of an operator as an integral over phase space:

$$\boxed{\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \int d^2\alpha \; W_{-\sigma}(\alpha)\, A_{\sigma}(\alpha)}$$

This important relation states a kind of duality relation. To calculate the expectation value, we averge the $-\sigma$-order representation of $\hat{\rho}$ with the $+\sigma$-representation of $\hat{A}$, e.g. normal ordered $\hat{\rho}$ with anti-normally ordered $\hat{A}$, vice-versa, or symmetric for both.

This is the major result: Any operator function of $\hat{X}$ and $\hat{P}$, or equivalently $\hat{a}$ and $\hat{a}^{+}$, has different representation in phase space. this can be used to calculate different operator-order products.

$$\langle \{ \hat{a}^{+n} \hat{a}^{m} \}_{sym} \rangle = \int d^2\alpha \, (\alpha^*)^n \alpha^m \, W(\alpha) \qquad \text{Wigner}$$

$$\langle (\hat{a}^{+})^n (\hat{a})^m \rangle = \int d^2\alpha \, (\alpha^*)^n (\alpha)^m \, P(\alpha) \qquad \text{Glauber}$$

$$\langle (\hat{a})^m (\hat{a}^{+})^n \rangle = \int d^2\alpha \, (\alpha^*)^n (\alpha^m) \, Q(\alpha) \qquad \text{Husmi}$$

As an example, consider Glauber's expression for n-fold coincidence. For a single mode, the expectation value is $\langle : \hat{n}^n : \rangle$. As this operator is <u>normally ordered</u>, we can calculate this expectation value by averaging it with the <u>antinormally-order</u> representation of $\hat{\rho}$, i.e. the P-function

$$: \hat{n}^n :_{\sigma=1} = \alpha^{*n} \alpha^n = |\alpha|^{2n} \qquad \hat{\rho}\,\Big|_{\sigma=-1} = P(\alpha)$$

$$\langle : \hat{n}^n : \rangle = \int d^2\alpha \; |\alpha|^{2n} \, P(\alpha)$$

In contrast, The Wigner function gives us the distribution function for calculating averages of quadratures which are <u>symmetric</u> functions of $\hat{a}$ and $\hat{a}^{+}$

e.g. $f(\hat{X}) = \hat{X}^2 = \frac{1}{2}(\hat{a} + \hat{a}^{+})^2 = \frac{1}{2}(\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a})$

$$\Rightarrow \langle f(\hat{X}) \rangle = \int dX\, dP \; W(X,P)\, f(X) = \int dX \; \underset{}{\overset{\text{marginal}}{\mathcal{P}(X)}}\, f(X)$$

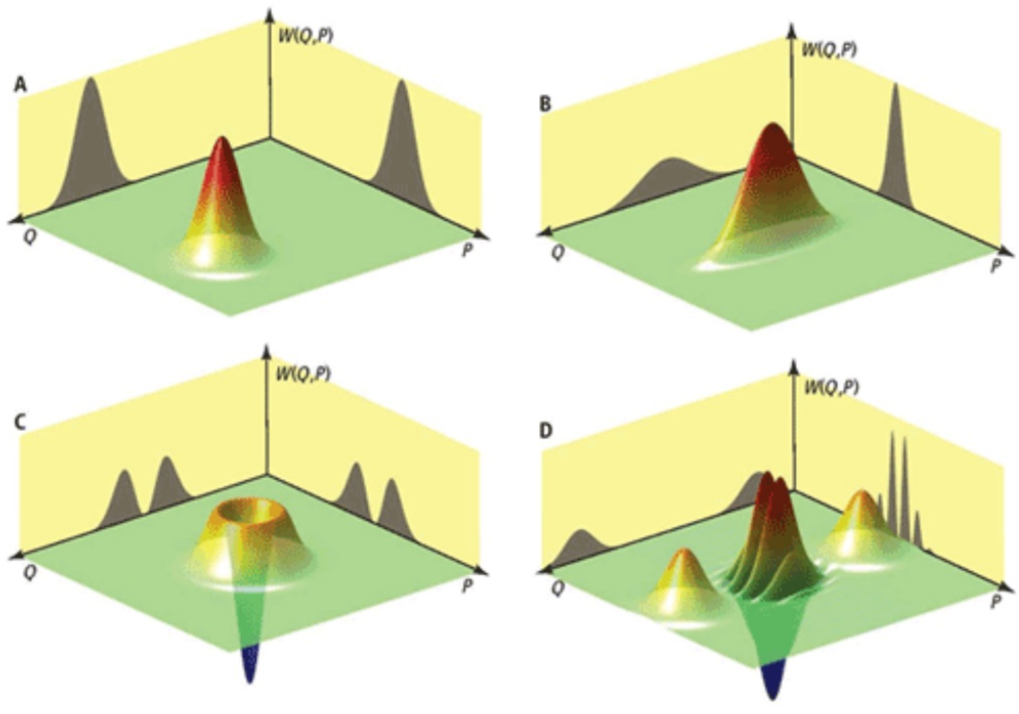
Properties of the quasiprobability functions

Real: $W_\sigma^*(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left(\hat\rho\, \hat{T}_{-\sigma}(\alpha)\right)^* = \frac{1}{\pi}\mathrm{Tr}\left(\hat\rho^\dagger\, \hat{T}^\dagger_{-\sigma}(\alpha)\right) = W_\sigma(\alpha)$

Normalization: $\mathrm{Tr}(\hat\rho) = \int d^2\alpha\; W_\sigma(\alpha)\, \underbrace{\mathrm{Tr}\left(\hat{T}_\sigma(\alpha)\right)}_{1} = \int d^2\alpha\; W_\sigma(\alpha) = 1$

Negativity:

While $W_\sigma(\alpha)$ looks like a probability on phase space, and is normalized like one, in general it is not. $W_\sigma(\alpha)$, can sometimes be negative. For this reason $W_\sigma(\alpha)$ is called a "quasi" probability distribution; true probability distributions can never be negative. The existence of "negative probabilities" in the quasiprobability distribution is the smoking gun of nonclassical behavior. It indicates the impossibility of assigning a joint-probability of measuring noncommuting operators $\hat{X}$ and $\hat{P}$. The existence of negativity in the quasiprobability representations is now seen as a resource for achieving information processing tasks that would not be possible solely with devices described by classical physics and classical statistics. Note, this negativity, or even singular behavior, might appear in the representation of the state, $W_\sigma(\alpha)$ or in the observable $A_{-\sigma}(\alpha)$. Only when all aspects of the system, state and observation relating to measurement, can be described by classical distribution can we call the system "classical."



Shown here are the Wigner functions for a variety of quantum states for one mode (taken from a paper by P. Grangier). The marginals $Pr(X)$ and $Pr(P)$ are shown as shadows on the two walls. A: Coherent State: $W(\alpha)$ = Gaussian wavepacket centered at α. The marginals are equal width Gaussian prob distribution. B: Squeezed State: Like the coherent state $W(\alpha)$ is everywhere positive; the marginals have unequal widths. From the perspective of the Wigner function, which described homodyne measurements, the squeezed state is classical. These are the only pure states that have this property.

- Hudson-Piquet Theorem: The Wigner function of a pure state is everywhere positive (and thus quasiclassical) iff it is Gaussian $\Leftrightarrow$ $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ for some α, ξ $\Rightarrow$ squeezed/coherent state. (To be proven later).

C: Fock state $|n\rangle = |1\rangle$. The Wigner function must be rotationally symmetric in phase space because it is an eigenstate of the number operator, the generator of rotations. The Wigner function is concentrated around the circle corresponding to the classical orbit with energy $(n+\frac{1}{2})\hbar\omega$ with $n=1$. The marginals are the squares of the familiar $n=1$ wavefunctions of the simple harmonic oscillator: 1st order Hermite polynomial times a Gaussian. These functions have one node, as the must. For this to be true, the Wigner function must be negative. Fock states are very nonclassical

D: "Schrödinger cat state": This is a superposition of two coherent states along X, $|\psi\rangle = \mathcal{N}(|\alpha = X\rangle + |\alpha = -X\rangle)$, separated by a distance large compared to the zero-point width. It is called a Schrödinger cat state because it represents a distinguishable superposition of two classical alternatives (really to be like Schrödinger's thought experiment, $|\alpha| \gg 1 \Rightarrow$ macroscopic). The marginal in X, shows the probability of being at X or $-X$. The marginal in P shows the interference between these alternatives. It is the double slit diffraction pattern. These interference fringes manifest in the Wigner function as seen, with large negative values. The Schrödinger cat state is very nonclassical. The larger the separation between X and $-X$, the larger and more rapidly these fringes that make the state more and more nonclassical.

Characteristic function

An important tool in classical statistics is the Fourier transform of the probability distribution, known as the Characteristic function:

$$\boxed{\chi(y) \equiv \int_{-\infty}^{\infty} dx\, e^{-iyx} P(x) = \overline{e^{-iyx}}}$$

The characteristic function is used to calculate moments of the distribution:

$$\overline{x^n} = \int_{-\infty}^{\infty} dx\, x^n P(x) = i^n \frac{d^n \chi}{dy^n}\Big|_{y=0}$$

For a joint probability distribution over multiple random variables $P(x_1, x_2, \dots, x_N) \equiv P(\vec{x})$

$$\chi(\vec{y}) \equiv \int d^N x\, e^{-i\vec{y}\cdot\vec{x}} P(\vec{x}),$$

$$\Rightarrow \overline{x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}} \equiv \frac{\partial^{n_1}}{\partial(-iy_1)^{n_1}} \frac{\partial^{n_2}}{\partial(-iy_2)^{n_2}} \cdots \frac{\partial^{n_N}}{\partial(-iy_N)^{n_N}} \chi(\vec{y})\Big|_{\vec{y}=0}$$

We define the characterist function of the quasiprobabilty distribution

$$\chi_\sigma(\beta) \equiv \int \frac{d^2\alpha}{\pi} W_\sigma(\alpha)\, e^{\beta\alpha^* - \beta^*\alpha} = \int \frac{d^2\alpha}{\pi} \mathrm{Tr}(\hat{\rho}\, \hat{T}_{-\sigma}(\alpha))\, e^{\beta\alpha^* - \beta^*\alpha}$$

$$= \mathrm{Tr}\left(\hat{\rho} \int \frac{d^2\alpha}{\pi} \hat{T}_{-\sigma}(\alpha)\, e^{\beta\alpha^* - \beta^*\alpha}\right) = \mathrm{Tr}(\hat{\rho}\, \hat{D}_{-\sigma}(\beta))$$

$$\Rightarrow \boxed{\chi_\sigma(\beta) = \langle \hat{D}_{-\sigma}(\beta) \rangle}$$

$$\langle \{\hat{a}^{\dagger n} \hat{a}^m\}_\sigma \rangle = \frac{\partial^n}{\partial\beta^n} \frac{\partial^m}{(-\partial\beta^*)^m} \chi_{-\sigma}\Big|_{\beta=0}$$

$$\Rightarrow W_\sigma(\alpha) = \int \frac{d^2\beta}{\pi} \chi_\sigma(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

Existence: So far we have not actually proven that any of the operator-ordered power series actually converge. To prove this we turn to the Fourier analysis.

The class of function for which the Fourier transform exists are the Square normalizable functions on phase space

$$\|f\|^2 \equiv \int \frac{d^2\alpha}{\pi} |f(\alpha)|^2 \quad : \text{Finite}$$

If $\|f\|$ is finite $\tilde{f}(\beta) = \int \frac{d^2\alpha}{\pi} f(\alpha)\, e^{\beta\alpha^* - \alpha^*\beta}$ exists, and

$$\|f\|^2 = \int \frac{d^2\beta}{\pi} |\tilde{f}(\beta)|^2$$

Bounded operators: $\|\hat{A}\|^2 \equiv (\hat{A}|\hat{A}) = \mathrm{Tr}(\hat{A}^\dagger \hat{A}) = \int \frac{d^2\alpha}{\pi} A^*_{\sigma}(\alpha)\, A_{-\sigma}(\alpha)$

(Hilbert-Schmidt norm)

$$\|\hat{A}\|^2 = \int \frac{d^2\alpha}{\pi} |A_0(\alpha)|^2 = \int \frac{d^2\beta}{\pi} |\tilde{A}_0(\beta)|^2 : \text{Bounded if finite}$$

Wigner function (W representation)

$\hat{\rho}$ is a bounded operator $\Rightarrow \mathrm{Tr}(\hat{\rho}^2) = \int \frac{d^2\beta}{\pi} |\chi_0(\beta)|^2 \leq 1$

$\Rightarrow$ The characteristic function of Wigner function is in $\mathcal{L}_2(\mathbb{R}^2)$

$\Rightarrow$ $\boxed{W(\alpha) \text{ always exists for a physical state}}$

More generally the symmetrically ordered Weyl symbol $A_0(\alpha)$ always exists

Husmi distribution (Q-representation)

$$Q(\alpha) = W_{+1}(\alpha) = \int \frac{d^2\beta}{\pi}\, \chi_{+1}(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{+1}(\beta) = \mathrm{Tr}(\hat{\rho}\, \hat{D}_{-1}(\beta)) = e^{-|\beta|^2/2}\, \chi_0(\beta)$$

$\Rightarrow$ The characteristic function of Q is always square normalizable

$\Rightarrow$ $\boxed{Q(\alpha) \text{ always exists for a physical state}}$

More generally the normally ordered Weyl symbol $\hat{A}_{+1}(\alpha)$ always exists

Glauber distribution (P-representation)

$$P(\alpha) = W_{-1}(\alpha) = \int \frac{d^2\beta}{\pi}\, \chi_{+1}(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{+1}(\beta) = \mathrm{Tr}(\hat{\rho}\, \hat{D}_{+1}(\beta)) = e^{+|\beta|^2/2}\, \chi_0(\beta) : \text{ Generally } \underline{\text{unbounded}}$$

$\Rightarrow$ $\boxed{P(\alpha) \text{ Only exists if } \chi_0(\beta) \text{ falls of at least as fast as fast as } e^{-|\beta|^2/2}.}$

More generally, the antisymmetrically ordered Weyl symbol $\hat{A}_{-1}(\alpha)$ doesnt exist as a "tempered" function.

Properties of Q, P, W

Husmi Q-representation

$$Q(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\,\hat{T}_{-1}(\alpha)\right) = \frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\,|\alpha\rangle\langle\alpha|\right)$$

$$\Rightarrow \boxed{Q(\alpha) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi} \geq 0 \Rightarrow \text{the Husimi distribution is true probability distribution}}$$

Note for a pure state $Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2$. This is the most natural way to define a probability distribution on phase space, as Husmi did in 1940

While $Q(\alpha)$ always exists and is always positive, it is rarely usely for making predictions. This is because

$$\langle\hat{A}\rangle = \int d^2\alpha\; Q(\alpha)\, A_{-1}(\alpha)$$ ← antisymmetric Weyl symbol

Because $A_{-1}(\alpha)$ rarely exists, this expression is not useful. That's what we expect, otherwise all of QM in phase space reduces to a classical model.

Glauber-Sudharshan P-representation

$$P(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\,\hat{T}_{+1}(\alpha)\right) \Rightarrow \hat{\rho} = \int d^2\alpha\; P(\alpha)\, \hat{T}_{-1}(\alpha)$$

$$\Rightarrow \hat{\rho} = \int d^2\alpha\; P(\alpha)\, |\alpha\rangle\langle\alpha| :$$ Statistical mixture of coherent states

$$\langle\hat{A}\rangle = \int d^2\alpha\; P(\alpha)\, \langle\alpha|\hat{A}|\alpha\rangle = \int d^2\alpha\; P(\alpha)\, A_{+1}(\alpha)$$ ← always exists

$\Rightarrow$ $P(\alpha)$ doesn't always exist, but when it does and ≥ 0 the state is essential classical

Wigner-Function W-representation

$$\langle \hat{A} \rangle = \int d^2\alpha \; W(\alpha)\, A_0(\alpha)$$

The Wigner function always exists, and it always is useful for making predictions. Those predictions are only equivalent to a classical statistical theory if $W(\alpha) \geq 0 \;\; \forall \alpha$.

Other properties: (Proved in homework)

- Wigner's original formula: $W(X,P) = \frac{1}{2} W(\alpha)$ Because $\alpha = \frac{1}{\sqrt{2}}(X+iP)$

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi}\, e^{-iPY} \langle X+\tfrac{Y}{2} |\, \hat{\rho}\, | X-\tfrac{Y}{2} \rangle = \int_{-\infty}^{\infty} \frac{dY}{2\pi}\, \psi^*(X+\tfrac{Y}{2})\, \psi(X-\tfrac{Y}{2})\, e^{-iPY}$$

- Marginals: (probability distribution of one variable in joint distrib)

$$P(X) \equiv \int_{-\infty}^{\infty} dP \; W(X,P) = \langle X | \hat{\rho} | X \rangle \;\; (= |\langle X | \psi \rangle|^2 \text{ pure state})$$

$$P(P) \equiv \int_{-\infty}^{\infty} dX \; W(X,P) = \langle P | \hat{\rho} | P \rangle \;\; (= |\langle P | \psi \rangle|^2)$$

Generally, $P(X_\theta) = \int_{-\infty}^{\infty} dP_\theta \; W(X,P) = \langle X_\theta | \hat{\rho} | X_\theta \rangle$ quadrature

Thus, $\langle \Delta \hat{X}_\theta^2 \rangle = \int d^2\alpha \; \Delta X_\theta^2 \; W(\alpha)$

$\Rightarrow$ moments of quadratures are the moments of the Wigner function.

Relations between P, Q, W

$$W(\alpha) = \frac{2}{\pi}\int P(\beta)\, e^{-2|\beta-\alpha|^2} d^2\beta$$

$$Q(\alpha) = \frac{2}{\pi}\int W(\beta)\, e^{-2|\beta-\alpha|^2} d\beta = \frac{1}{\pi}\int P(\beta)\, e^{-|\alpha-\beta|^2} d\beta$$

$\Rightarrow$ W is a "smoothed" version of P, Q is a "smoothed" version of W. Q is the most "coarse grained" (least rippled).

Examples

- Coherent State $|\alpha_0\rangle$; $\rho = |\alpha_0\rangle\langle\alpha_0|$

$$\boxed{Q(\alpha) = \frac{1}{\pi}|\langle\alpha_0|\alpha\rangle|^2 = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2} \Rightarrow \frac{1}{2\pi} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} e^{-\frac{(p-p_0)^2}{2\sigma_p^2}} , \quad \sigma_x^2 = \sigma_p^2 = 1}$$

There are many routes to determining other distributions
We can read off $P(\alpha)$ by eye

$$\hat{\rho} = \int d\alpha\, P(\alpha)|\alpha\rangle\langle\alpha| = |\alpha_0\rangle\langle\alpha_0| \Rightarrow \boxed{P(\alpha) = \delta(\alpha-\alpha_0)}$$

$$W(\alpha) = 2\int \frac{d^2\beta}{\pi} P(\beta)\, e^{-2|\beta-\alpha|^2}$$

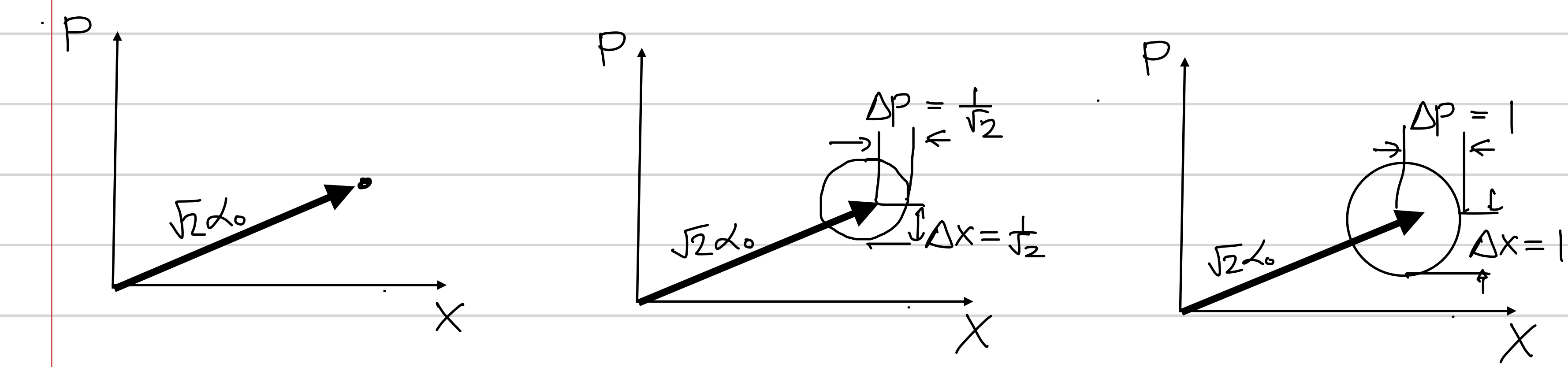
$$\Rightarrow \boxed{W(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2} \Rightarrow \frac{1}{\sqrt{2\pi\Delta x^2}} e^{-\frac{(x-x_0)^2}{2\Delta x^2}} \frac{1}{\sqrt{2\pi\Delta p^2}} e^{-\frac{(p-p_0)^2}{2\Delta p^2}} \quad \Delta x = \Delta p = \frac{1}{\sqrt{2}}}$$

Alternative: Characteristic function

$$\chi_0(\beta) = Tr(\hat{\rho}\,\hat{D}(\beta)) = \langle\alpha_0|\hat{D}(\beta)|\alpha_0\rangle = e^{-|\beta|^2/2}\, e^{\beta\alpha_0^* - \alpha_0^*\beta}$$

$$W_0(\alpha) = \int \frac{d^2\beta}{\pi} \chi_0(\beta)\, e^{\alpha\beta^* - \alpha^*\beta^2} \quad \text{Gaussian integral}$$

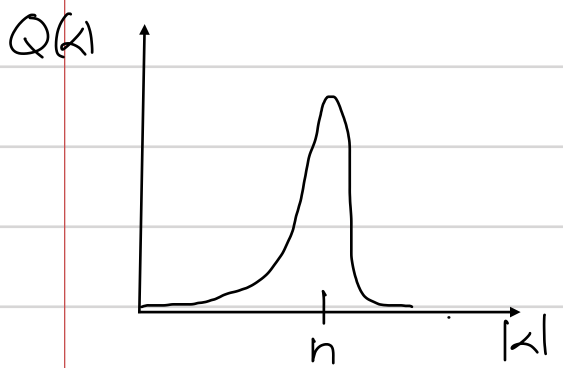
The P, W, Q are Gaussians of different widths

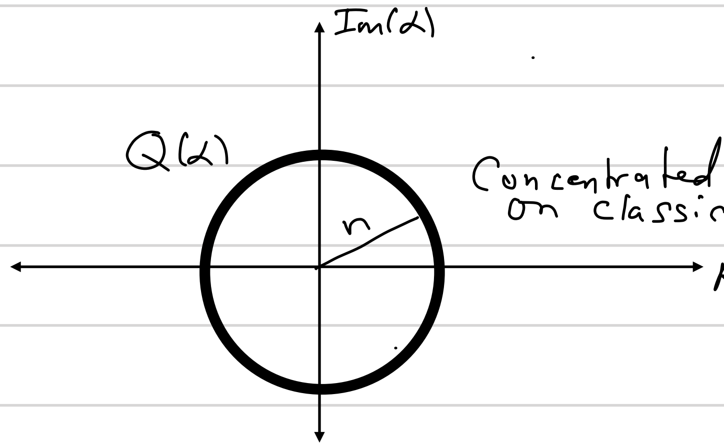


These "uncertainty bubbles" we have been drawing can be interpreted as the contors of the wigner function.

- Fock State

$$Q(\alpha) = \frac{1}{\pi}|\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{(|\alpha|^2)^n}{\pi\, n!}$$





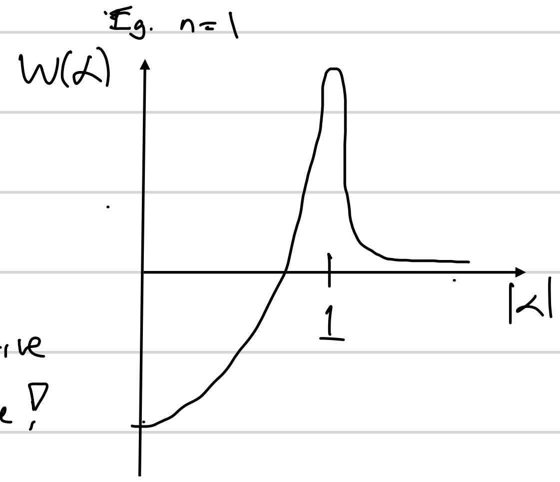
Concentrated on classical energy surface

Symmetric Characteristic function:

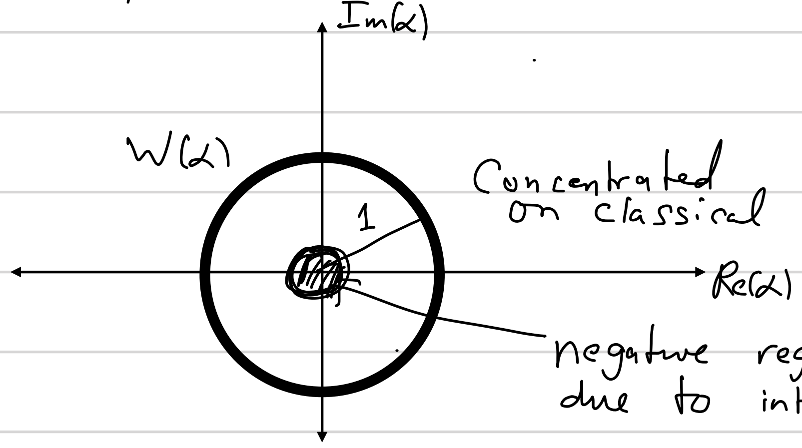
$$\chi_0(\beta) = \langle n|\hat{D}(\beta)|n\rangle = e^{-|\beta|^2/2}\,\mathcal{L}_n(|\beta|^2)$$ ← Laguerre polynomial

$$\Rightarrow W(\alpha) = \int\frac{d^2\beta}{\pi}\,\chi_0(\beta)\,e^{\alpha\beta^* - \beta^*\alpha} = \frac{2(-1)^n}{\pi}\,e^{-2|\alpha|^2}\mathcal{L}_n(4|\alpha|^2)$$

Eg. n=1



negative value!



Concentrated on classical energy surface

negative region due to interference!