Physics 581: Quantum Optics II
Problem Set #1
Due Thursday Jan. 30, 2020

Problem 1: Boson Algebra (30 points)
This problem is to give you some practice manipulating the boson algebra. A great source is the classic "Quantum Statistical Properties of Radiation", by W. H. Louisell, reprinted by "Wiley Classics Library", ISBN 0-471-52365-8.

(a) Gaussian integrals in phase-space are used all the time. Show that

$$\int \frac{d^2\beta}{\pi} e^{-\gamma|\beta|^2} e^{\alpha\beta^* - \beta\alpha^*} = \frac{1}{\gamma} e^{-|\alpha|^2/\gamma} .$$

(b) Prove the completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1}$$ (Hint: Expand in number states).

(c) The "quadrature" operators in optics are the analogs of X and P, $\hat{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}}$. Show

$$\begin{aligned} \hat{U}^\dagger(\theta)\hat{X}\hat{U}(\theta) &= \cos\theta\, \hat{X} + \sin\theta\, \hat{P} \\ \hat{U}^\dagger(\theta)\hat{P}\hat{U}(\theta) &= \cos\theta\, \hat{P} - \sin\theta\, \hat{X} \end{aligned}, \quad \text{where} \quad \hat{U}(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}} .$$

Interpret in phase space.

(d) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\{i\,\mathrm{Im}(\alpha\beta^*)\}$$

(e) Show that the displacement operators has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle \alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2} e^{i\,\mathrm{Im}\left(\alpha\alpha_2^* - \alpha_1\alpha^* - \alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2} \mathsf{L}_n\left(|\alpha|^2\right)$, where L_n is the Laguerre polynomial of order n

Problem 2: Thermal Light (25 points)

Consider a single mode field in thermal equilibrium at temperature T, Boltzmann factor $\beta = 1/k_B T$. The state of the field is described by the "canonical ensemble",
$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$, $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$ is the Hamiltonian and $Z = Tr\left(e^{-\beta \hat{H}}\right)$ is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- $\langle n \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$ (the Planck spectrum)
- $P_n = \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}}$ (the Bose-Einstein distribution).

(b) Make a list-plot of P_n for both the thermal state and the coherent state on the same graph as a function of n, for each of the following: $\langle n \rangle = 0.1, 1, 10, 100$.

(c) Using the number-state representation, show that

- $\Delta n^2 = \langle n \rangle + \langle n \rangle^2$. How does this compare to a coherent state?
- $\langle \hat{a} \rangle = 0 \Rightarrow \langle \vec{E} \rangle = 0$. How does this compare to a coherent state?

(d) Show that the Glauber-Sudharshan distribution of this state, $P(\alpha) = \frac{1}{\pi \langle n \rangle} e^{-|\alpha|^2/\langle n \rangle}$,

satisfies $\int d^2\alpha \, P(\alpha) |\alpha\rangle\langle\alpha| = \sum_n \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}} |n\rangle\langle n|$. Sketch $P(\alpha)$ in the phase plane.