Lecture 18
(Angular Velocity and Acceleration, Equations of Angular Motion)

Physics 160-02 Spring 2017

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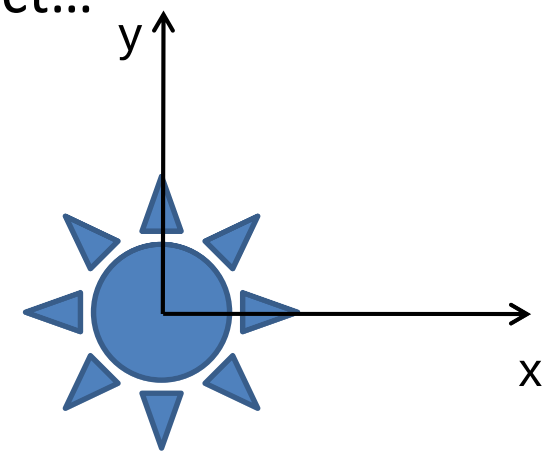
Rotation of Rigid Bodies

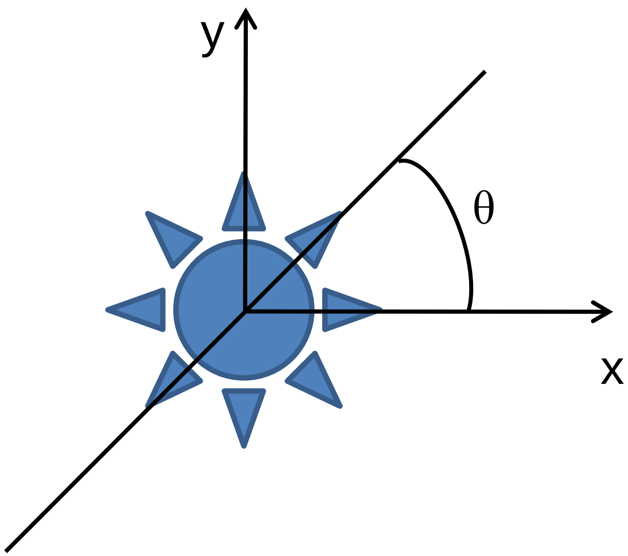
- We know how to describe the motion of the COM of an object…



- Now, we want to understand how to describe the motion of an object ***about*** its COM.
- We could look at each piece of the object as a separate particle and calculate forces and then calculate accelerations…
- Yuk!!!

Rotation of Rigid Bodies

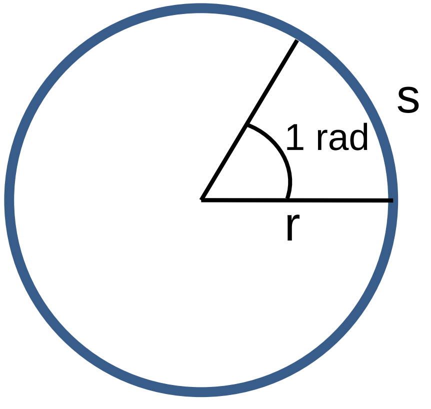
- We need a new information set first:



- So, we can describe the motion about the COM via an angle defined to some point on the body.
- Now, let's define the units for angles…

Radians

- One radian is the angle when the arc length is the same as the radius.



$$\theta = \frac{s}{r}$$

- All the way around, the arc length is $2\pi r$, so there are 2π radians in a complete revolution.

Rotational Variables

- Angular displacement.

$$\Delta\theta = \theta_f - \theta_i$$

- Angular velocity.

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}, \quad \omega = \frac{d\theta}{dt}$$

- Angular acceleration.

$$\alpha = \frac{d\omega}{dt}$$

Constant angular acceleration

- Same as constant linear acceleration:

$$y_f = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

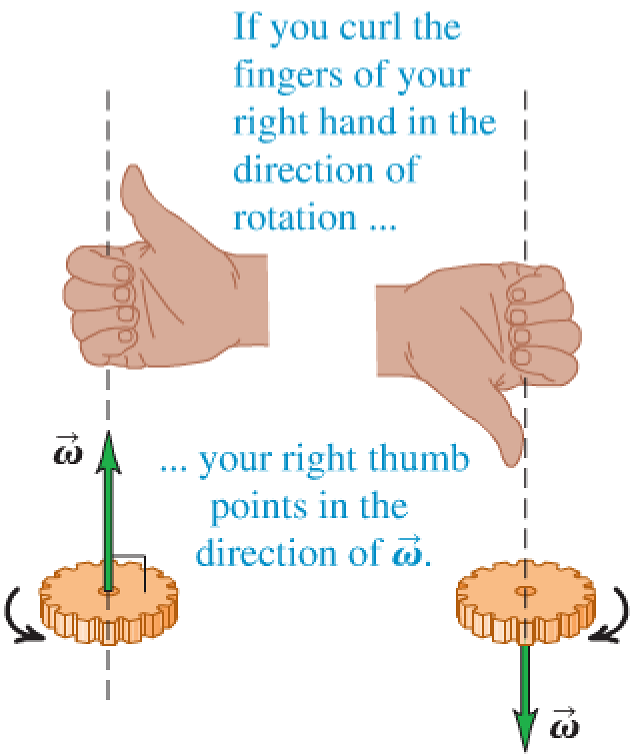
$$v_{fy} = v_{0y} + a_y t$$

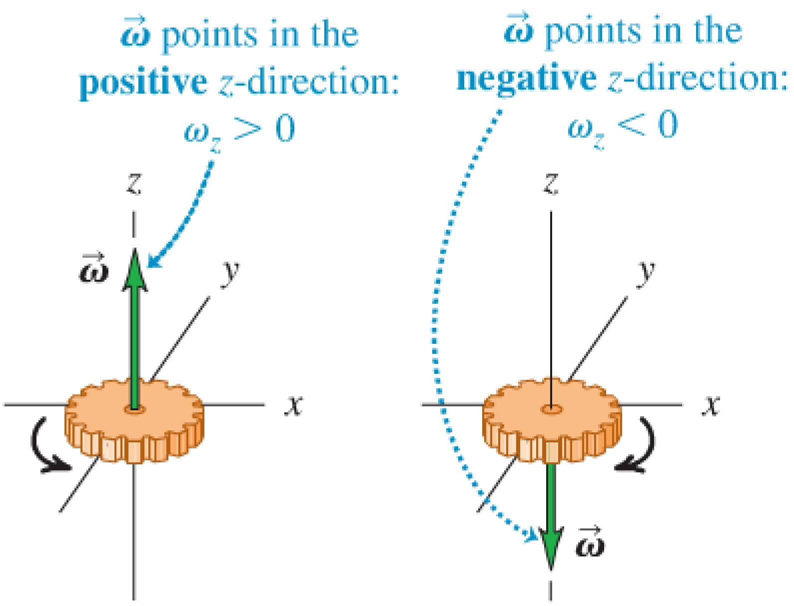
$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega_f = \omega_0 + \alpha t$$

Vector Definitions

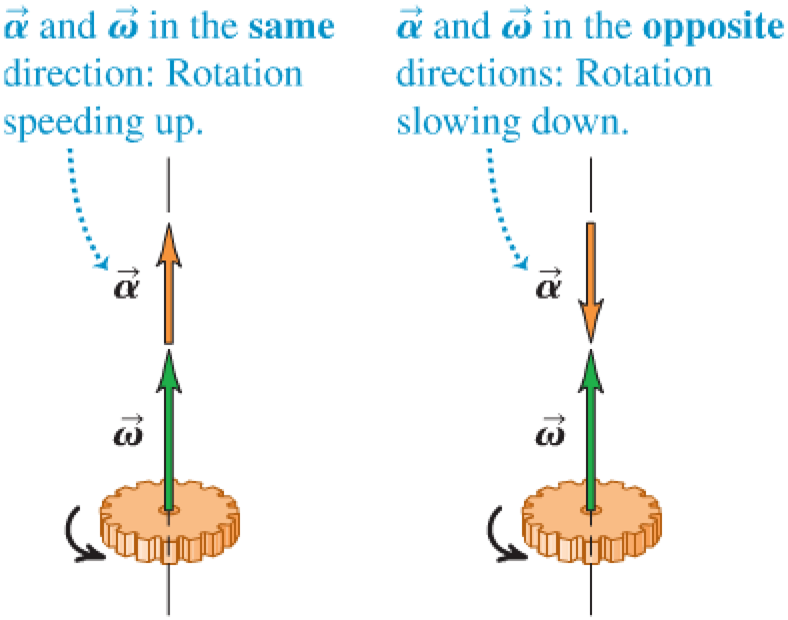
- We will use the right hand rule to define a direction for the angular velocity and the angular acceleration:





Vector Definitions

- and the angular acceleration:



Example

9.8. A wheel is rotating about an axis that is in the z-direction. The angular velocity ω_z is -6.00 rad/s at $t = 0$, increases linearly with time, and is $+8.00$ m/s at $t = 7.00$ s. We have taken counterclockwise rotation to be positive. (a) Is the angular acceleration during this time interval positive or negative? (b) During what time interval is the speed of the wheel increasing? Decreasing? (c) What is the angular displacement of the wheel at $t = 7.00$ s?

Relation to Translational Variables

- Sometimes we may want to relate rotational motion to linear motion…

$$\theta = \frac{s}{r} \Rightarrow s = \theta r$$

$$v = \frac{ds}{dt} = \frac{d\theta}{dt} r = \omega r \Rightarrow v = \omega r$$

$$a_{\text{radial}} = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r \Rightarrow a_{\text{radial}} = \omega^2 r$$

$$a_{\text{tangential}} = \frac{dv}{dt} = \frac{d(\omega r)}{dt} = \alpha r \Rightarrow a_{\text{tangential}} = \alpha r$$

Exercise 9.3

9.3 • **CP CALC** The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B). (a) What are the units of A and B if ω_z is in rad/s? (b) What is the angular acceleration of the wheel at (i) $t = 0.00$ and (ii) $t = 5.00$ s? (c) Through what angle does the flywheel turn during the first 2.00 s?

Exercise 9.7

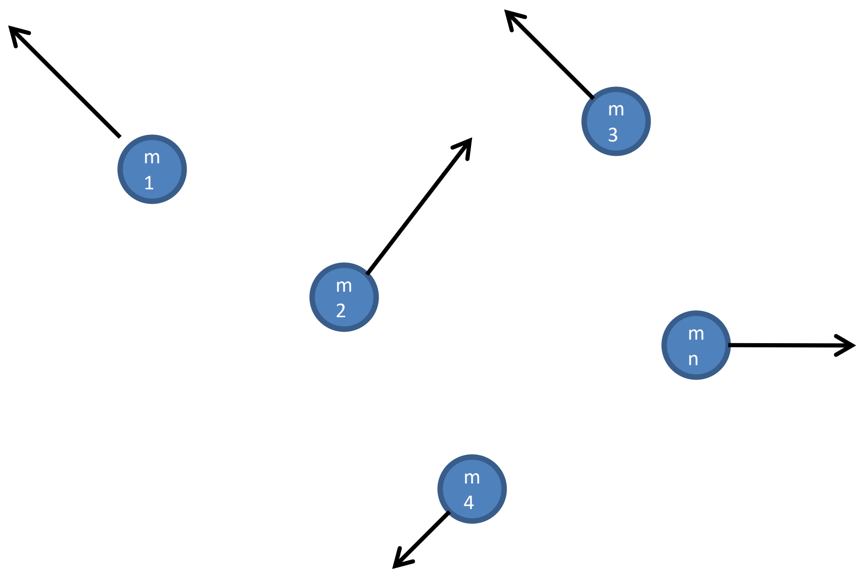
9.7 • **CALC** The angle θ through which a disk drive turns is given by $\theta(t) = a + bt - ct^3$, where a, b, and c are constants, t is in seconds, and θ is in radians. When $t = 0$, $\theta = \pi/4$ rad and the angular velocity is 2.00 rad/s, and when $t = 1.50$ s, the angular acceleration is 1.25 rad/s^2. (a) Find a, b, and c, including their units. (b) What is the angular acceleration when $\theta = \pi/4$ rad? (c) What are θ and the angular velocity when the angular acceleration is 3.50 rad/s^2?

Problem 9.61

9.61 • **CP CALC** A flywheel has angular acceleration $\alpha_z(t) = 8.60\ \text{rad/s}^2 - (2.30\ \text{rad/s}^3)t$, where counterclockwise rotation is positive. (a) If the flywheel is at rest at $t = 0$, what is its angular velocity at 5.00 s? (b) Through what angle (in radians) does the flywheel turn in the time interval from $t = 0$ to $t = 5.00$ s?

Rotational Kinetic Energy

- Let's examine a system of n particles:



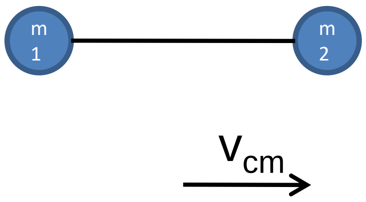
- The total kinetic energy of the system is:

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots \frac{1}{2}m_nv_n^2$$

$$= \sum_{i=1}^{n} \frac{1}{2}m_iv_i^2$$

Rotational Kinetic Energy

- But, for a rigid body, there are constraints on what the velocities can be.
- Let's examine a system of 2 constrained particles (not rotating):



- The total kinetic energy of the system is:

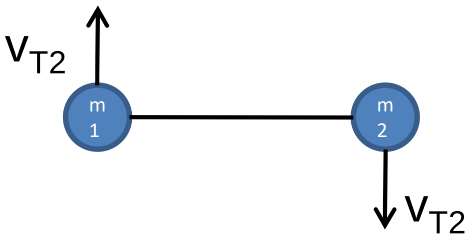
$$KE_{cm} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$but, \quad v_1 = v_2 = v_{cm}$$

$$KE_{cm} = \frac{1}{2}(m_1 + m_2)v_{cm}^2 = \frac{1}{2}Mv_{cm}^2$$

Rotational Kinetic Energy

- Now let's examine a system of 2 constrained particles (no cm motion, but rotating):



- Then the total kinetic energy of the system is:

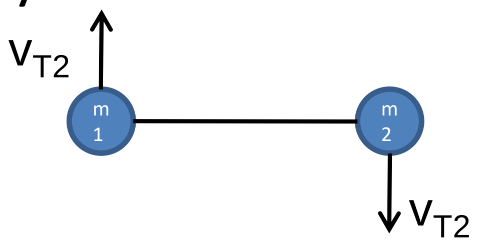
$$KE_{rot} = \frac{1}{2}m_1 v_{T1}^2 + \frac{1}{2}m_2 v_{T2}^2$$

$$KE_{rot} = \frac{1}{2}m_1\left(\omega r_1\right)^2 + \frac{1}{2}m_2\left(\omega r_2\right)^2$$

$$= \frac{1}{2}\omega^2\left(m_1 r_1^2 + m_2 r_2^2\right)$$

Rotational Kinetic Energy

- Notice that the term in parenthesis is only a function of the geometry of the system!



$$KE_{rot} = \frac{1}{2}\omega^2\left(m_1 r_1^2 + m_2 r_2^2\right) = \frac{1}{2}I\omega^2,$$

$$where,$$

$$I = \left(m_1 r_1^2 + m_2 r_2^2\right)$$

- Or, in general,

$$I = \sum_i m_i r_i^2$$

- This is called the moment of inertia.

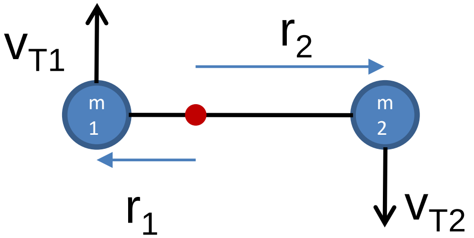
Rotational Kinetic Energy

- But what is r_i ? Remember where it came from:

$$KE_{rot} = \frac{1}{2} m_1 v_{T1}^2 + \frac{1}{2} m_2 v_{T2}^2$$

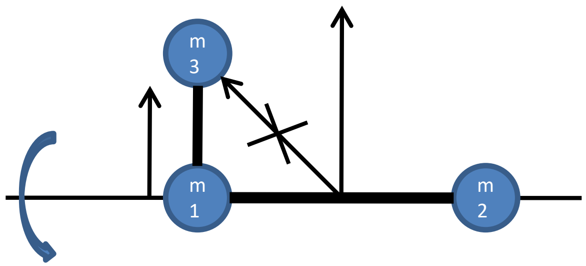
$$KE_{rot} = \frac{1}{2} m_1 \left(\omega r_1\right)^2 + \frac{1}{2} m_2 \left(\omega r_2\right)^2$$

$$= \frac{1}{2} \omega^2 \left(m_1 r_1^2 + m_2 r_2^2\right)$$

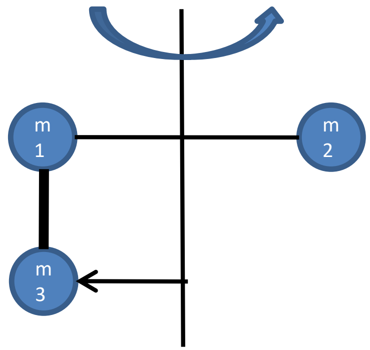


Rotational Kinetic Energy

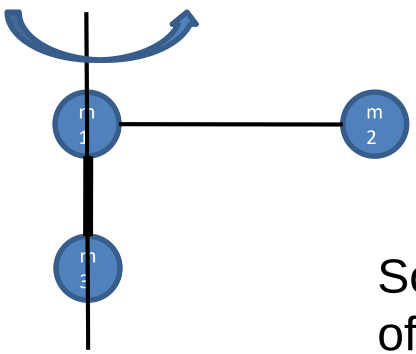
- So the r_i are the distance of the masses to the axis of rotation



$$I = \sum_i m_i r_i^2 = m_1 (0)^2 + m_2 (0)^2 + m_3 (d)^2$$
$$= m_3 d^2$$



$$I = \sum_i m_i r_i^2 = m_1 (d)^2 + m_2 (d)^2 + m_3 (d)^2$$
$$= (m_1 + m_2 + m_3) d^2$$



$$I = \sum_i m_i r_i^2 = m_1 (0)^2 + m_2 (2d)^2 + m_3 (0)^2$$
$$= (m_2) 4d^2$$

So, the same object can have a different moment of inertia depending upon the axis of rotation! (You can actually formalize this using matrices…)

CPS Question 23-1

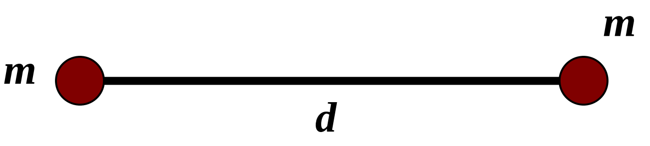
- **Two point masses of mass *m* are attached to a long massless rod of length *d*. What is the moment of inertia that you would use in the calculation of the system's kinetic energy?**

A) $2md^2$.

B) md^2.

C) $\frac{1}{2}\omega^2 md^2$.

D) Not enough information to solve.



CPS Question 23-2

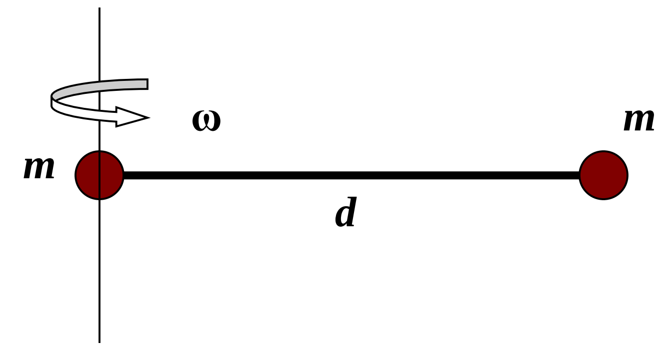
- **Two point masses of mass *m* are attached to a long massless rod of length *d*. The system is rotating about one end. What is the moment of inertia that you would use in the calculation of the system's kinetic energy?**

A) $2md^2$.

B) md^2.

C) ½ $\omega^2 md^2$.

D) Not enough information to solve.



Continuous Distributions

- Generalization of the sum

$$I = \sum_i m_i r_i^2 \rightarrow \int r^2 dm$$

- But what is dm???

$$dm = \rho dV$$

- In Cartesian coordinates then we can write

$$I = \int r^2 \rho dV = \int r^2 \rho dxdydz$$