# Lecture 19
# (Energy in Rotational Motion & Moments of Inertia)

Physics 160-02 Spring 2017

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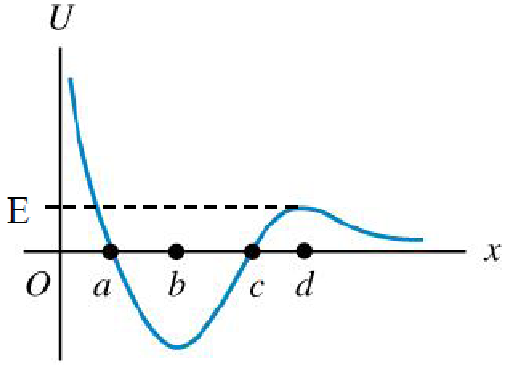
**2)** A marble moves along the x-axis. The potential-energy function is shown below.   Describe the force (magnitude and direction) on the marble at the four labeled points.



# Review

- Linear
  - For const a:

$$x_f = x_i + v_i t + \frac{1}{2} a t^2$$

$$v_f = v_i + at$$

  - Kinetic energy:

$$KE = \frac{1}{2} M v^2$$

- Rotational
  - For const α:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2$$

$$\omega_f = \omega_i + \alpha t$$

  - Kinetic energy:

$$KE = \frac{1}{2} I \omega^2$$

# Continuous Distributions

- Generalization of the sum

$$I = \sum_i m_i r_i^2 \rightarrow \int r^2 dm$$

- But what is dm???

$$dm = \rho dV$$

- In Cartesian coordinates then we can write

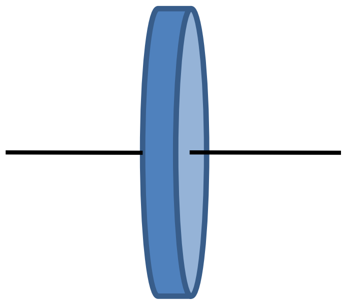
$$I = \int r^2 \rho dV = \int r^2 \rho dxdydz$$

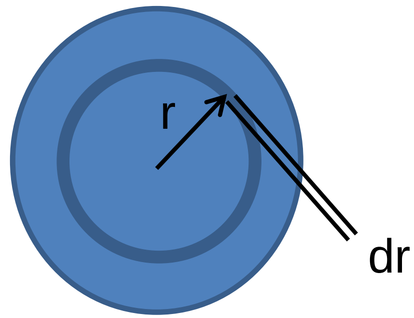
# Densities

- In physics we use a "standard notation" for densities of different types:
  - Linear density $\lambda$ = mass/unit length
  - Area or surface density $\sigma$ = mass/unit area
  - Volume density $\rho$ = mass/unit volume
- These can be constant throughout an object, or can be a function of position.
- Examples…

# Example

- Thin disk of uniform density (mass/unit area)






$$dm = \sigma dA = \frac{M}{A} 2\pi r dr$$

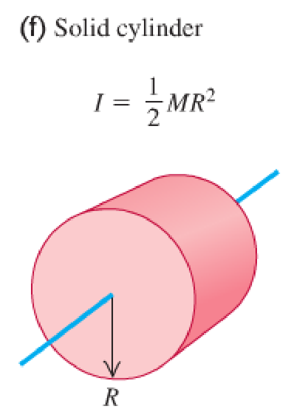
$$I = \int r^2 dm = \int r^2 \frac{M}{A} 2\pi r dr = 2\pi \frac{M}{A} \int r^3 dr$$

$$= 2\pi \frac{M}{A} \left[ \frac{r^4}{4} \right]_0^R = 2\pi \frac{M}{A} \frac{R^4}{4}$$

$$but,$$

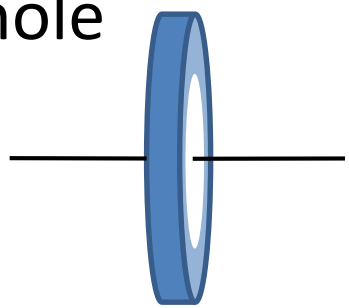
$$A = \pi R^2$$

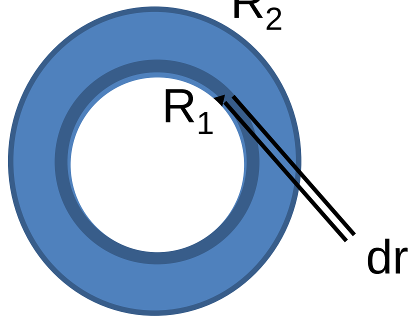
$$I = 2\pi \frac{M}{\pi R^2} \frac{R^4}{4} = \frac{1}{2} MR^2$$



# Example

- Thin disk of uniform density (mass/unit area) with hole






$$dm = \sigma dA = \frac{M}{A} 2\pi r dr$$

$$I = \int r^2 dm = \int r^2 \frac{M}{A} 2\pi r dr = 2\pi \frac{M}{A} \int r^3 dr$$

$$= 2\pi \frac{M}{A} \left[ \frac{r^4}{4} \right]_{R_1}^{R_2} = 2\pi \frac{M}{A} \frac{R_2^4 - R_1^4}{4}$$

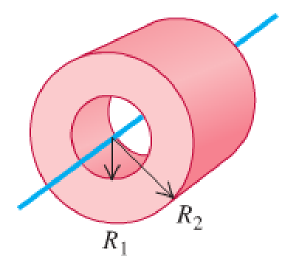
$$but,$$

$$A = \pi \left( R_2^2 - R_1^2 \right)$$

$$I = 2\pi \frac{M}{A} \frac{R_2^4 - R_1^4}{4} = 2\pi \frac{M}{A} \frac{\left( R_2^2 - R_1^2 \right)\left( R_2^2 + R_1^2 \right)}{4} = \frac{1}{2} M \left( R_2^2 + R_1^2 \right)$$

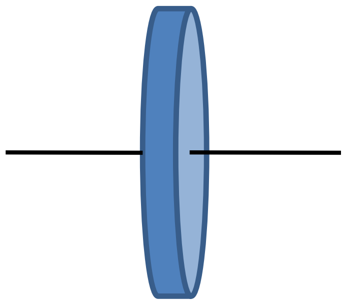
(e) Hollow cylinder

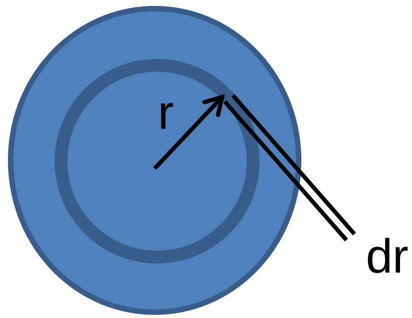
$$I = \frac{1}{2} M (R_1{}^2 + R_2{}^2)$$



# Example

- Thin disk of NON-uniform density (mass/unit area):






$$\sigma(r) = \sigma_0 + cr$$

$$dm = \sigma dA = (\sigma_0 + cr)2\pi r dr$$

$$I = \int r^2 dm = \int r^2 (\sigma_0 + cr) 2\pi r dr = 2\pi \int (\sigma_0 + cr) r^3 dr$$

$$I = 2\pi\sigma_0 \left[\frac{r^4}{4}\right]_0^R + 2\pi c \left[\frac{r^5}{5}\right]_0^R = 2\pi\left[\sigma_0 \frac{R^4}{4} + c\frac{R^5}{5}\right]$$

# Example

- Sphere of uniform density (mass/unit volume)

$$dm = \rho dV = \rho \pi r^2 dx = \rho \pi \left(R^2 - x^2\right) dx$$

$$dI = \frac{1}{2} r^2 dm = \frac{1}{2}\left(R^2 - x^2\right)\rho\pi\left(R^2 - x^2\right)dx$$

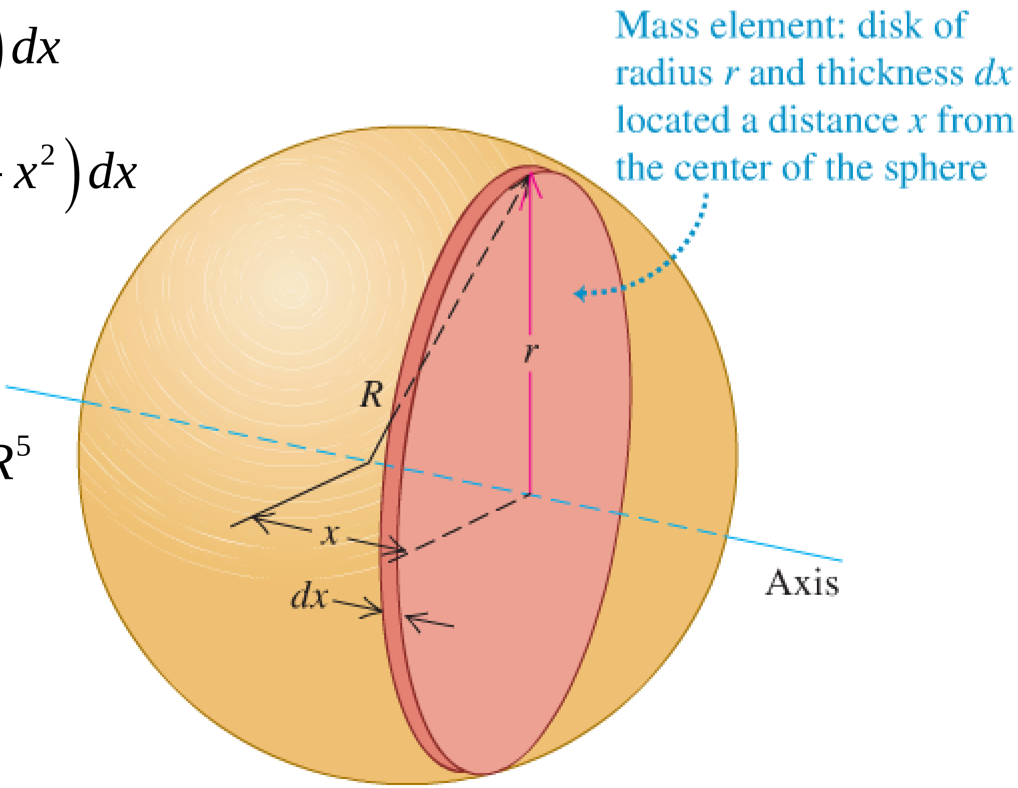
$$= \frac{\rho\pi}{2}\left(R^2 - x^2\right)^2 dx \Rightarrow$$

$$I = (2)\frac{\rho\pi}{2}\int_0^R \left(R^2 - x^2\right)^2 dx = \frac{8\pi\rho}{15} R^5$$

$$but,$$

$$M = \rho V = \rho \frac{4}{3}\pi R^3 \Rightarrow$$

$$I = \frac{2}{5} M R^2$$



# Moments of Inertia

**Table 9.2** Moments of Inertia of Various Bodies

(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$

(b) Slender rod, axis through one end

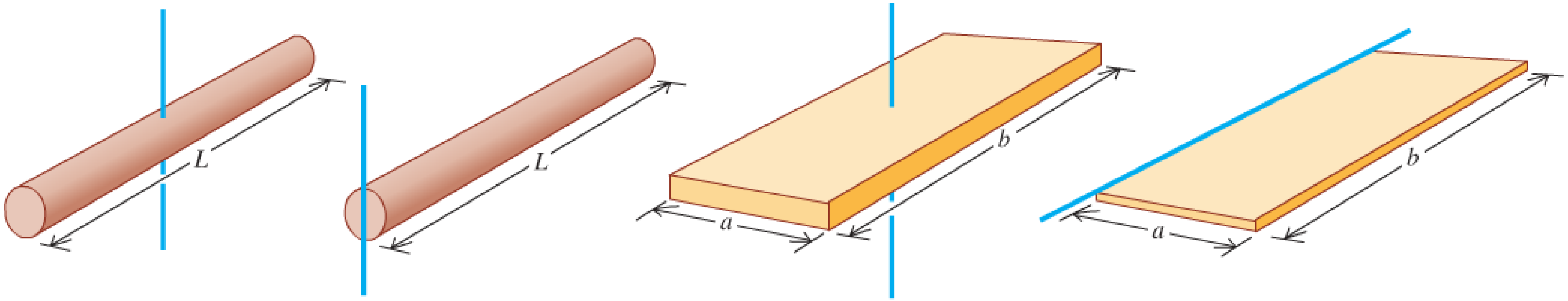
$$I = \frac{1}{3}ML^2$$

(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$

(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$




(e) Hollow cylinder

$$I = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$$

(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$

(g) Thin-walled hollow cylinder

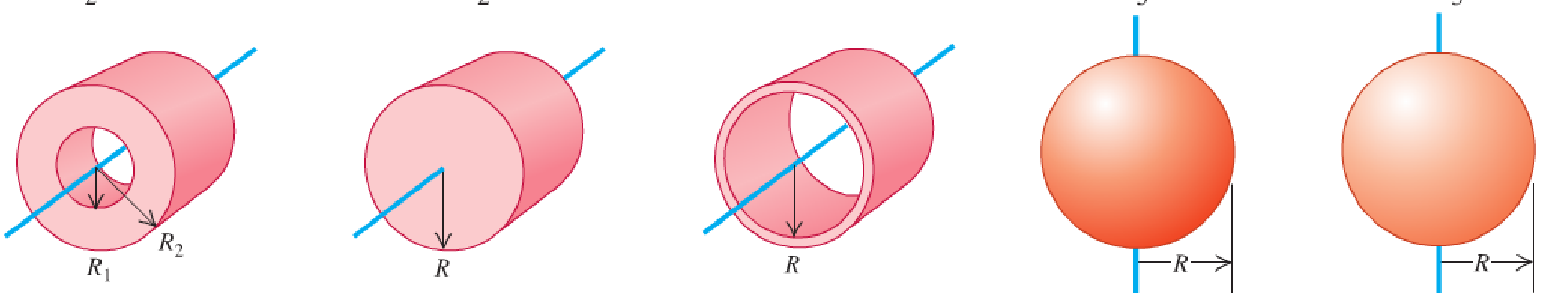
$$I = MR^2$$

(h) Solid sphere

$$I = \frac{2}{5}MR^2$$

(i) Thin-walled hollow sphere

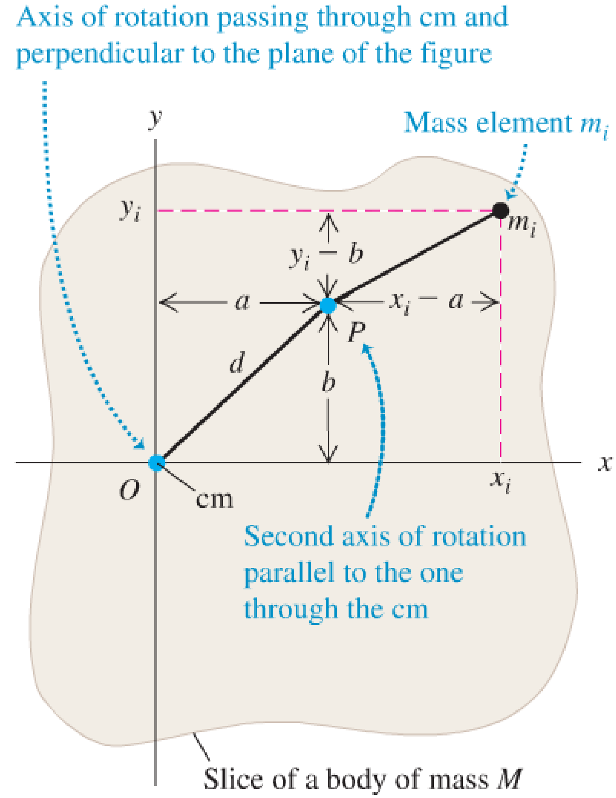
$$I = \frac{2}{3}MR^2$$



# Parallel Axis Theorem

- If we know the moment of inertia about an axis that passes through the center of mass of an object, then the moment of inertia about any axis parallel to that a distance d away is given by:

$$I_P = I_{cm} + Md^2$$




$$I_{cm} = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

$$I_P = \sum_i m_i \left[\left(x_i - a\right)^2 + \left(y_i - b\right)^2\right]$$

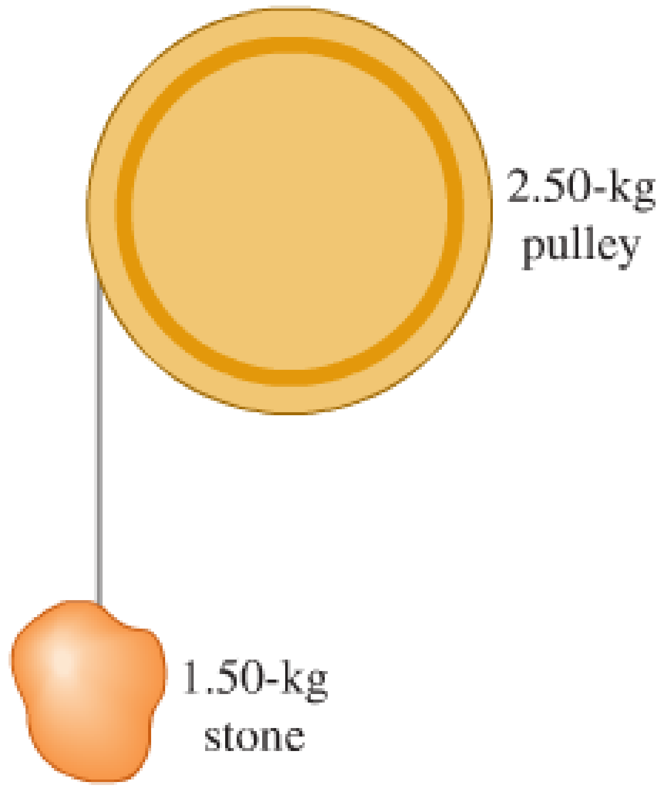
$$= \sum_i m_i \left(x_i^2 + y_i^2\right) - 2a\cancelto{0}{\sum_i m_i x_i} - 2b\cancelto{0}{\sum_i m_i y_i} + \cancelto{d^2}{\left(a^2 + b^2\right)}\cancelto{M}{\sum_i m_i}$$

# Problem 9.49

**9.49.** A frictionless pulley has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50-kg stone is attached to a very light wire that is wrapped around the rim of the pulley (Fig. 9.32), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

**9.50.** A bucket of mass $m$ is tied

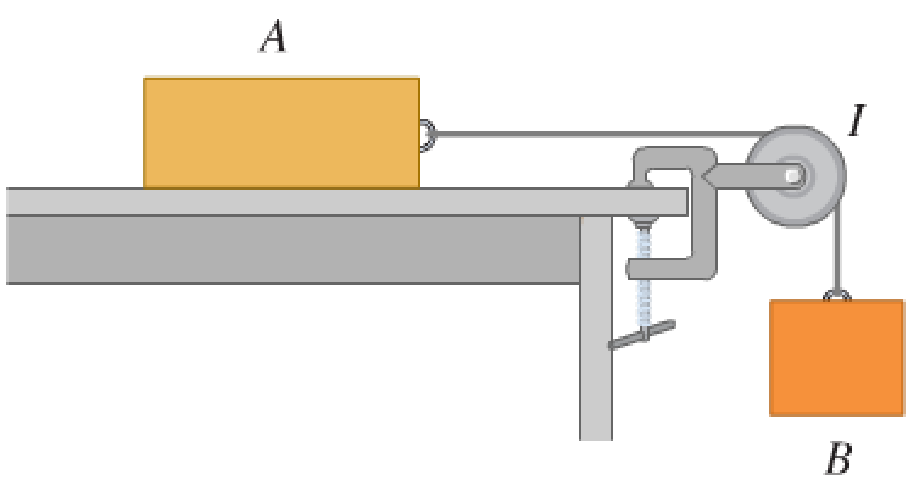
**Figure 9.32** Exercise 9.49.



# Problem 9.85

**9.85.** The pulley in Fig. 9.35 has radius $R$ and a moment of inertia $I$. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block $A$ and the tabletop is $\mu_k$. The system is released from rest, and block $B$ descends. Block $A$ has mass $m_A$ and block $B$ has mass $m_B$. Use energy methods to calculate the speed of block $B$ as a function of the distance $d$ that it has descended.

**Figure 9.35** Problem 9.85.



# Problem 9.89

**9.89.** Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center (Fig. 9.37). (a) What is the total moment of inertia of the two disks? (b) A light string is wrapped around the edge of the smaller disk, and a 1.50-kg block is suspended from the free end of the string. If the block is released from rest at a distance of 2.00 m above the floor, what is its speed just before it strikes the floor? (c) Repeat the calculation of part (b), this time with the string wrapped around the edge of the larger disk. In which case is the final speed of the block the greatest? Explain why this is so.

**Figure 9.37** Problem 9.89.