Lecture 23
(Gravitation, Potential Energy and Gauss's Law; Kepler's Laws)

Physics 160-02 Spring 2017

Douglas Fields

Gravitational Force

- Up until now, we have said that the gravitational force on a mass m is just mg.
- But remember that we always said that there is a condition on this, that we are at the earth's surface.
- What is the general form of the force due to gravity?

$$F_G = G\frac{m_1 m_2}{r^2}$$

- That is, two particles with mass will attract each other proportionately to their masses and inverse proportionately to the square of the distance between them
- The proportionality constant, G, is known as the universal gravitational constant.

Gravitational Force

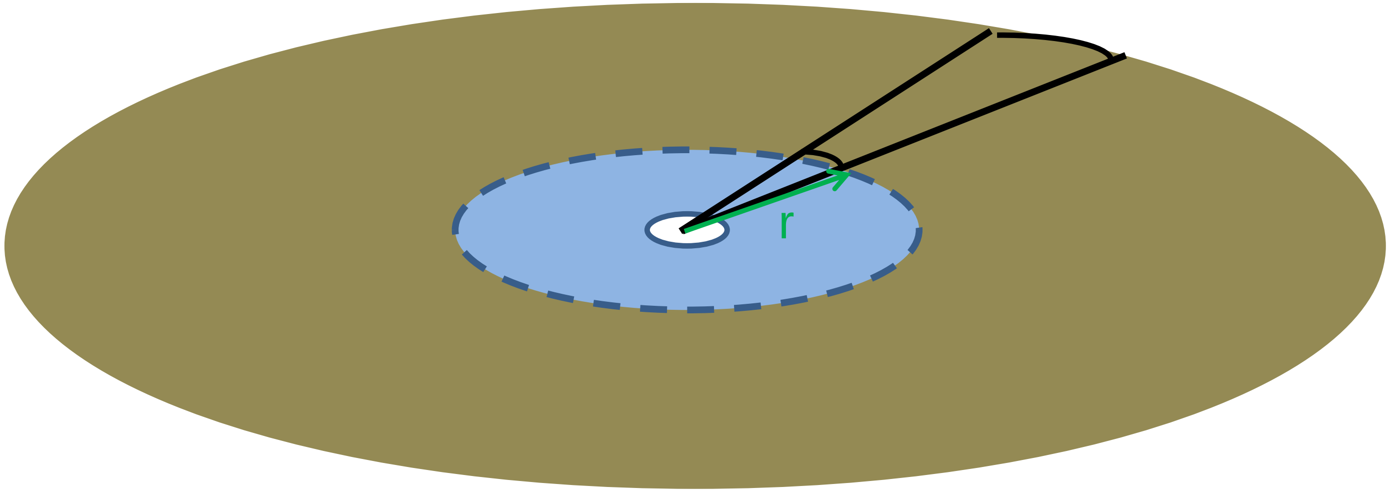
- The gravitational force is very weak (?)…
- For two masses each of 1kg, separated by 1m:

$$F_G = G\frac{m_1 m_2}{r^2} = G\frac{1kg \cdot 1kg}{(1m)^2} = 6.6742 \times 10^{-11} N$$

- That is, the proportionately constant, G = $6.6742 \times 10^{-11} Nm^2/kg^2$.
- So why is it that we feel such a strong force on us?
- The earth's radius is 6.37×10^6m!
- But, the earth's mass = 5.98×10^{24}kg!!

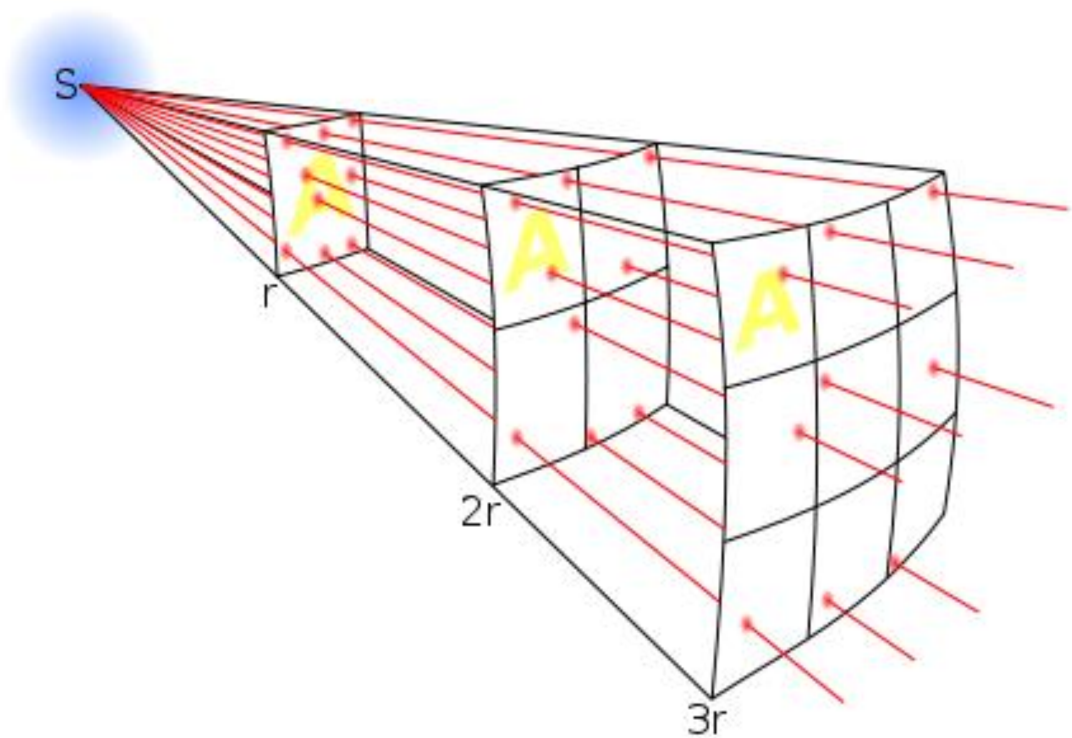
Why $1/r^2$?

- Consider water flowing out of a hole in a level surface, and spreading out evenly along the surface…



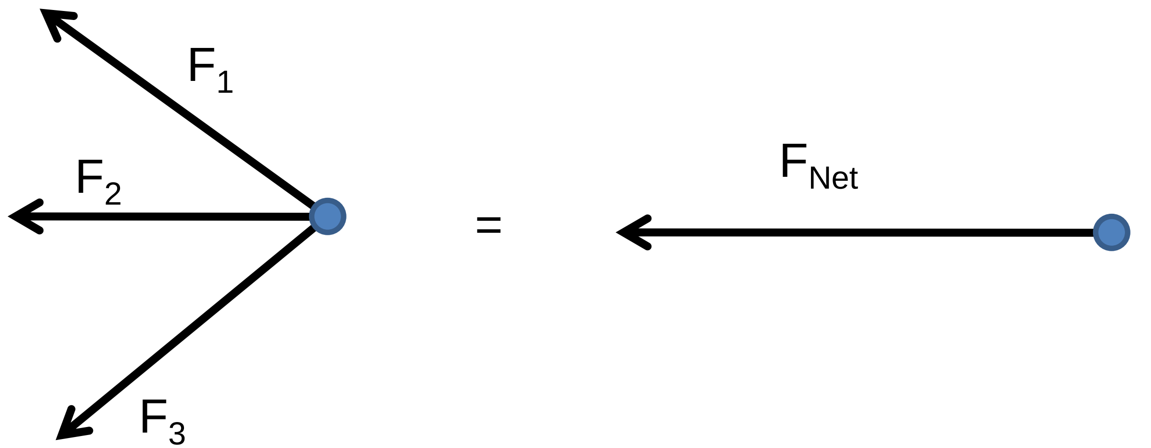
Why $1/r^2$?

- Now, in three dimensions, we examine the flux passing through the surface of a sphere…



Superposition of Force

- Remember that if two (or more) forces are acting on a body, the net force is just the (vector) sum of all the forces:

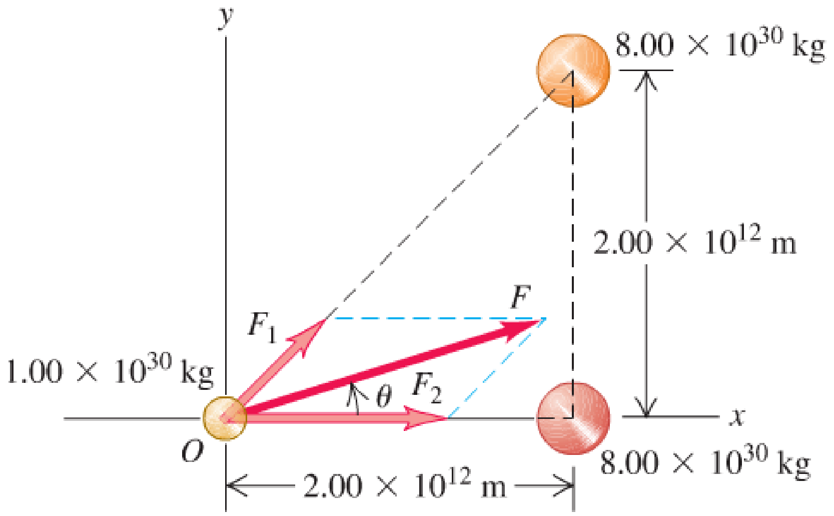


- The same is true for gravitational forces.

Example

$$F_1 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}) \\ \times (8.00 \times 10^{30}\ \mathrm{kg})(1.00 \times 10^{30}\ \mathrm{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \mathrm{m})^2 + (2.00 \times 10^{12}\ \mathrm{m})^2}$$
$$= 6.67 \times 10^{25}\ \mathrm{N}$$

$$F_2 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}) \\ \times (8.00 \times 10^{30}\ \mathrm{kg})(1.00 \times 10^{30}\ \mathrm{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \mathrm{m})^2}$$
$$= 1.33 \times 10^{26}\ \mathrm{N}$$



$$F_{1x} = (6.67 \times 10^{25}\ \mathrm{N})(\cos 45^\circ) = 4.72 \times 10^{25}\ \mathrm{N}$$
$$F_{1y} = (6.67 \times 10^{25}\ \mathrm{N})(\sin 45^\circ) = 4.72 \times 10^{25}\ \mathrm{N}$$
$$F_{2x} = 1.33 \times 10^{26}\ \mathrm{N}$$
$$F_{2y} = 0$$

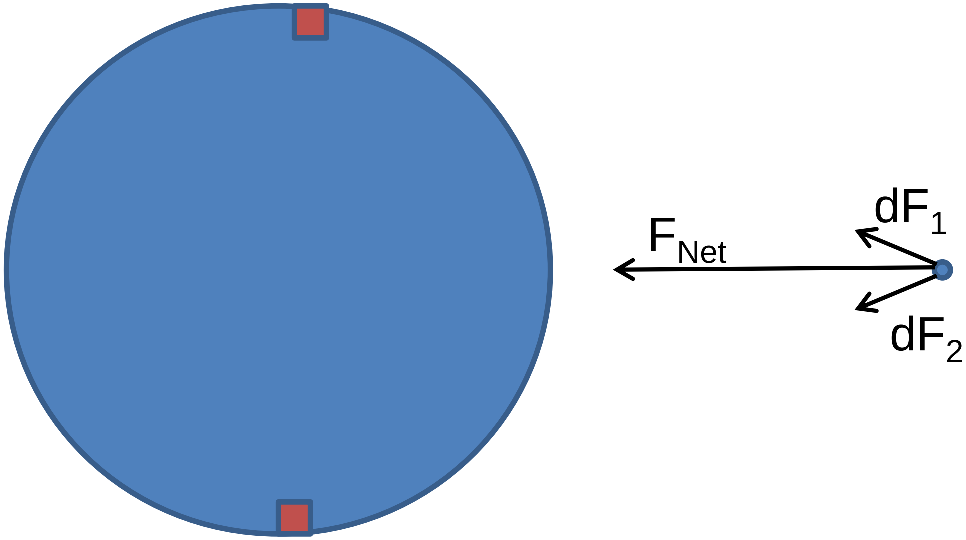
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(1.81 \times 10^{26}\ \mathrm{N})^2 + (4.72 \times 10^{25}\ \mathrm{N})^2}$$
$$= 1.87 \times 10^{26}\ \mathrm{N}$$

$$F_x = F_{1x} + F_{2x} = 1.81 \times 10^{26}\ \mathrm{N}$$
$$F_y = F_{1y} + F_{2y} = 4.72 \times 10^{25}\ \mathrm{N}$$

$$\theta = \arctan\frac{F_y}{F_x} = \arctan\frac{4.72 \times 10^{25}\ \mathrm{N}}{1.81 \times 10^{26}\ \mathrm{N}} = 14.6^\circ$$

Spherically Symmetric Bodies

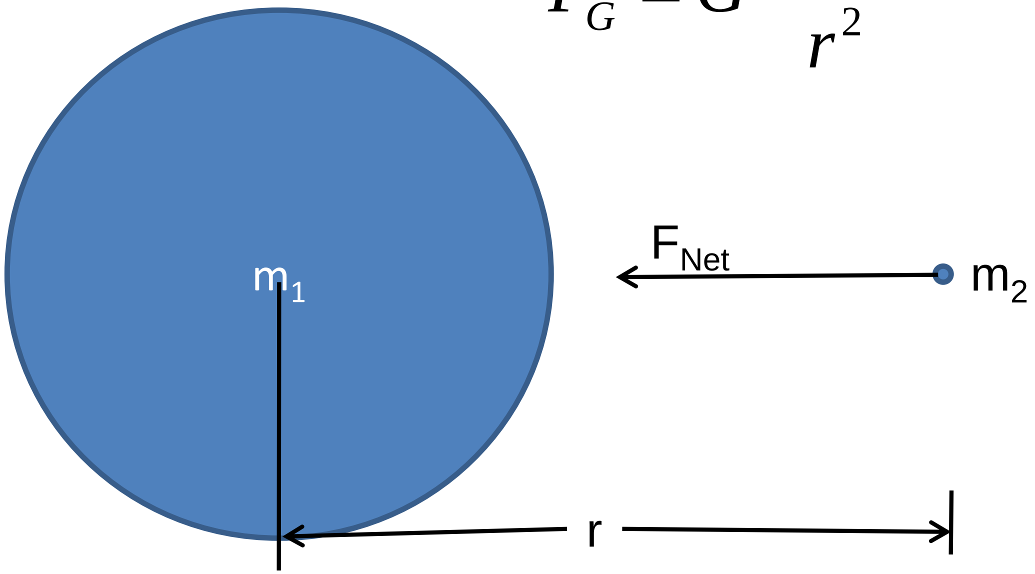
- We can do the same thing for continuous distributions of mass.



Spherically Symmetric Bodies

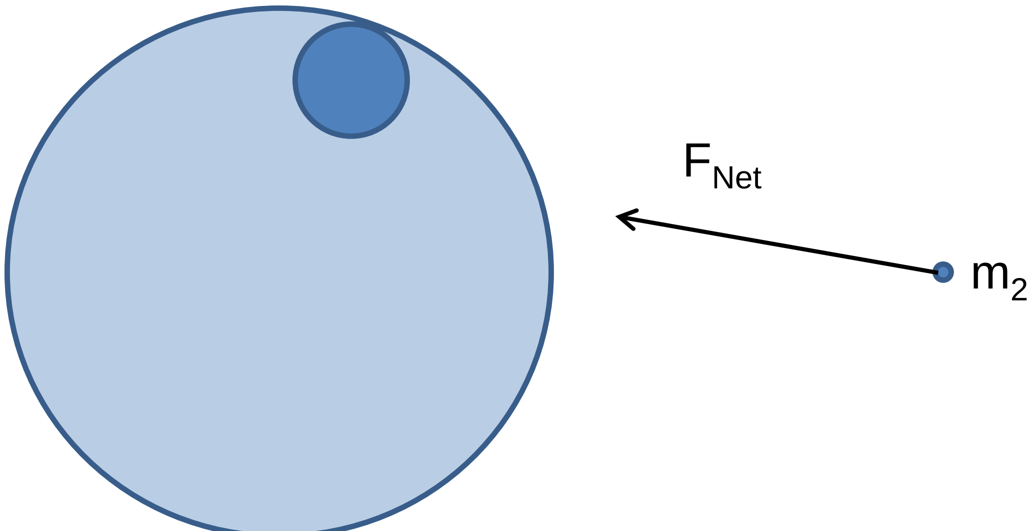
- For a spherically symmetric distribution, the net force is pointed to the center, and has the magnitude as if all the mass was located at the center:

$$F_G = G\frac{m_1 m_2}{r^2}$$



Search For Oil

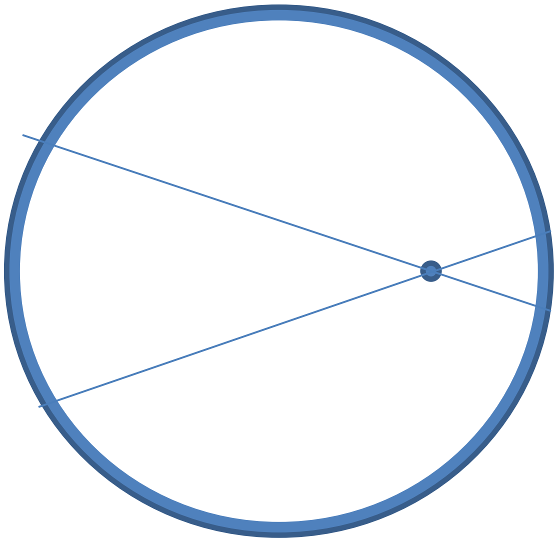
- If the mass is not spherically symmetric, this is no longer the case:



- This can be used to look for non-uniform densities in the earth's crust (oil, uranium, etc.).

Shell Theorem

$$F_G = G\frac{m_1 m_2}{r^2}$$



$F_{Net} = 0!$ inside a shell.

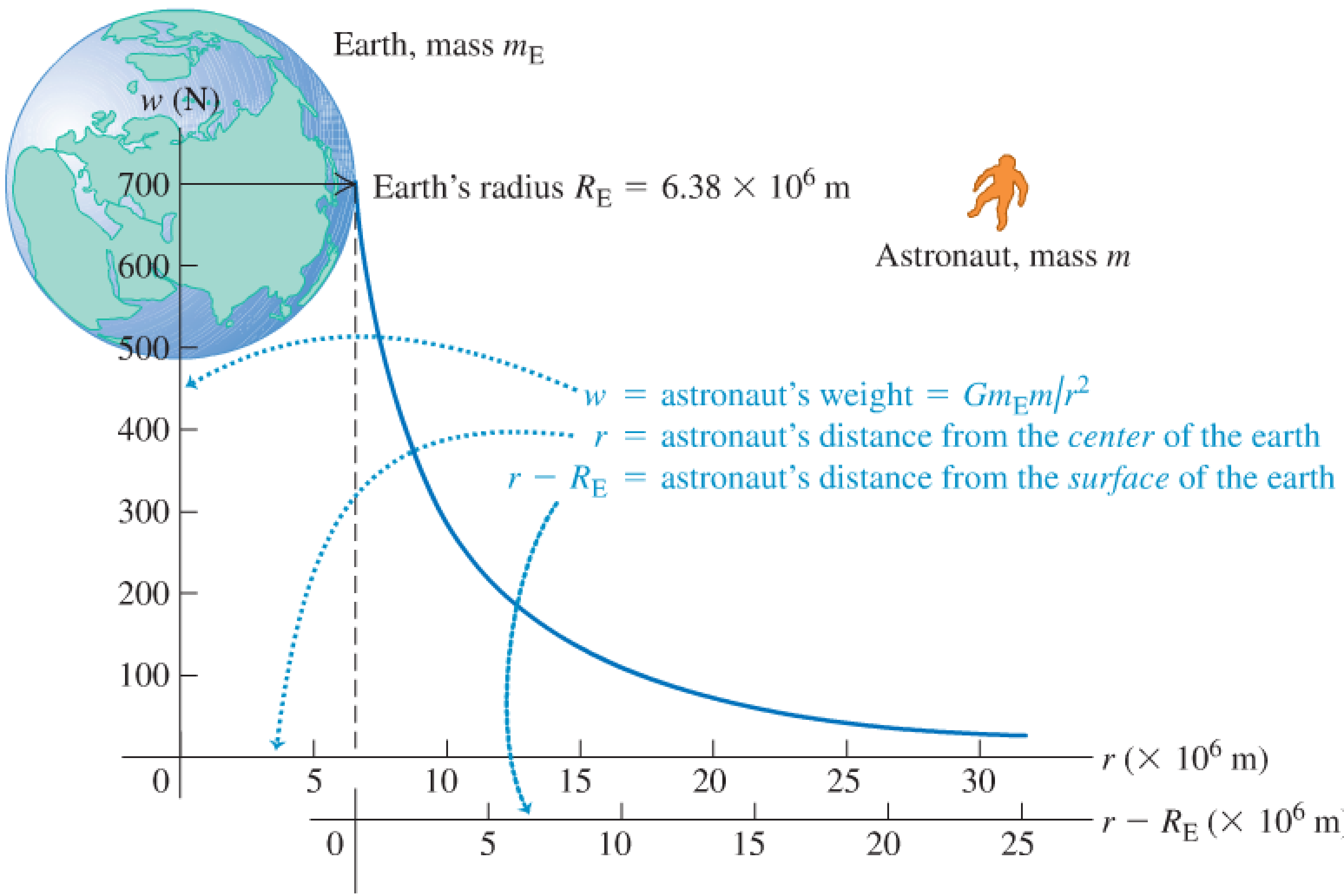
Weight and "Little g"

- So, what is the force due to gravity on a mass m, at the surface of the earth?

$$F_G = G\frac{m_E m}{r_E^2} = \frac{Gm_E}{r_E^2}m = \frac{\left(6.67\times10^{-11}\,N\cdot m^2/kg^2\right)\left(5.98\times10^{24}\,kg\right)}{\left(6.38\times10^{6}\,m\right)^2}m = \left(9.8\frac{N}{kg}\right)m$$

- Recognize the factor in front of the object's mass?
- Also notice that as the object goes farther from the earth's center, the force of gravity from the earth gets less…

Astronaut's Weight

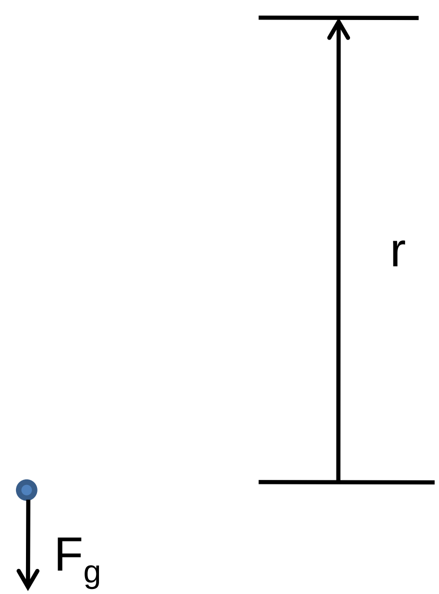


Gravitational Potential Energy

- Remember that we defined the gravitational potential energy as being the work done by gravity when an object is moved from one point to another:

$$\Delta U_g = -W_g = -\int_1^2 \vec{F}_g \cdot d\vec{r}$$

Near the earth's surface, you can take the force to be constant = mg, so the change in potential is just mg $(r_2 - r_1)$ = mgh



Gravitational Potential Energy

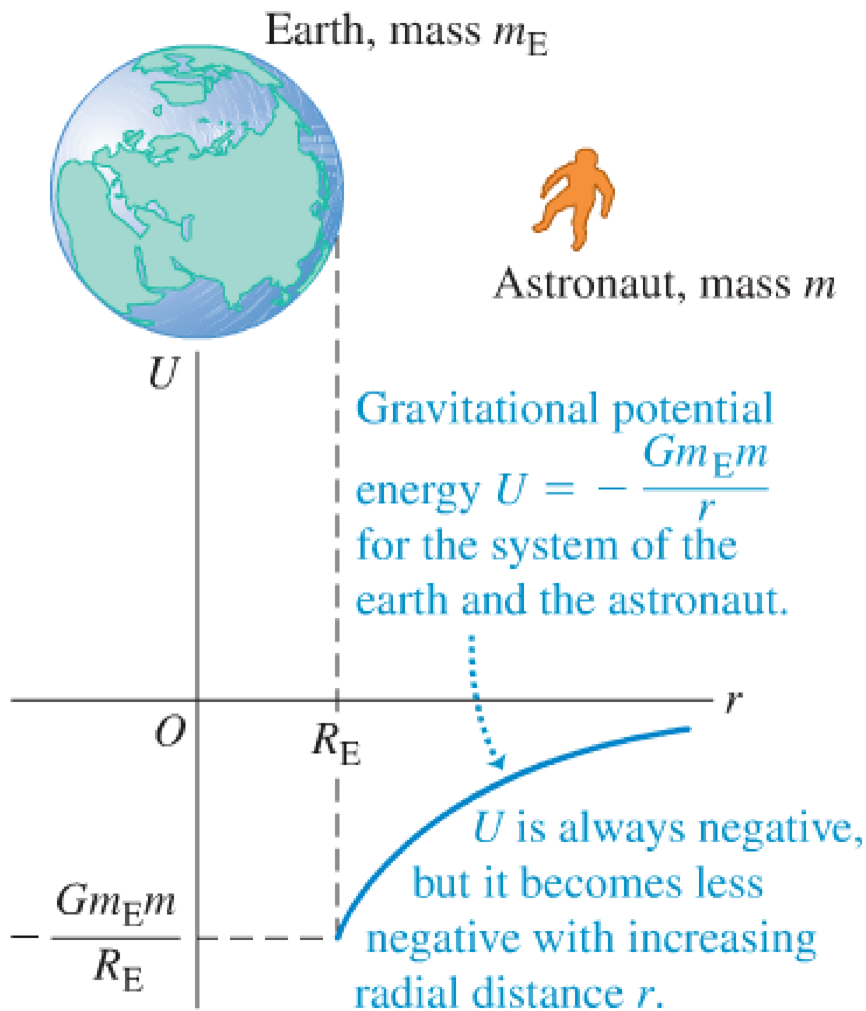
- Now, we have a force that varies with distance:

$$\Delta U_g = -W_g = \int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}\,dr = \frac{Gm_1m_2}{r_1} - \frac{Gm_1m_2}{r_2}$$

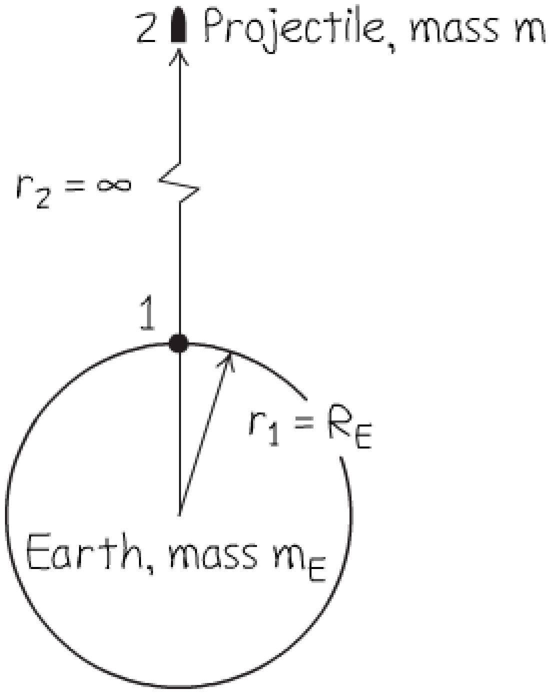
- If we define the zero of the potential now to be at infinity, we can set values for the potential:

$$U_g = -\frac{Gm_1m_2}{r}$$

Gravitational Potential Energy



Escape Velocity



From Conservation of energy:

$$\frac{1}{2}mv_1^{\,2} + \left(-\frac{Gm_E m}{R_E}\right) = 0 + 0$$

$$v_1 = \sqrt{\frac{2Gm_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}}$$

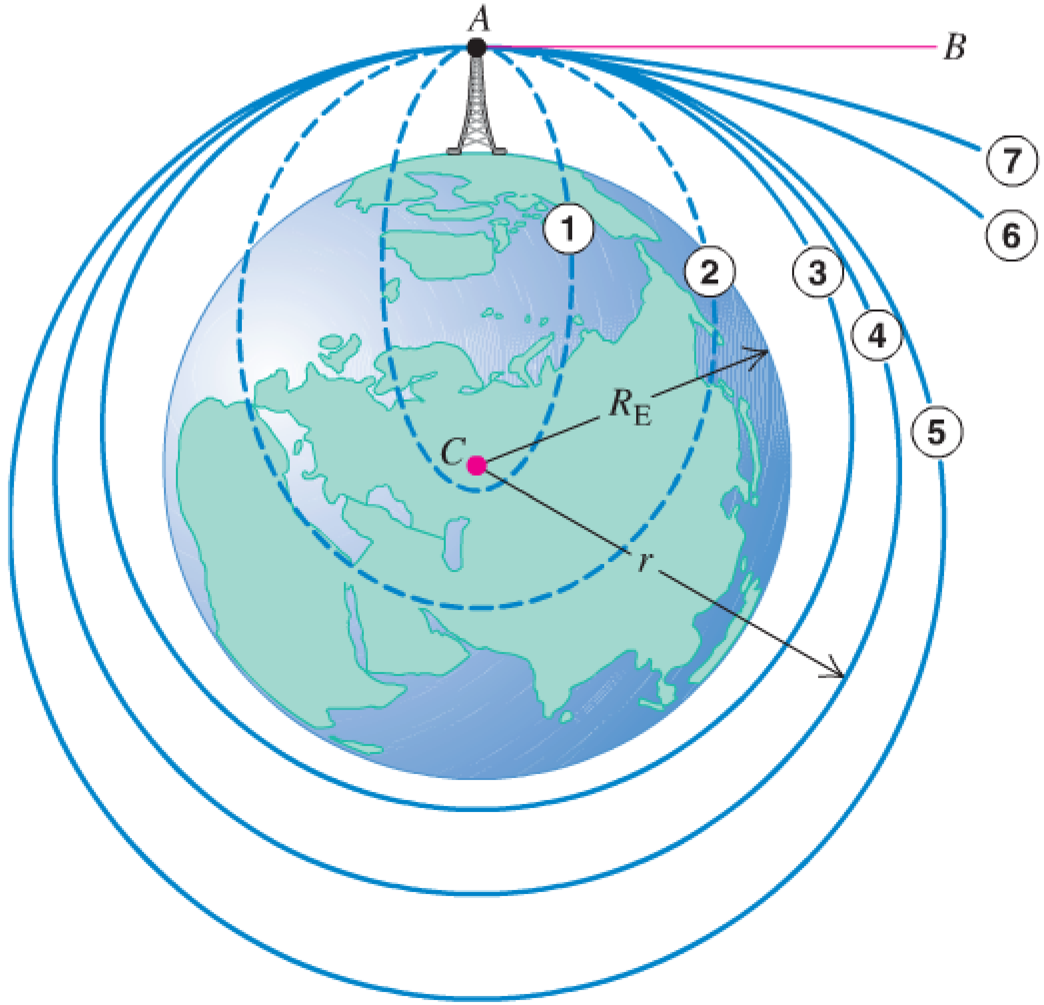
$$= 1.12 \times 10^4\ \mathrm{m/s}\,(= 40{,}200\ \mathrm{km/h} = 25{,}000\ \mathrm{mi/h})$$

Gravitational Potential Energy From More Than One Object

- Since potential energy is just a scalar, it adds just like any other quantity adds:

$$U_{g1} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_1m_4}{r_{14}} \ldots$$

Motion of Satellites

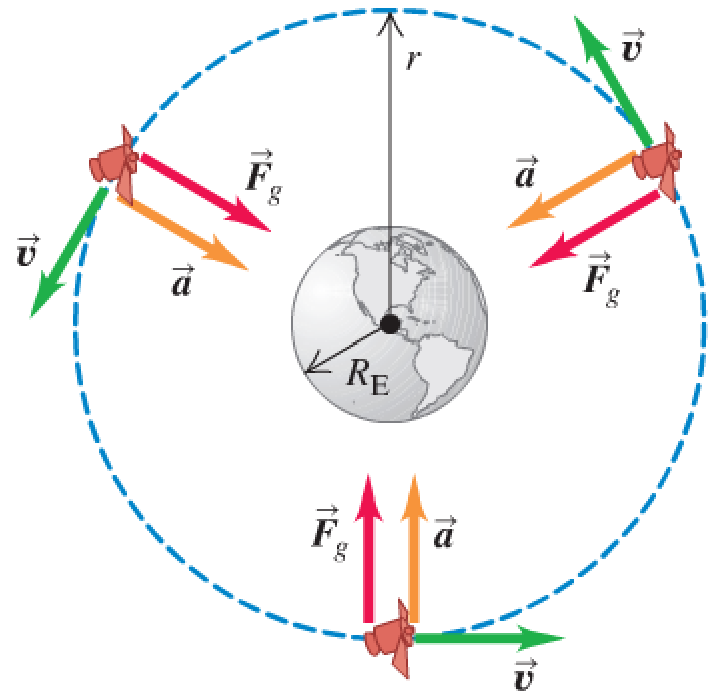


Motion of Satellites (Circular Orbit)

$$\sum \vec{F} = m\vec{a} \Rightarrow$$

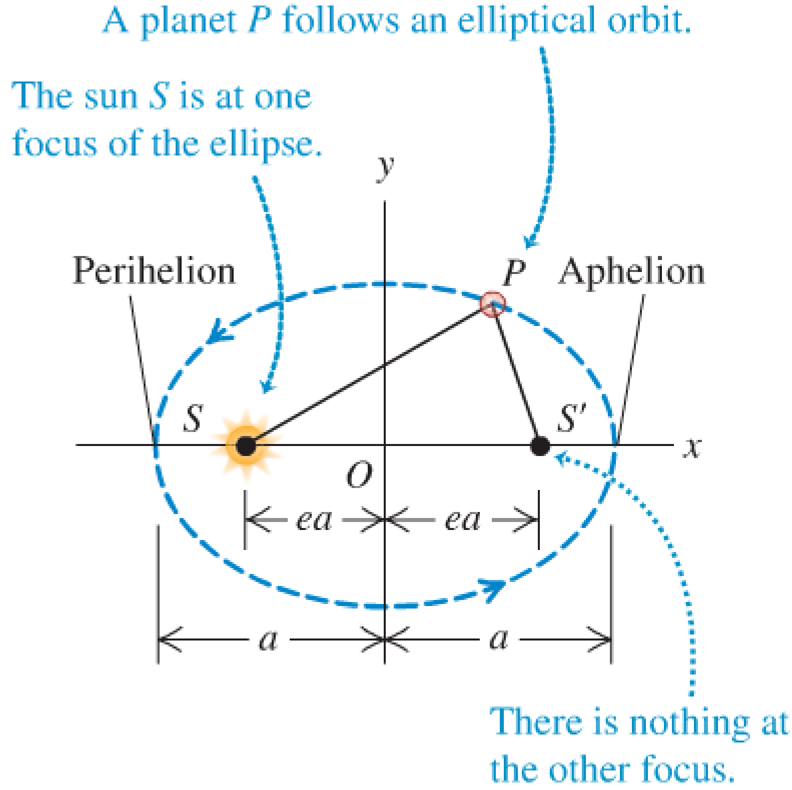
$$\frac{Gm_E m}{r^2} = m\frac{v^2}{r} \Rightarrow$$

$$v = \sqrt{\frac{Gm_E}{r}}$$



Kepler's Laws

1. **Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.**
2. **A line from the sun to a given planet sweeps out equal areas in equal times.**
3. **The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.**



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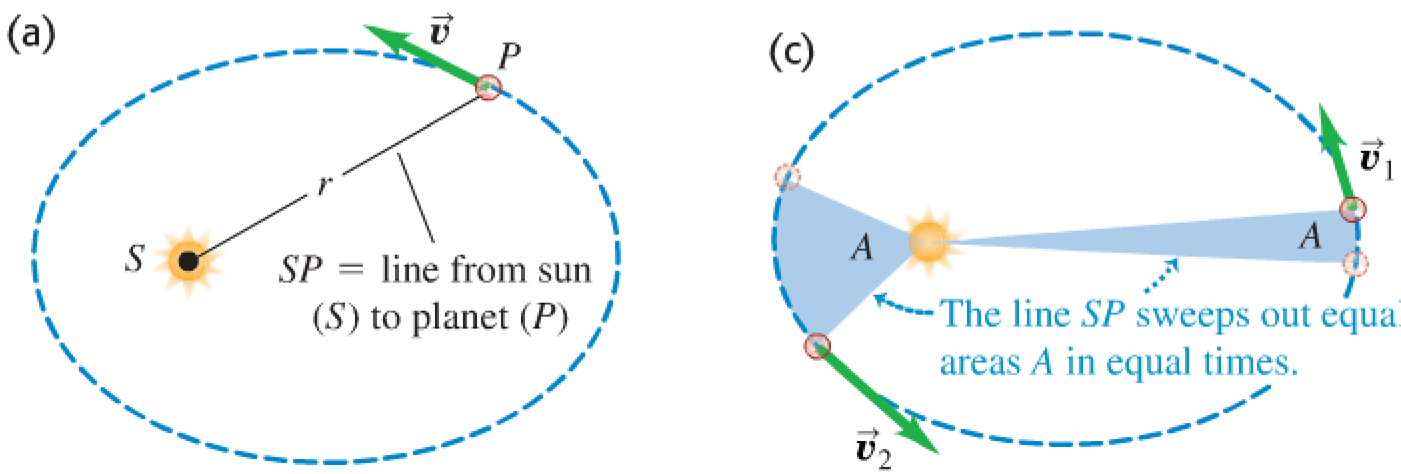
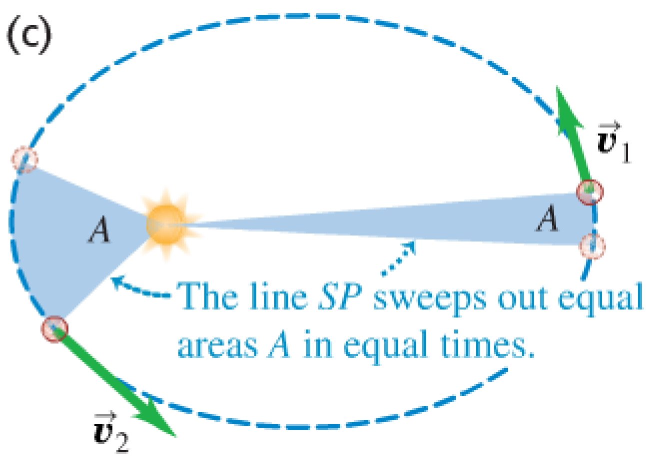
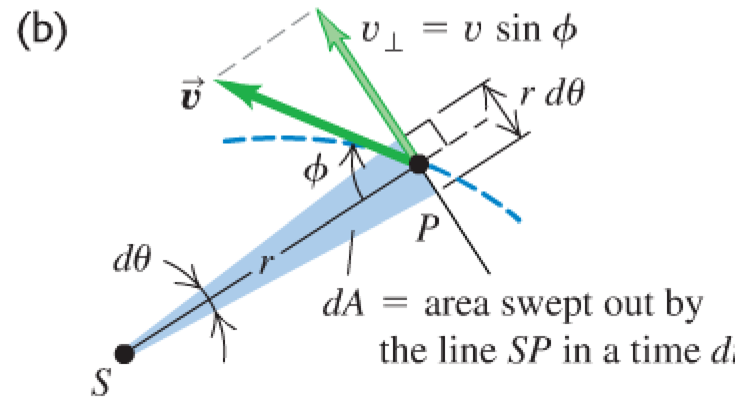


Kepler's Laws





$$dA = \frac{1}{2} r^2 d\theta \Rightarrow$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r \left(r \frac{d\theta}{dt} \right) = \frac{1}{2} r v_\perp = \frac{1}{2m} r m v_\perp = \frac{1}{2m} \vec{r} \times \vec{p} = \frac{1}{2m} \vec{L}$$

But, since the force acts along the line r, L is conserved, so dA/dt is constant!

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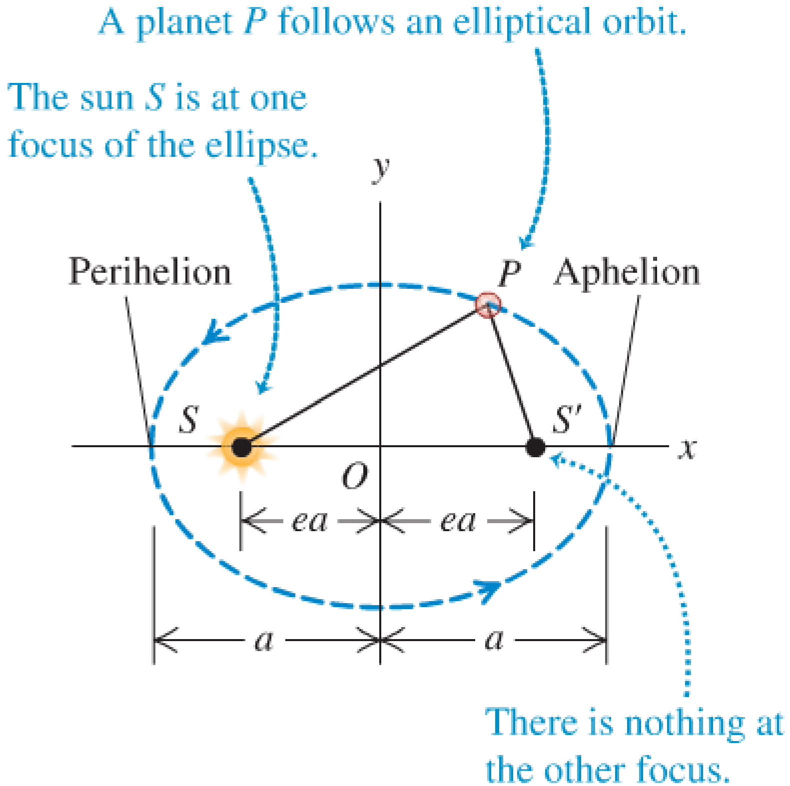


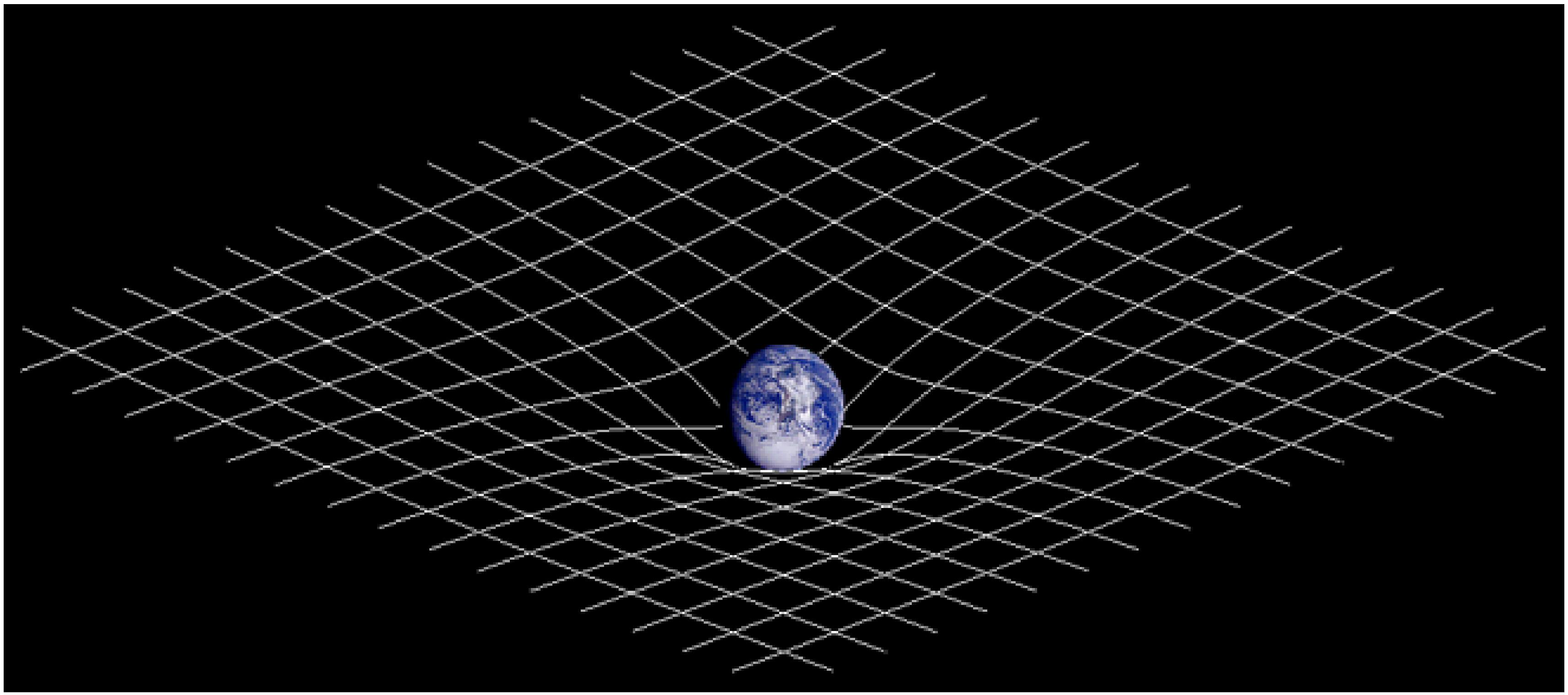
$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}}$$

Einstein's Gravity

- General relativity is a metric theory of gravitation. At its core are Einstein's equations, which describe the relation between the geometry of a four-dimensional, semi-Riemannian manifold representing spacetime on the one hand, and the energy-momentum contained in that spacetime on the other.[31] Phenomena that in classical mechanics are ascribed to the action of the force of gravity (such as free-fall, orbital motion, and spacecraft trajectories), correspond to inertial motion within a curved geometry of spacetime in general relativity; there is no gravitational force deflecting objects from their natural, straight paths. Instead, gravity corresponds to changes in the properties of space and time, which in turn changes the straightest-possible paths that objects will naturally follow.[32] The curvature is, in turn, caused by the energy-momentum of matter. Paraphrasing the relativist John Archibald Wheeler, spacetime tells matter how to move; matter tells spacetime how to curve.[33]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Space Time Curvature

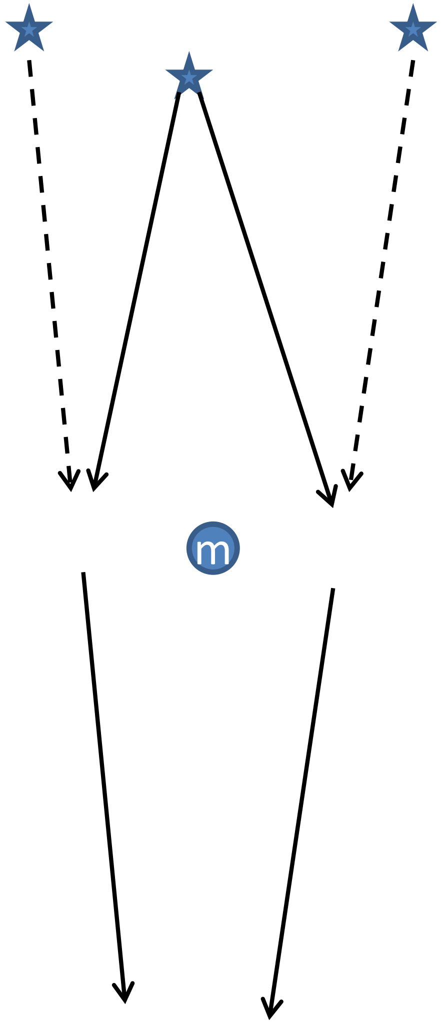


Gravity Affects Light Too!

- Light is also affected by gravity and can get pulled into an massive object.

Gravitational Lensing (simulation)

Gravitational Lensing



Black Holes

- Remember that we discussed the escape velocity of an object from a massive body (planet, sun, etc.).

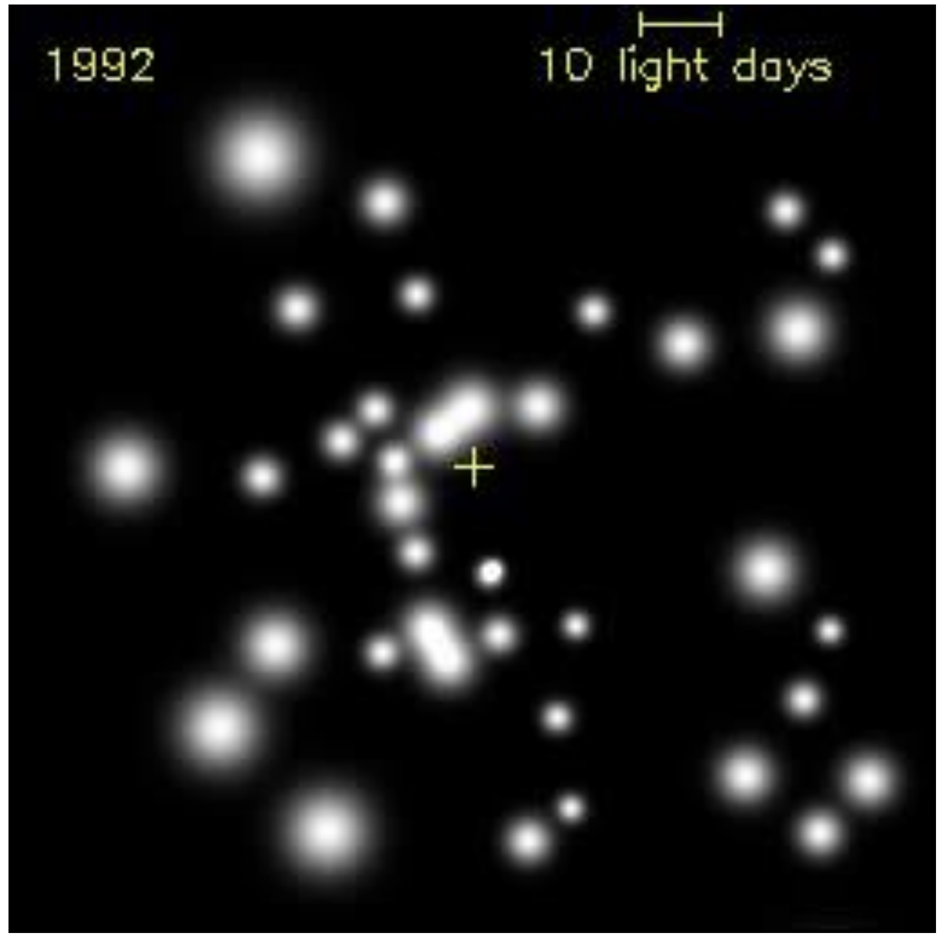
$$v = \sqrt{\frac{2Gm}{R}}$$

- We will learn later that light has a characteristic velocity, c = 3×10^8 m/s.
- What happens when the mass/radius of an object becomes so large that $v > c$?
- It's called a black hole.

Stars Around our Black Hole

- How do we know black holes exist?
- Look for objects that have orbits which require massive objects.

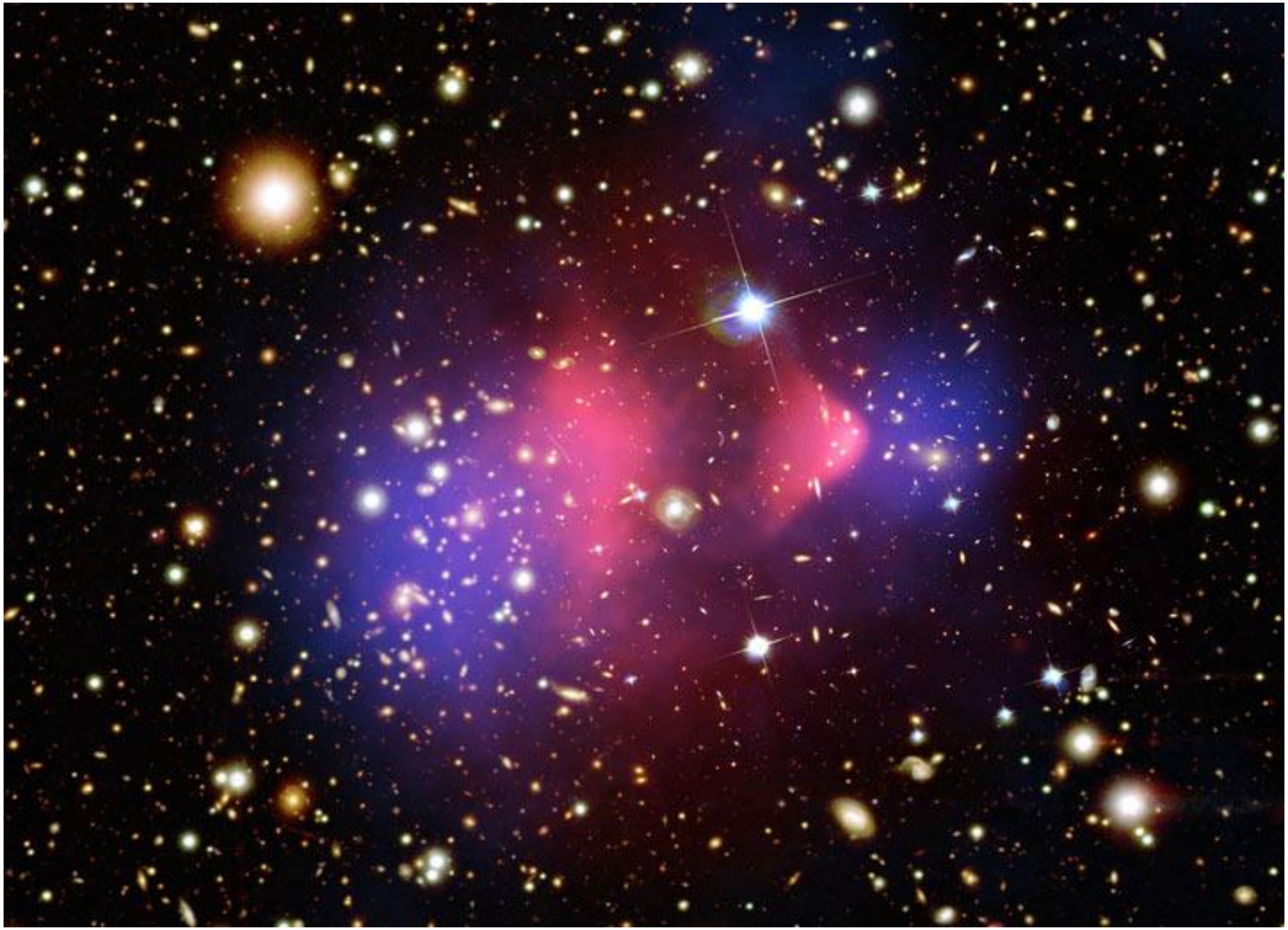
$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm}}$$



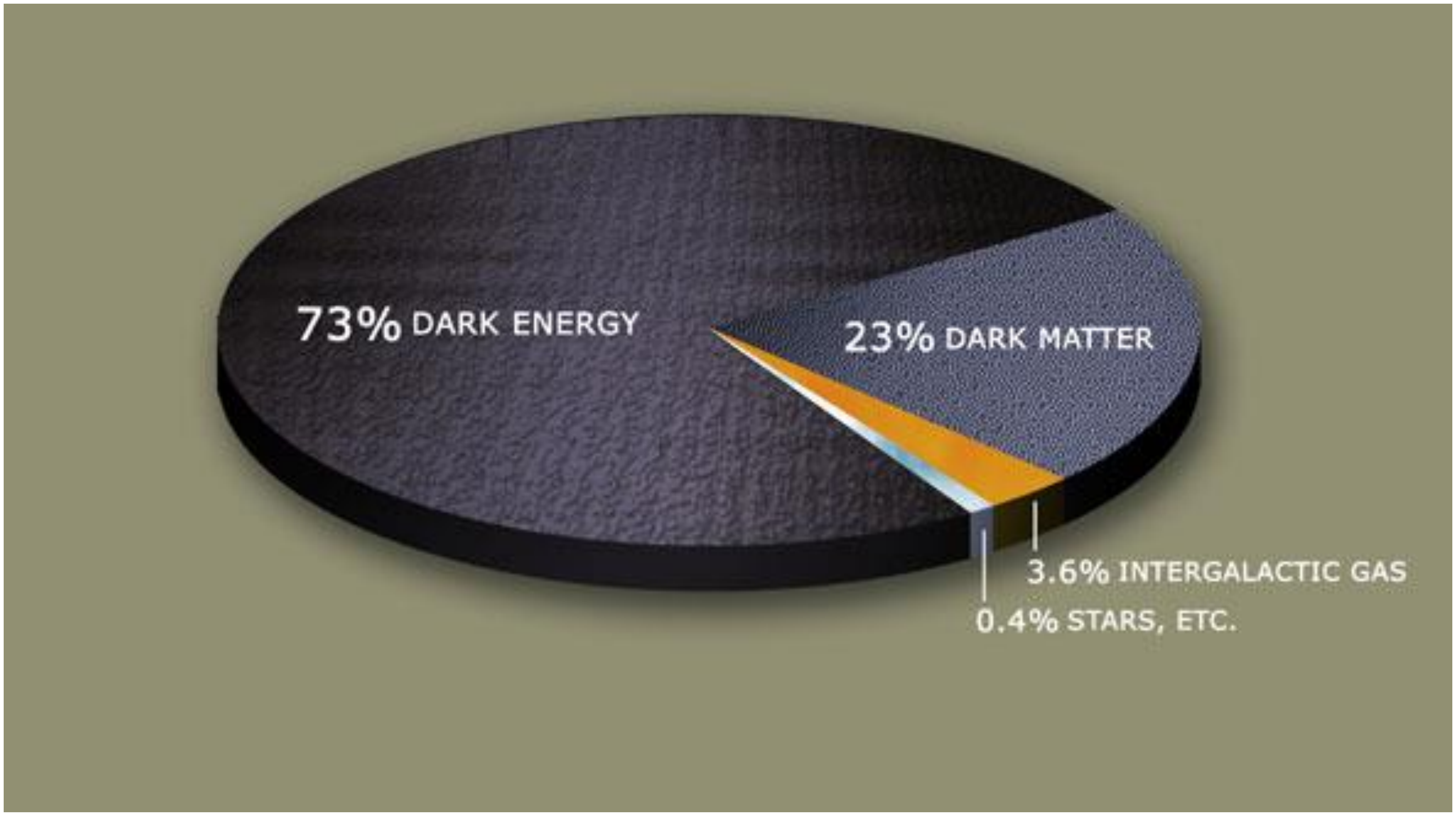
M = 3.7million times the mass of the sun!!!

Dark Matter

Dark Matter



Dark Energy


73% DARK ENERGY
23% DARK MATTER
3.6% INTERGALACTIC GAS
0.4% STARS, ETC.