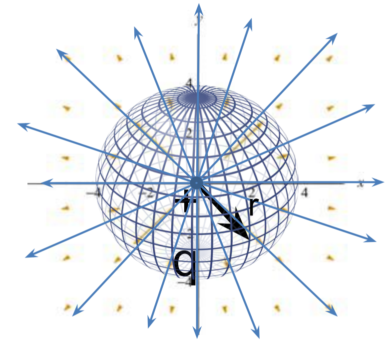
Lecture 10
(Maxwell's Equations)

Physics 2310-01 Spring 2020

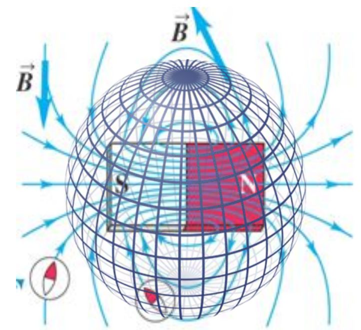
Douglas Fields

E&M Equations So Far

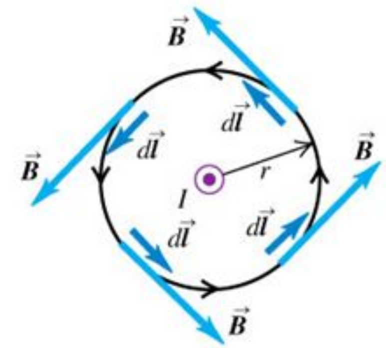
- Gauss's Law for E-Field $$\oint_{\partial V} \vec{E} \cdot d\vec{S} = \frac{q_{\text{enc}}}{\epsilon_0}$$



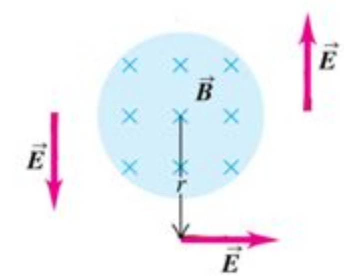
- Gauss's Law for B-Field $$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0$$



- Ampere's Law $$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{enc}}$$

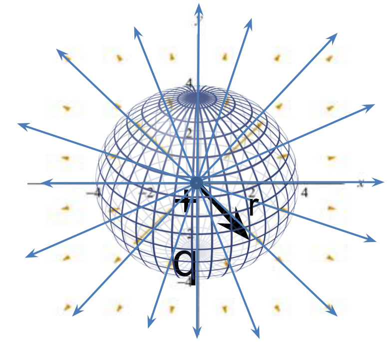


- Faraday's Law $$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t}\Phi_B$$



In Vacuum…

- Gauss's Law for E-Field $$\oint_{\partial V} \vec{E} \cdot d\vec{S} = 0$$



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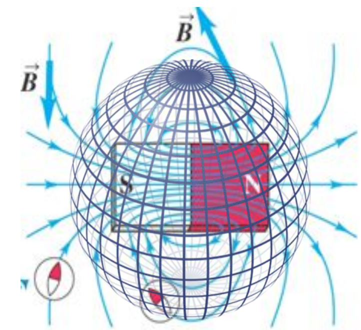
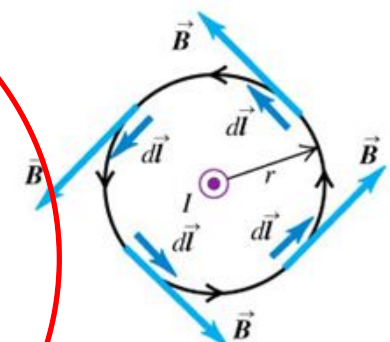
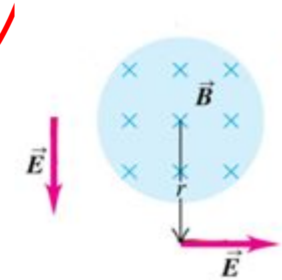


- Ampere's Law $$\oint_{\partial S} \vec{B} \cdot d\vec{r} = 0$$



- Faraday's Law $$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t}\Phi_B$$



Fixing Ampere's Law

- In 1865 (right after the American Civil War), James Clerk Maxwell was examining Ampere's Law and found a fundamental flaw.
- Fixing the flaw led to a fundamental shift in the way we understood nature.
- Because of that, all of the E&M equations were renamed in his honor.

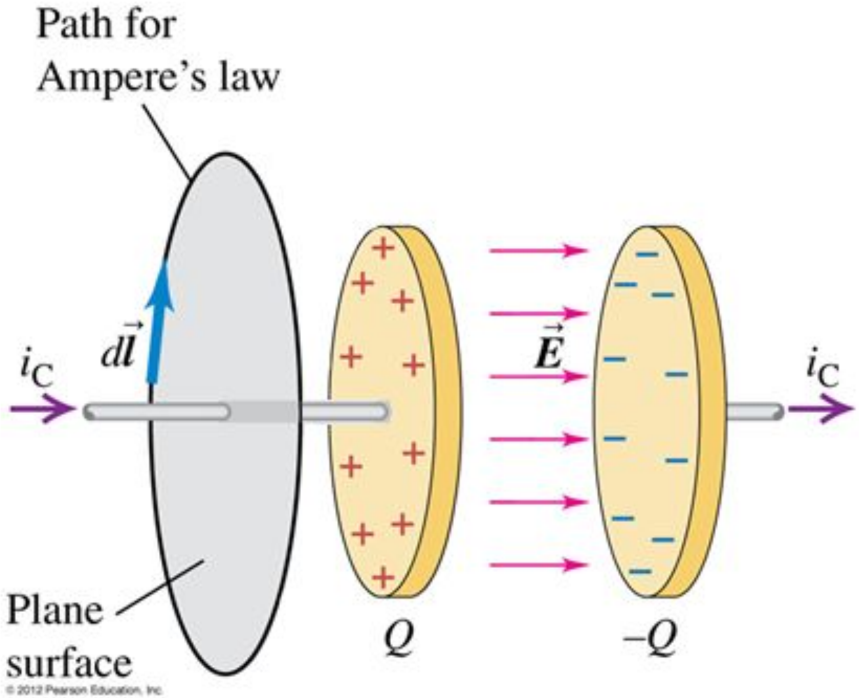
Fixing Ampere's Law

- Let's use Ampere's Law to examine the magnetic field around a wire with a current that is leading to a charging capacitor.
- Remember the English translation of Ampere's Law:
- The integral of the magnetic field components along a path, times the differential path lengths around a closed path bounding a surface is equal to a constant (μ_0) times the current which passes through that surface.

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{enc}}$$

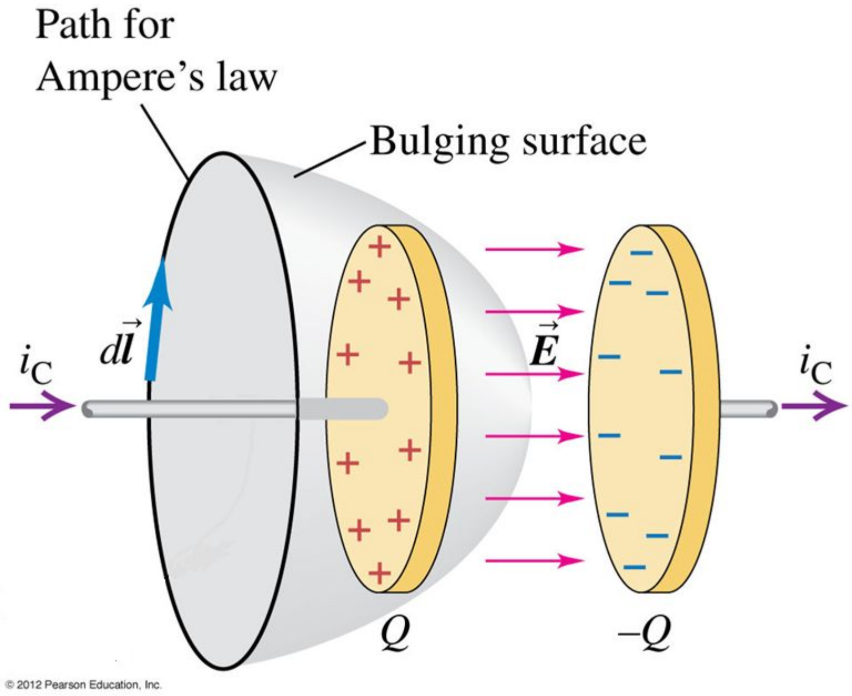
$$B 2\pi r = \mu_0 i_C$$

$$B = \frac{\mu_0 i_C}{2\pi r}$$



Fixing Ampere's Law

- Now, Maxwell realized that the surface could be any surface whose bound was still the same closed path.
- So, what about a bulging surface as shown below?
- NO CURRENT passes through this surface, but it has the same bound, so one would expect the same field on the bound as before…



Fixing Ampere's Law

- But, there IS something passing through the surface – a changing electric field:

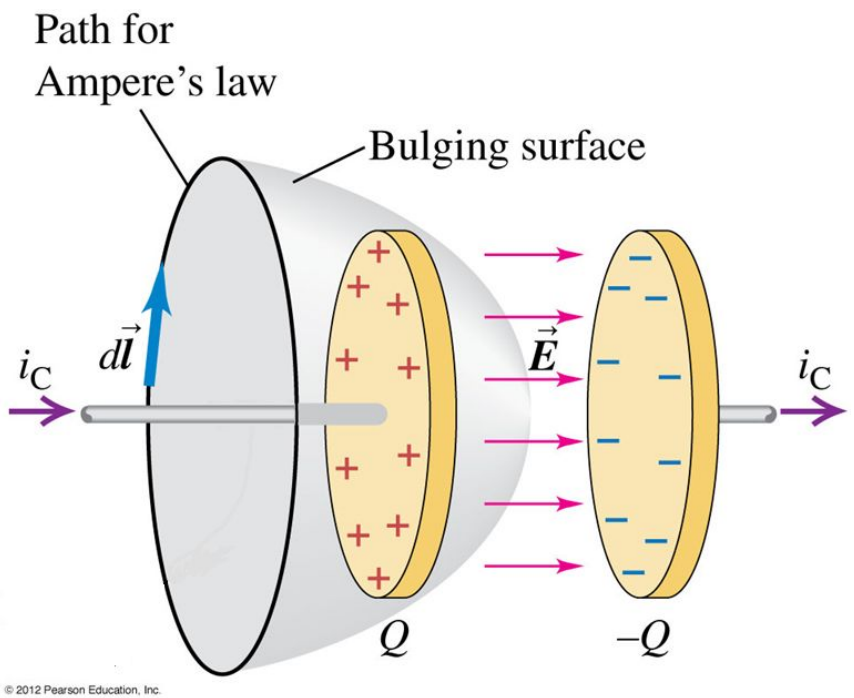
$$q(t) = CV(t) = \varepsilon_0 \frac{A}{d} V(t) = \varepsilon_0 \frac{A}{d} E(t) d \Rightarrow$$

$$q(t) = \varepsilon_0 E(t) A = \varepsilon_0 \Phi_E(t)$$

- Now, let's define a "current" analogous to the current in the wire, i_C, which Maxwell called the displacement current i_D:

$$q(t) = \varepsilon_0 \Phi_E(t) \Rightarrow$$

$$i_D = \frac{dq}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$



Fixing Ampere's Law

- Then we can rescue Ampere's Law by adding another "current" term:

$$q(t) = \varepsilon_0 \Phi_E(t) \Rightarrow$$

$$i_D = \frac{dq}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$

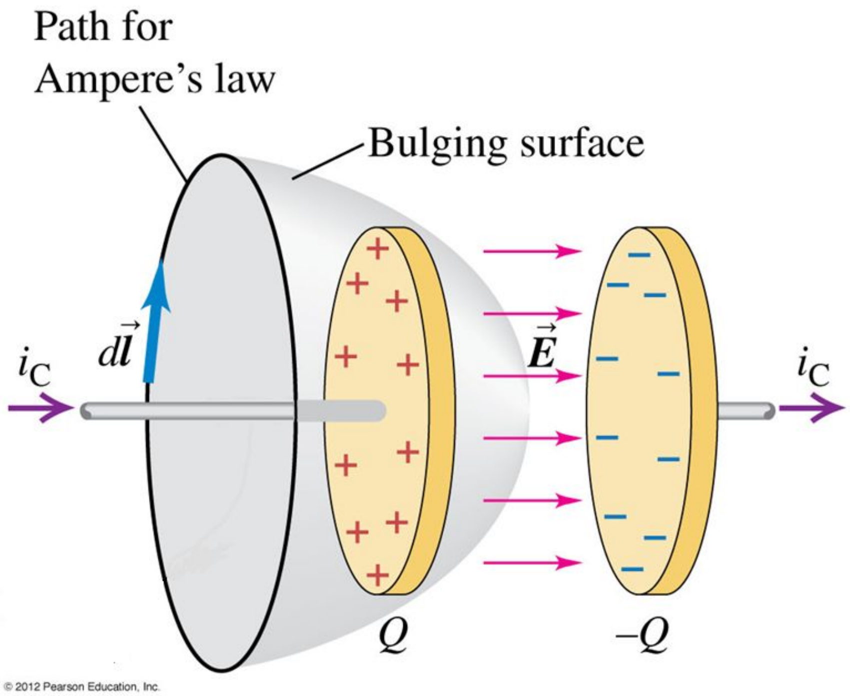
$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 (i_C + i_D)$$

$$B 2\pi r = \mu_0 \left(0 + \epsilon_0 \frac{d\Phi_E}{dt}\right)$$

$$B 2\pi r = \mu_0 \left(0 + \frac{dq}{dt}\right)$$

$$B 2\pi r = \mu_0 \left(i_C\right)$$

$$B = \frac{\mu_0 i_C}{2\pi r}$$



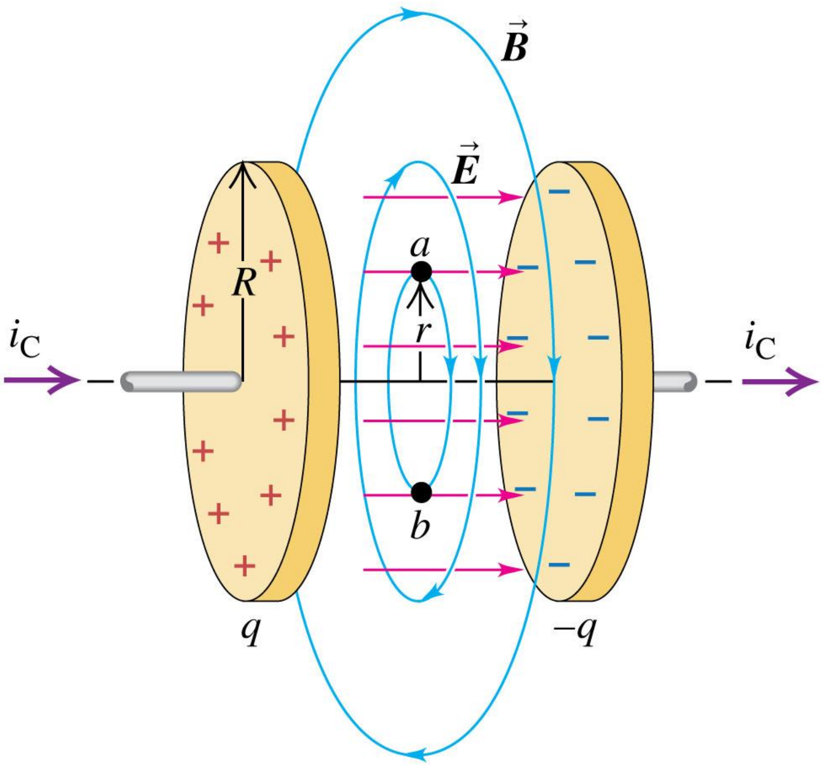
Fixing Ampere's Law

$$i_D = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{dE}{dt} A \Rightarrow$$

$$J_D = \varepsilon_0 \frac{dE}{dt}$$

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 (i_C + i_D) \Rightarrow$$

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 \oint (\vec{J}_C + \vec{J}_D) \cdot d\vec{S}$$



No, REALLY!

$$q\left(t\right)=\varepsilon_0 E\left(t\right)A\Rightarrow$$

$$E\left(t\right)=\frac{q\left(t\right)}{\varepsilon_0 A}\Rightarrow \quad \frac{dE}{dt}=\frac{1}{\varepsilon_0 A}i_C$$

$$i_D=\varepsilon_0\frac{d\Phi_E}{dt}=\varepsilon_0\frac{dE}{dt}A\Rightarrow$$

$$J_D=\varepsilon_0\frac{dE}{dt}$$

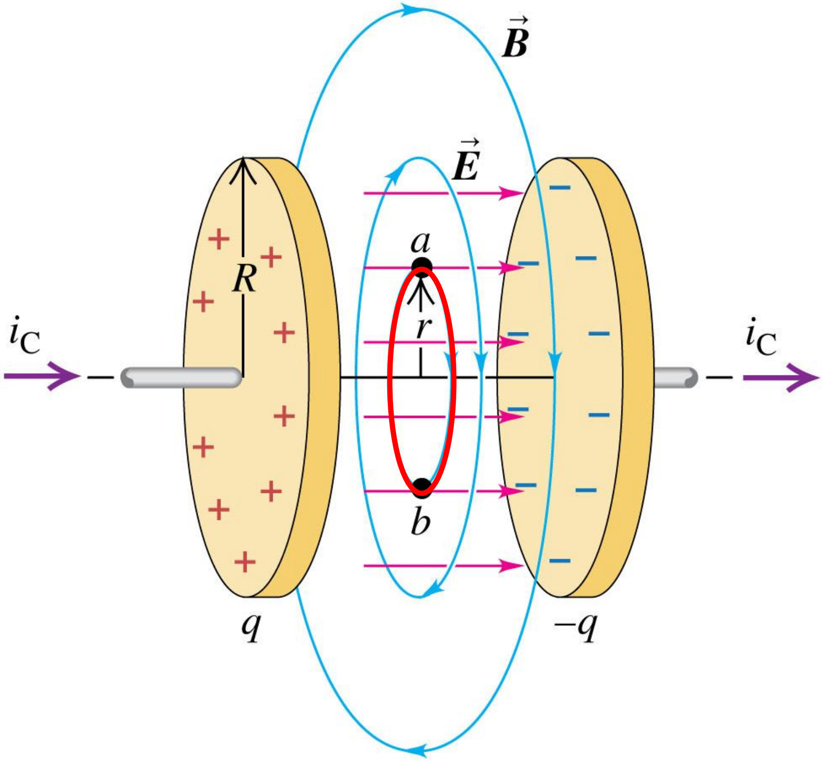
$$\oint_{\partial S}\vec{B}\cdot d\vec{r}=\mu_0\oint(\vec{J}_C+\vec{J}_D)\cdot d\vec{S}$$

$$B2\pi r=\mu_0\left(0+\epsilon_0\frac{dE}{dt}\right)\pi r^2$$

$$B=\frac{\mu_0\epsilon_0 r}{2}\frac{dE}{dt}$$

$$B=\frac{\mu_0\epsilon_0 r}{2}\frac{1}{\epsilon_0\pi R^2}\frac{dq}{dt}$$

$$B=\frac{\mu_0 r}{2\pi R^2}i_C$$



No, REALLY!

$$q(t) = \varepsilon_0 E(t) A \Rightarrow$$

$$E(t) = \frac{q(t)}{\varepsilon_0 A} \Rightarrow \quad \frac{dE}{dt} = \frac{1}{\varepsilon_0 A} i_C$$

$$i_D = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{dE}{dt} A \Rightarrow$$

$$J_D = \varepsilon_0 \frac{dE}{dt}$$

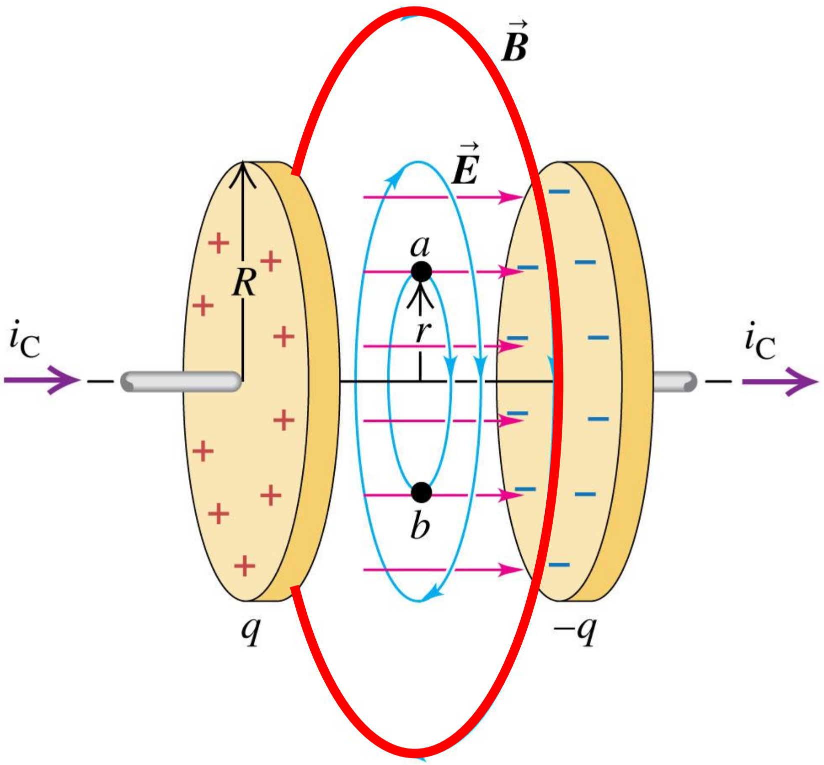
$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 \oint (\vec{J}_C + \vec{J}_D) \cdot d\vec{S}$$

$$B 2\pi r = \mu_0 \left(0 + \epsilon_0 \frac{dE}{dt}\right) \pi R^2$$

$$B = \frac{\mu_0 \epsilon_0 \pi R^2}{2\pi r} \frac{dE}{dt}$$

$$B = \frac{\mu_0 \epsilon_0 \pi R^2}{2\pi r} \frac{1}{\epsilon_0 \pi R^2} \frac{dq}{dt}$$

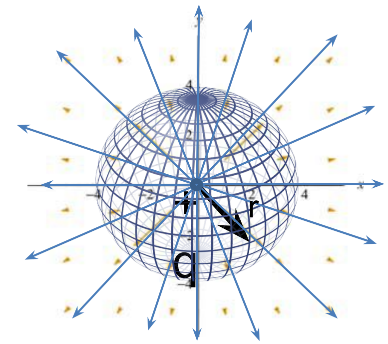
$$B = \frac{\mu_0}{2\pi r} i_C$$



Maxwell's Equations

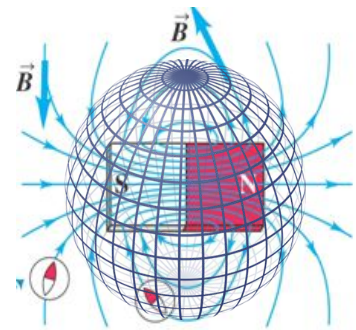
- Gauss's Law for E-Field

$$\oint_{\partial V} \vec{E} \cdot d\vec{S} = \frac{q_{\text{enc}}}{\epsilon_0}$$



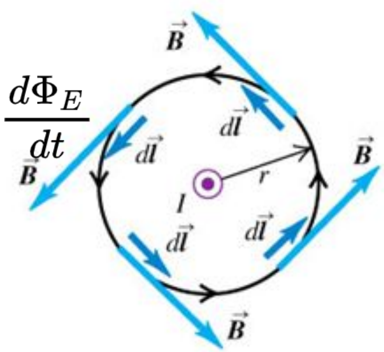
- Gauss's Law for B-Field

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0$$



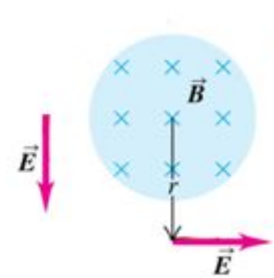
- Ampere's Law

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$



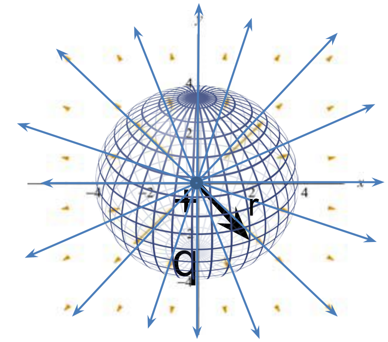
- Faraday's Law

$$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t}\Phi_B$$

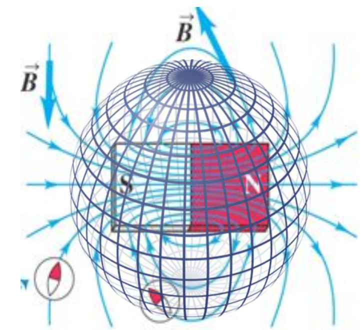


In Vacuum (no charges or currents)

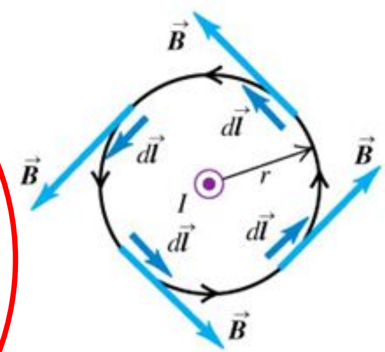
- Gauss's Law for E-Field $$\oint_{\partial V} \vec{E} \cdot d\vec{S} = 0$$



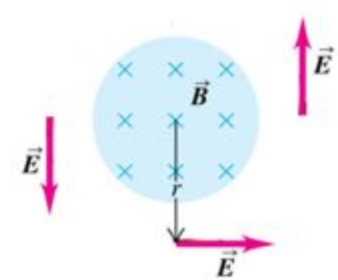
- Gauss's Law for B-Field $$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0$$



- Ampere's Law $$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$



- Faraday's Law $$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \Phi_B$$



Mathematical Aside

- We will need to use Stokes' Theorem: $\oint_{\partial S} \vec{V} \cdot d\vec{l} = \oint_{\partial V} \vec{\nabla} \times \vec{V} \cdot d\vec{S}$
- But what is $\vec{\nabla}$? And what does $\vec{\nabla} \times \vec{V}$ mean?
- Del: $\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) = \sum_{i=1}^{n} \vec{e_i} \frac{\partial}{\partial x_i}$
- In 3D: $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \vec{e_x} \frac{\partial}{\partial x} + \vec{e_y} \frac{\partial}{\partial y} + \vec{e_z} \frac{\partial}{\partial z}$
- But what is its meaning and utility?

Mathematical Aside (from Wikipedia)

- Grad operator: $$\text{grad } f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z = \nabla f$$
  - It always points in the direction of greatest increase of *f*, and it has a magnitude equal to the maximum rate of increase at the point.
- Divergence: $$\text{div } \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot \vec{v}$$
  - The divergence is roughly a measure of a vector field's increase in the direction it points; but more accurately, it is a measure of that field's tendency to converge toward or repel from a point.

Mathematical Aside (from Wikipedia)

- Curl: $$\mathrm{curl}\,\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{e_x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{e_y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{e_z} = \nabla \times \vec{v}$$
  - The curl at a point is proportional to the on-axis torque to which a tiny pinwheel would be subjected if it were centered at that point [and the vector field were a force].
- Laplacian: $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2$$

Back to Maxwell's Equations

- We can rewrite Ampere's Law using Stokes' Theorem:

$$\oint_{\partial S} \vec{V} \cdot d\vec{l} = \oint_{\partial V} \vec{\nabla} \times \vec{V} \cdot d\vec{S}$$

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \Rightarrow$$

$$\oint_{\partial V} \vec{\nabla} \times \vec{B} \cdot d\vec{S} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$= \mu_0 \epsilon_0 \frac{d}{dt} \oint_{\partial V} \vec{E} \cdot d\vec{S} \Rightarrow$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Back to Maxwell's Equations

- We have:

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \Rightarrow$$

- Then, take the curl of both sides:

$$\vec{\nabla} \times \left[\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \right] \Rightarrow$$

$$\nabla \left(\cancel{\vec{\nabla} \cdot \vec{B}} \right) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E} \Rightarrow$$

$$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E}$$

Back to Maxwell's Equations

- We can now do the same thing with Faraday's Law:

$$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t}\Phi_B$$

- Leads to (by applying Stokes' Theorem):

$$\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}$$

- Then, putting this into the result from the previous page,

$$-\nabla^2 B = \mu_0 \varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E} = -\mu_0 \varepsilon_0 \frac{d^2 B}{dt^2}$$

$$\nabla^2 B = \mu_0 \varepsilon_0 \frac{d^2 B}{dt^2}$$

And then, a miracle occurs…

- Wave Equation

$$y(x,t) = A\cos(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y$$

$$but, \quad \omega = vk \Rightarrow v^2 = \frac{\omega^2}{k^2} \Rightarrow$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\nabla^2 B = \mu_0\varepsilon_0 \frac{d^2 B}{dt^2} \quad \text{and} \quad \nabla^2 E = \mu_0\varepsilon_0 \frac{d^2 E}{dt^2}$$

- EM wave velocity

$$v = \frac{1}{\sqrt{(\mu_0\varepsilon_0)}} = \frac{1}{\sqrt{\left(4\pi \times 10^{-7}\, H/m\right) \times \left(8.854 \times 10^{-12}\, F/m\right)}} = 2.998 \times 10^8\, m/s = c$$