Lecture 12 (Poynting Vector and Standing Waves)

Physics 2310-01 Spring 2020

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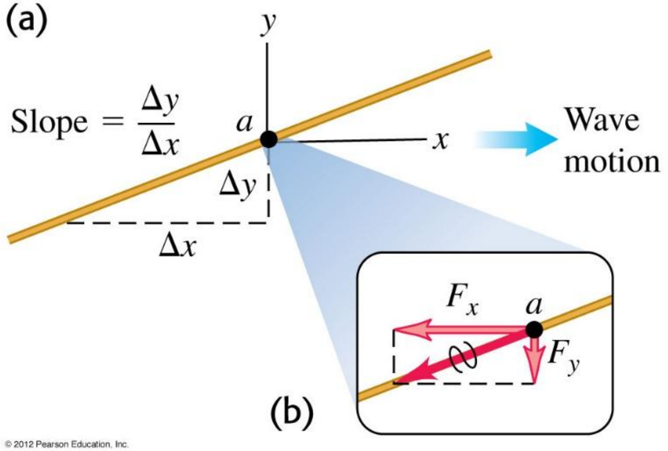
Review of Power in Waves

- Let's examine a wave moving to the right on a string.
- At point a, the force from the string to the left, F, can be broken down into the horizontal and vertical components using the slope of the string:



$$F_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}$$

- And then the power at a is just given by the force times the velocity of the piece of string at that point:

$$P(x,t) = F_y(x,t)v_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial t}$$

- Since, $y(x,t) = A\cos(kx-\omega t)$,
- Then,

$$\frac{\partial y(x,t)}{\partial x} = -kA\sin(kx-\omega t) \quad \frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t)$$

- And the power is given by: $$P(x,t) = -Fk\omega A^2\sin^2(kx-\omega t)$$

- Thus, the time averaged power is given by: $$P_{\text{Avg}} = \frac{1}{2}Fk\omega A^2$$
- Notice that it depends on the amplitude squared

Power in an EM Wave

- Let's review what we know about the energy stored in fields:

$$u_E = \frac{1}{2}\varepsilon_0 E^2, \quad u_B = \frac{1}{2}\frac{1}{\mu_0}B^2$$

- So, at any point in the electromagnetic wave,

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0}B^2$$

- But, $E = cB \Rightarrow B = \frac{E}{c}$

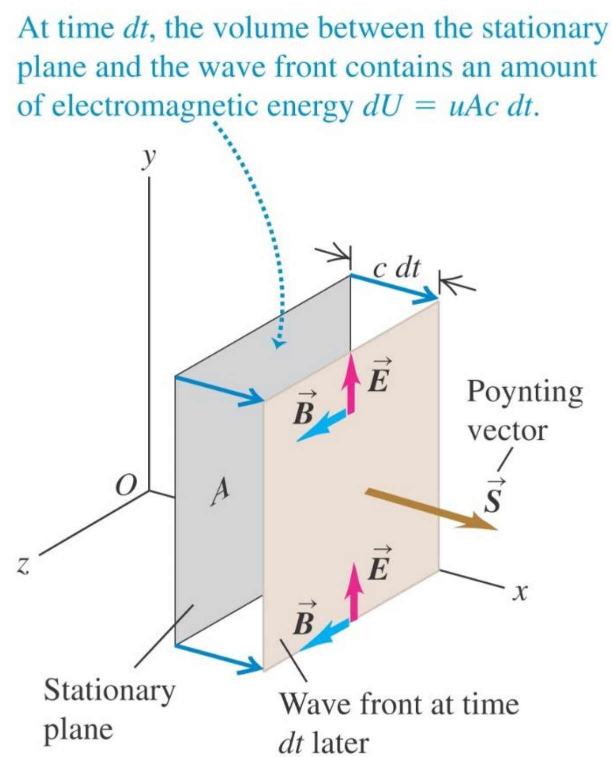
- So

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0 c^2}E^2 = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2$$

$$\mu_0 c^2 = \frac{\mu_0}{\mu_0 \varepsilon_0} = \frac{1}{\varepsilon_0}$$

Power in an EM Wave

- But this is the energy density at a localized point in space. We want to describe how this energy moves with the wave.
- Let's examine the energy (dU) in a plane wave that passes through a certain area (A) within a certain time (dt).



$$dU = udV = uA(cdt) \Rightarrow$$
$$dU = (\varepsilon_0 E^2) Acdt$$

Poynting Vector

$$dU = \left(\varepsilon_0 E^2\right) Acdt$$

- Now, let us define the ***energy per unit time, per unit area*** as:

$$S = \frac{1}{A}\frac{dU}{dt} = c\varepsilon_0 E^2$$

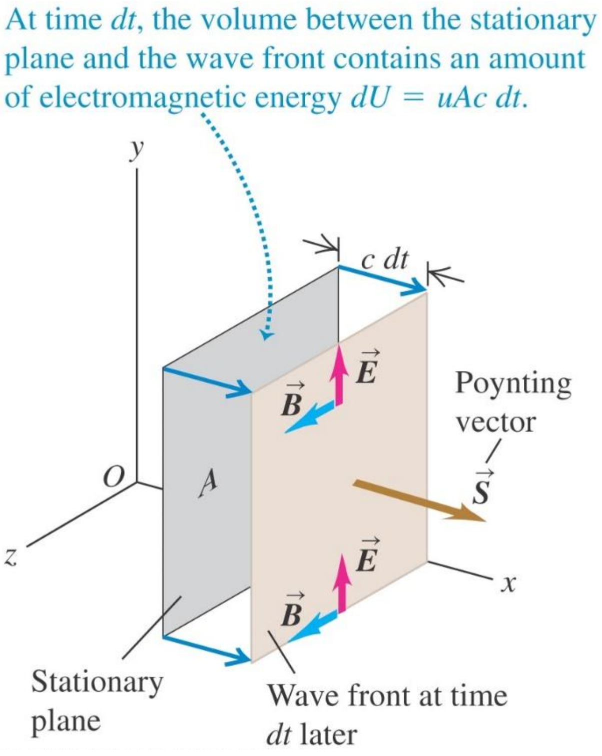
- Notice that it scales with the square of the field (wave amplitude).
- Now, just for fun, let's rewrite this using E = cB:

$$S = c^2\varepsilon_0 EB = \frac{1}{\mu_0} EB$$

- And put it in vector form to denote the direction of energy transmission:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

Poynting Vector



Poynting Vector

- Remember, the Poynting vector represents the ***energy per time per unit area*** instantaneously passing through an area at a particular point in space.

$$S = \frac{\text{energy}}{\text{time} \cdot \text{area}} = \frac{\text{power}}{\text{area}}$$

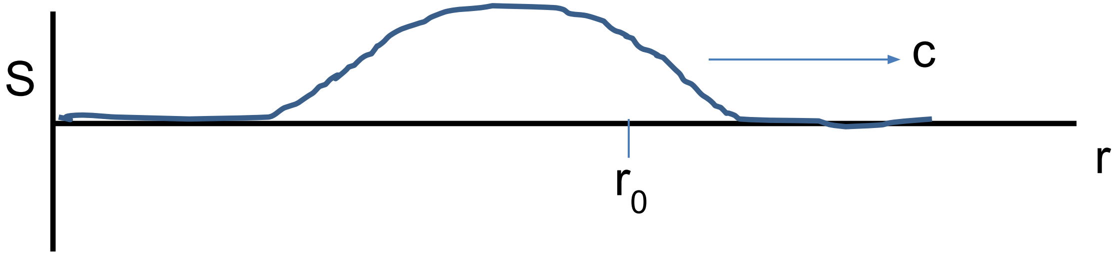
- So, if we want to know the entire *instantaneous* power output from a source, we have to integrate the Poynting vector over a closed surface containing the source:

$$P = \oint \vec{S} \cdot d\vec{A}$$

- We'll return to this in a moment, but let's first get rid of that annoying term: instantaneous.

Average Power (per unit area)

- The time average of the Poynting vector will depend on the its time dependence, and over what length of time you average.
- For instance, look at the following pulse:



- If we integrate over the time from when the pulse just reaches r_0 until the maximum reaches r_0, we will get one answer.
- If we integrate over all time, we will get an answer that will approach zero…
- So, when we speak of average power, we either have to be precise in our description of the time integration, or…

Average Power (per unit area)

- Assume a never-ending sinusoidal plane wave.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}(x,t)\times\vec{B}(x,t)$$
$$= \frac{1}{\mu_0}\left[\hat{j}E_{\text{Max}}\sin(kx-\omega t)\right]\times\left[\hat{k}B_{\text{Max}}\sin(kx-\omega t)\right]$$
$$= \frac{\hat{i}}{\mu_0}E_{\text{Max}}B_{\text{Max}}\sin^2(kx-\omega t)$$

- Then, because the average of $\sin^2$ is ½,

$$I = \left|\vec{S}_{\text{Avg}}\right| = \frac{1}{2\mu_0}E_{\text{Max}}B_{\text{Max}} = \frac{E_{\text{Max}}^2}{2\mu_0 c} = \frac{1}{2}\varepsilon_0 c E_{\text{Max}}^2$$

- Where, I is the intensity of the EM wave.

Example

- Consider a source of EM waves, say a light bulb that puts out 100W spread equally in all directions. What is the value of the intensity of light 10m from the bulb?

$$\frac{P}{A(r)} = \frac{100W}{4\pi(10m)^2} = \frac{1W}{4\pi m^2} = S_{\text{Avg}} = I$$

- But remember that the intensity is proportional to the amplitude squared:

$$I = \left|\vec{S}_{\text{Avg}}\right| = \frac{1}{2\mu_0} E_{\text{Max}} B_{\text{Max}} = \frac{E_{\text{Max}}^2}{2\mu_0 c} = \frac{1}{2}\varepsilon_0 c E_{\text{Max}}^2$$

- So, as pointed out earlier, for spherical waves.

$$\left|\vec{E}(r,t)\right| = \frac{E_0}{r}\sin(kr - \omega t)$$

Electromagnetic Momentum Flow

- Electromagnetic waves also carry momentum*, with a momentum density:

$$\frac{dp}{dV} = \frac{EB}{\mu_0 c^2} = \frac{S}{c^2}$$

- And you can then consider a momentum flow rate (momentum passing through unit area in a period of time dt):

$$\frac{1}{A}\frac{dp}{dt} = \frac{EB}{\mu_0 c} = \frac{S}{c}$$

- Since $dV = Acdt$

*see http://farside.ph.utexas.edu/teaching/em/lectures/node90.html for a discussion of how we know this.

Radiation Pressure

- And the average momentum flow is just:

$$\left\langle \frac{1}{A}\frac{dp}{dt} \right\rangle = \frac{S_{\text{Avg}}}{c} = \frac{I}{c}$$

- Now, remember Newton's second law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

- And since pressure is just force per area,

$$\left\langle \frac{1}{A}\frac{dp}{dt} \right\rangle = \left\langle \frac{F}{A} \right\rangle = \left\langle p_{rad} \right\rangle = \frac{S_{\text{Avg}}}{c} = \frac{I}{c}$$

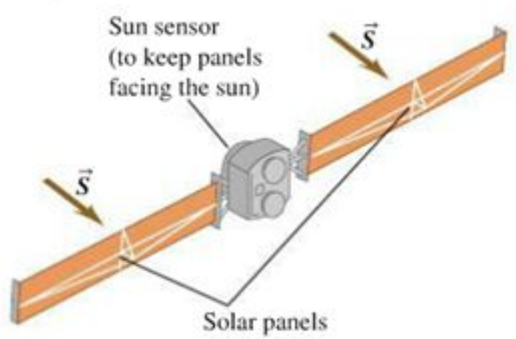
- If the wave is reflected, it's twice this.

Example

Example 32.5 Power and pressure from sunlight

An earth-orbiting satellite has solar energy–collecting panels with a total area of 4.0 m^2 (Fig. 32.21). If the sun's radiation is perpendicular to the panels and is completely absorbed, find the average solar power absorbed and the average radiation-pressure force.

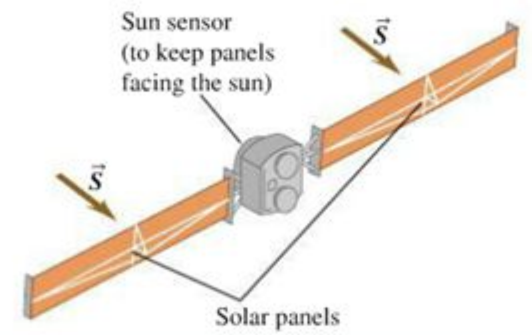
32.21 Solar panels on a satellite.



The intensity I (power per unit area) is $1.4 \times 10^3\ W/m^2$.

Example

32.21 Solar panels on a satellite.

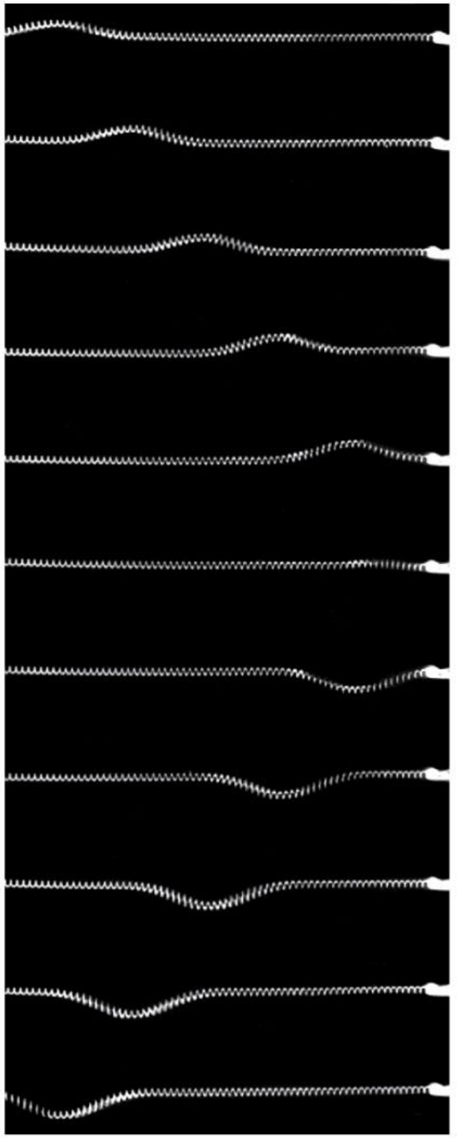


The intensity I (power per unit area) is $1.4 \times 10^3\ \mathrm{W/m^2}$.

$$p_{rad} = \frac{I}{c} = \frac{1.4\times10^3\ W/m^2}{3.0\times10^8\ m/s} = 4.7\times10^{-6}\,\mathrm{Pa}$$

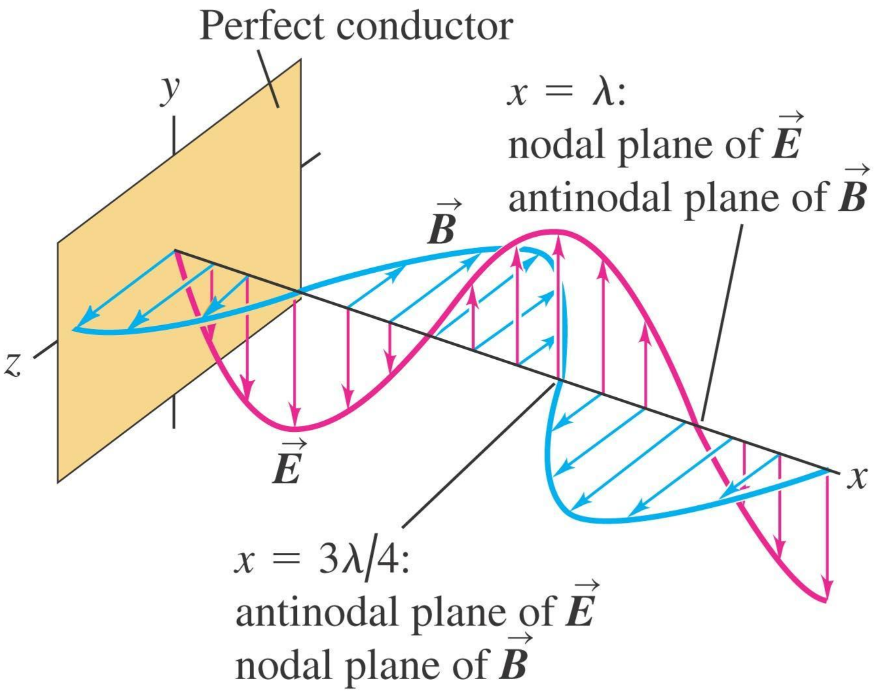
Reflections at Conducting Surfaces

- What happens when a transverse matter wave strikes a fixed boundary?
- The wave is reflected and inverted.
- A similar thing happens when an EM wave strikes a conducting boundary…



Reflections at Conducting Surfaces

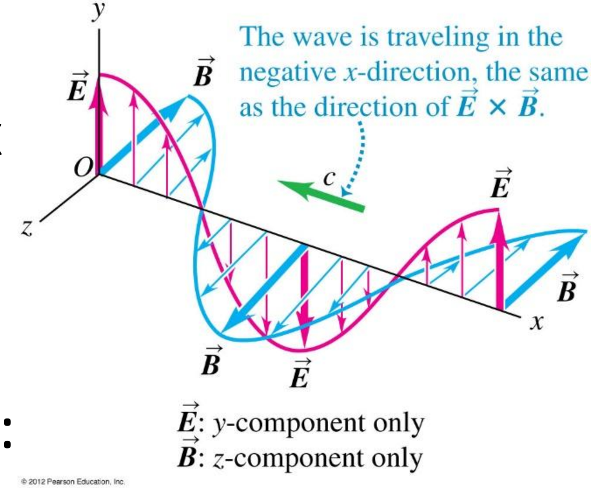
- The electric field of the wave causes charges to move on the surface of the conductor.
- The net effect of their motion is to keep the electric field inside the conductor **zero**.
- The moving charges also cause a magnetic field.
- Hence, these charges create an electromagnetic wave which is inverted from the original and travels in the opposite direction, away from the conductor.



Standing EM Waves

- Take the incoming wave (moving in the –x direction) as:

$$\vec{E}_{in}(x,t)=\hat{j}E_{\text{Max}}\cos(kx+\omega t),\quad \vec{B}_{in}(x,t)=-\hat{k}B_{\text{Max}}\cos(kx+\omega t)$$



- Then the reflected wave must be given by:

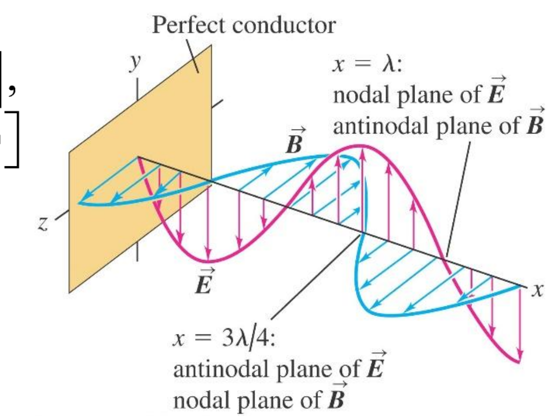
$$\vec{E}_{ref}(x,t)=-\hat{j}E_{\text{Max}}\cos(kx-\omega t),\quad \vec{B}_{ref}(x,t)=-\hat{k}B_{\text{Max}}\cos(kx-\omega t)$$

- The resulting superposition of these two is:

$$\vec{E}_{tot}(x,t)=\vec{E}_{in}(x,t)+\vec{E}_{ref}(x,t)=\hat{j}E_{\text{Max}}\left[\cos(kx+\omega t)-\cos(kx-\omega t)\right],$$
$$\vec{B}_{tot}(x,t)=\vec{B}_{in}(x,t)+\vec{B}_{ref}(x,t)=-\hat{k}B_{\text{Max}}\left[\cos(kx+\omega t)+\cos(kx-\omega t)\right]$$

- Or, simplifying:

$$\vec{E}_{tot}(x,t)=-\hat{j}2E_{\text{Max}}\sin(kx)\sin(\omega t),$$
$$\vec{B}_{tot}(x,t)=-\hat{k}2B_{\text{Max}}\cos(kx)\cos(\omega t)$$



Standing EM Waves

- There are then nodes in both the electric field at:

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots$$

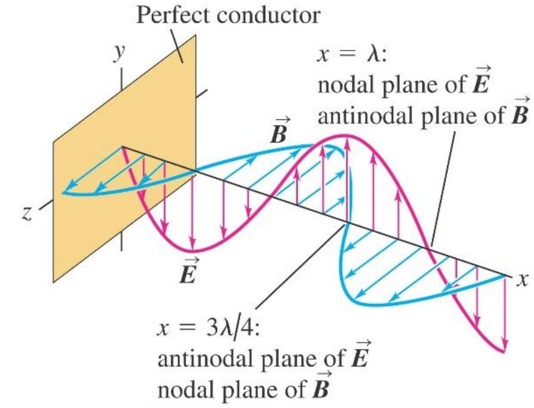
- And in the B-field at:

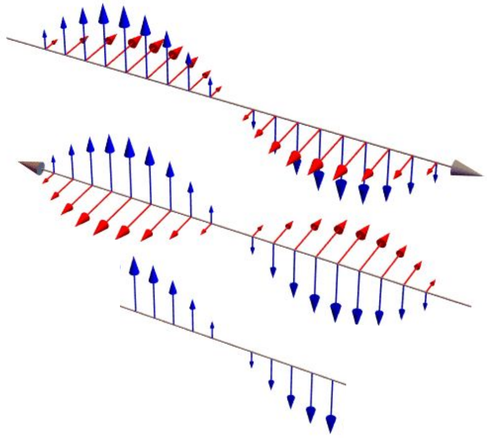
$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots$$

- Note also that the electric and magnetic field are out of phase *in time*

$$\vec{E}_{tot}(x,t) = -\hat{j}2E_{\text{Max}}\sin(kx)\sin(\omega t),$$
$$\vec{B}_{tot}(x,t) = -\hat{k}2B_{\text{Max}}\cos(kx)\cos(\omega t)$$

- When ωt is such that the electric field is zero everywhere, the magnetic field is maximum everywhere!





B is blue, E is red

Standing EM Waves

- What if we put in another conducting plane, we create an electromagnetic cavity that will sustain a resonance for EM waves of wavelength:

$$\lambda_n = \frac{2L}{n}, \quad (n = 1, 2, 3...)$$