Lecture 2
(Simple Harmonic Motion)

Physics 2310-01 Spring 2020
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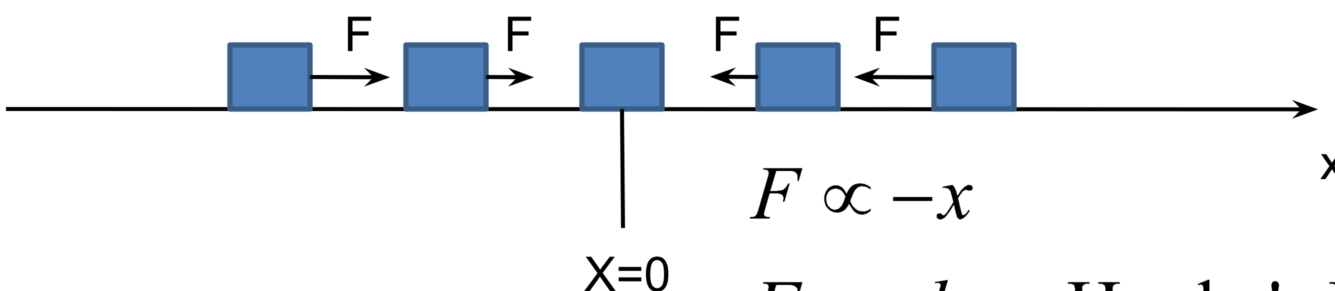
Simple Harmonic Motion

- A type of periodic motion with a very explicit definition:
- Motion about an equilibrium point with a restoring force proportional to the distance away from the equilibrium point.

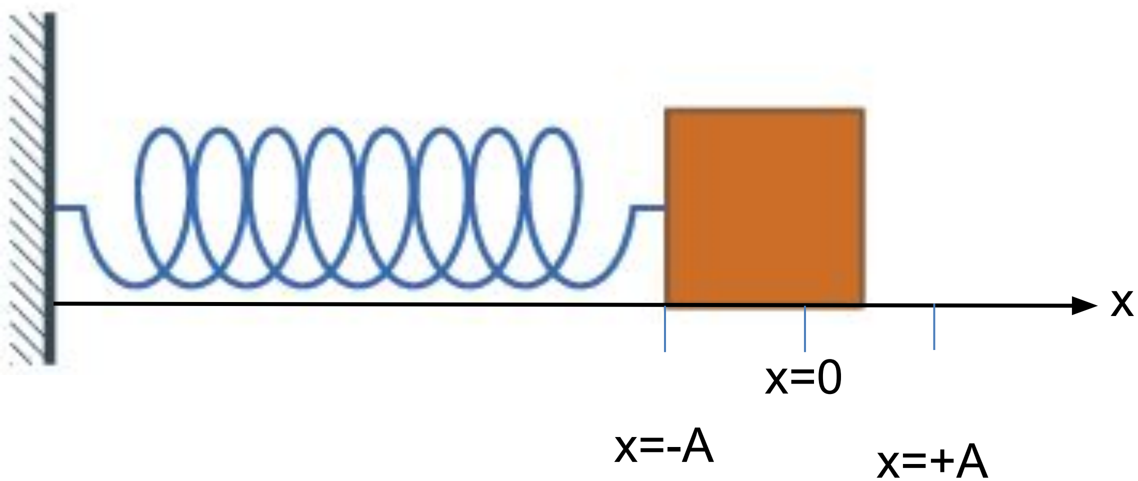


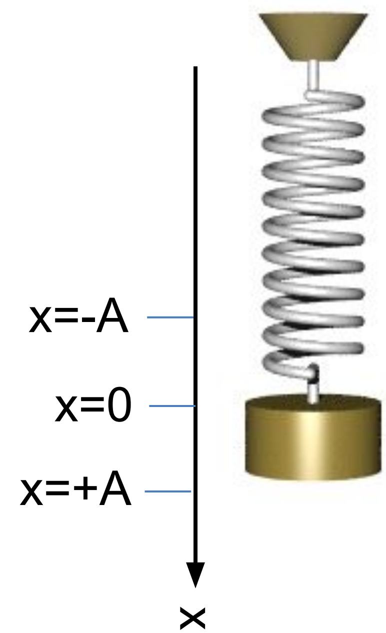
$$F \propto -x$$

$$F = -kx \quad \text{Hooke's Law}$$

Simple Harmonic Motion

$$F_{Net} = -kx$$





Simple Harmonic Motion

- Analyze:

$$F = -kx \Rightarrow \quad \text{Hooke's Law}$$

$$ma = -kx \Rightarrow \quad \text{Newton's 2nd Law}$$

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow \quad \text{Definition of acceleration}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{Divide both sides by m}$$

- Differential equation relating the changing acceleration to the position.
- Try non-periodic solutions:

$$x(t) = C \Rightarrow \frac{d^2x}{dt^2} = 0 \neq -\frac{k}{m}x(t) \quad \text{unless } C = 0$$

$$x(t) = e^{\sqrt{\frac{k}{m}}t} \Rightarrow \frac{dx}{dt} = \sqrt{\frac{k}{m}}e^{\sqrt{\frac{k}{m}}t} \Rightarrow \frac{d^2x}{dt^2} = \frac{k}{m}e^{\sqrt{\frac{k}{m}}t} \neq -\frac{k}{m}e^{\sqrt{\frac{k}{m}}t}$$

Simple Harmonic Motion

- Try a periodic solution:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$x(t) = \cos(ct) \Rightarrow \frac{dx}{dt} = -c\sin(ct) \Rightarrow \frac{d^2x}{dt^2} = -c^2\cos(ct) = -\frac{k}{m}\cos(ct)$$

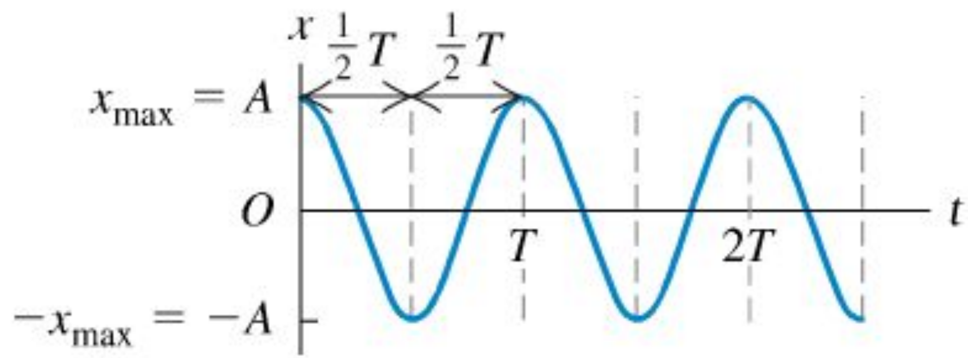
$$\text{if } c^2 = \frac{k}{m}$$

- The general solution is:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$$

or, equivalently,

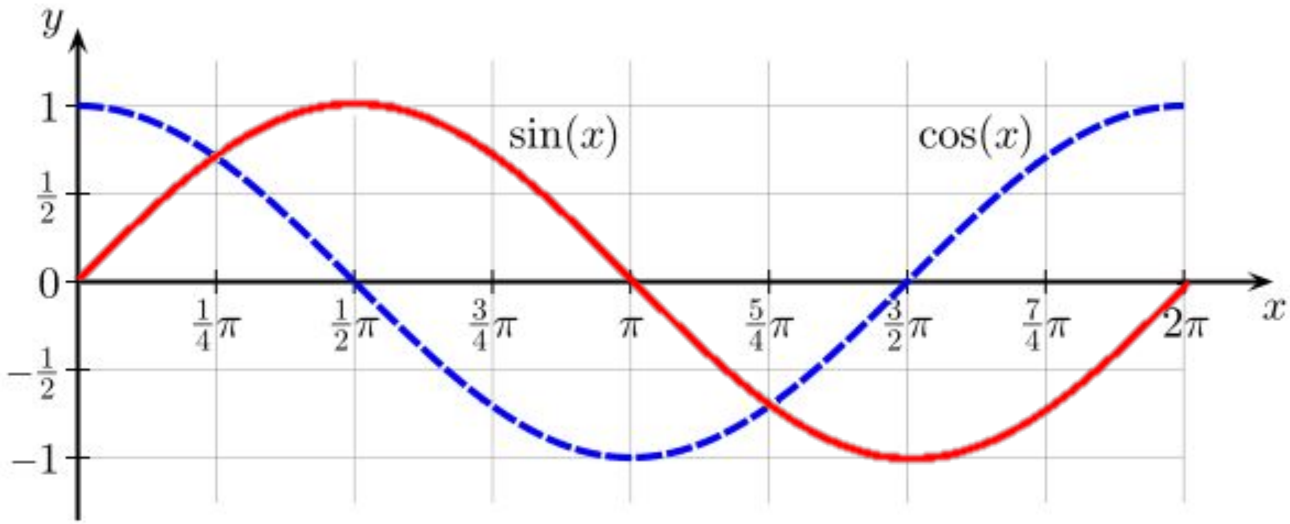
$$x(t) = B\cos\left(\sqrt{\frac{k}{m}}\,t\right) + C\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$



For φ = 0

Phase

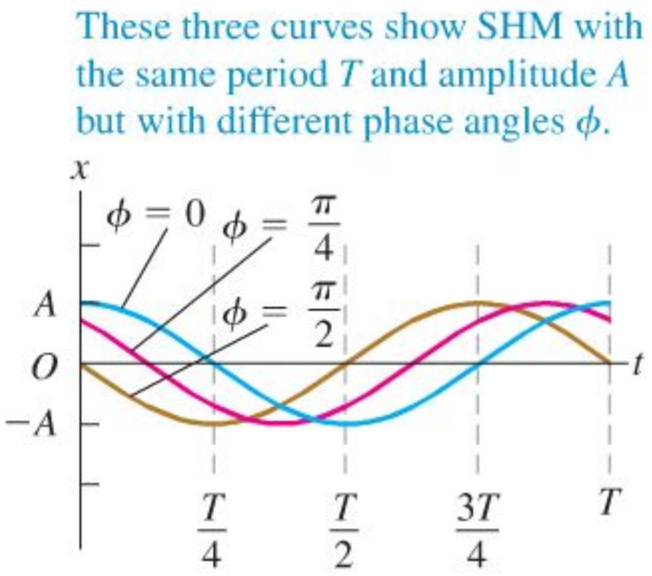
Note that the functions sin and cos repeat every 2π. The argument of these functions is called the phase.



Simple Harmonic Motion

- The phase ***constant*** determines the value of x at t=0:

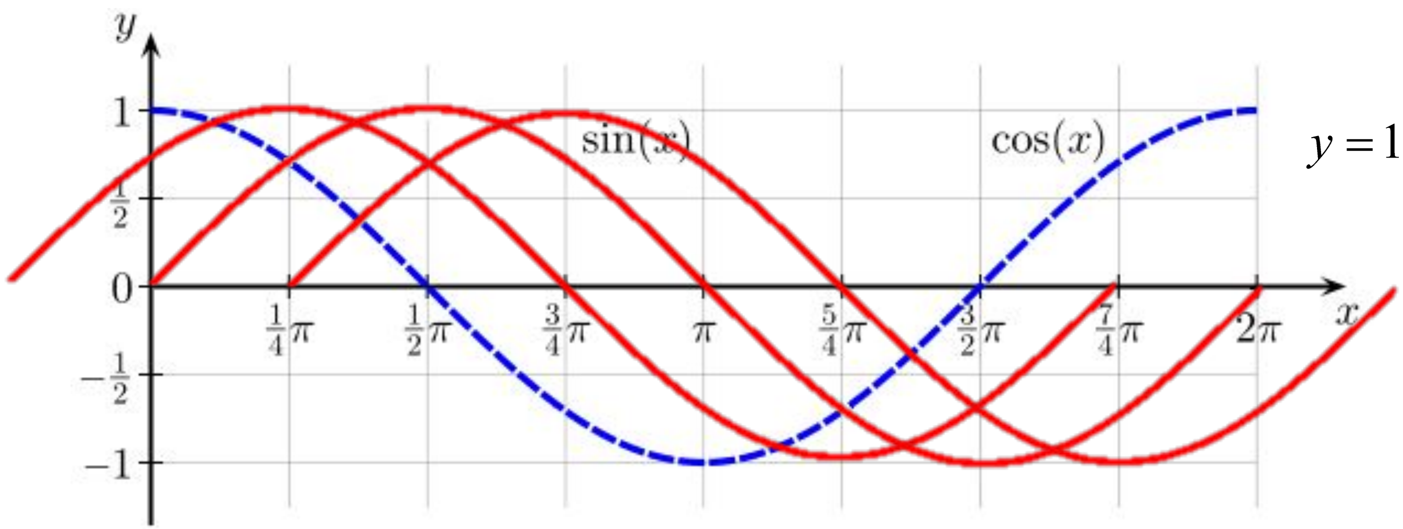
$$x(t) = A\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}$$



$$\phi = \text{phase angle that cycle is moved to left}$$
$$= 2\pi \times \text{ fraction of period moved to left}$$

Phase Constant

What if we want a sine function, but would like it to start at ¼ π instead of zero?



$$y = 1 \cdot \sin\left(x - \frac{1}{4}\pi\right)$$

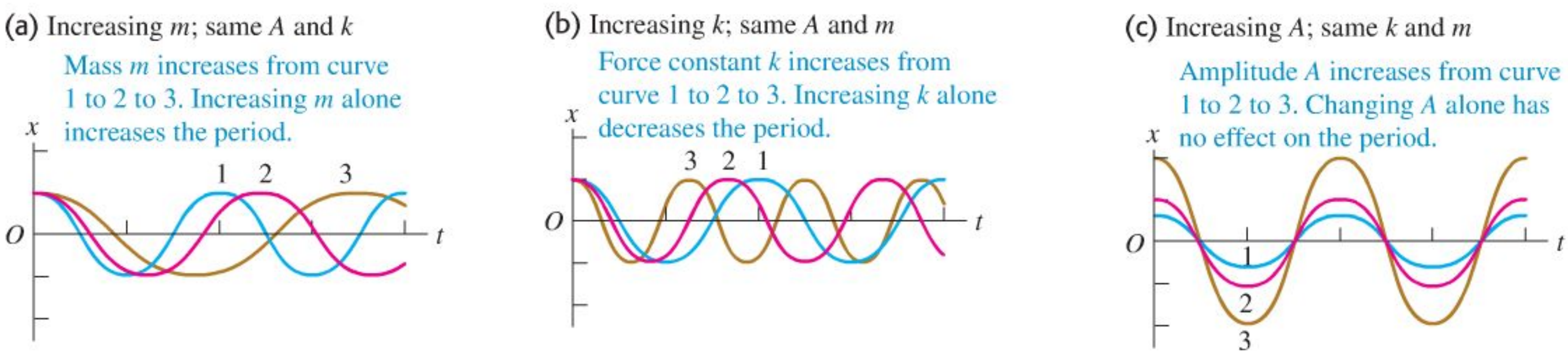
What if we want a sine function, but would like it to start at -¼ π instead of zero?

$$y = 1 \cdot \sin\left(x + \frac{1}{4}\pi\right)$$

Simple Harmonic Motion

- The factor in front of time sets the (angular) frequency of oscillations, so:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right) = A\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}$$



For $\varphi = 0$

Frequency, Angular Frequency and Period

- There is sometimes confusion about these quantities.

$$x(t) = A\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}$$

- ω is called the ***angular frequency***.
- The function x(t) returns to its starting point when ωt = 2π, so the period (amount of time to complete one cycle), is:

$$T = \frac{2\pi}{\omega}$$

- The ***frequency*** (number of cycles per second) is just:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \Rightarrow \quad \omega = 2\pi f$$

Position, Velocity and Acceleration

- We can differentiate to get the velocity

$$x(t) = A\cos(\omega t + \phi) \Rightarrow$$

$$v(t) = \frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi)$$

- And again to get acceleration

$$v(t) = -\omega A \sin(\omega t + \phi) \Rightarrow$$

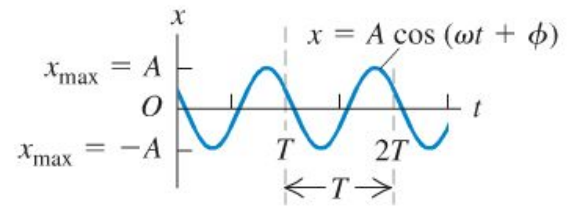
$$a(t) = \frac{dv(t)}{dt} = -\omega^2 A \cos(\omega t + \phi)$$

- Note that:

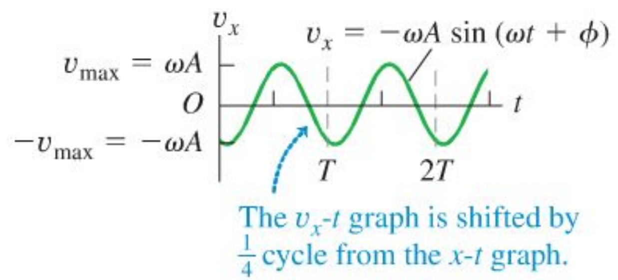
$$a(t) = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x(t) = -\frac{k}{m} x(t) \Rightarrow$$

$$ma(t) = -kx(t)$$

(a) Displacement x as a function of time t



(b) Velocity v_x as a function of time t



Energy in Simple Harmonic Motion

- Without any other forces (friction), we can describe the energy of a spring-mass system by the kinetic energy:

$$KE = \frac{1}{2} m v^2(t) = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} m \left(\sqrt{\frac{k}{m}} \right)^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2} k A^2 \sin^2(\omega t + \phi)$$

- And the potential energy is:

$$U_{el} = \frac{1}{2} k x^2(t) = \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

Energy in Simple Harmonic Motion

- So, the total energy is the sum of these:

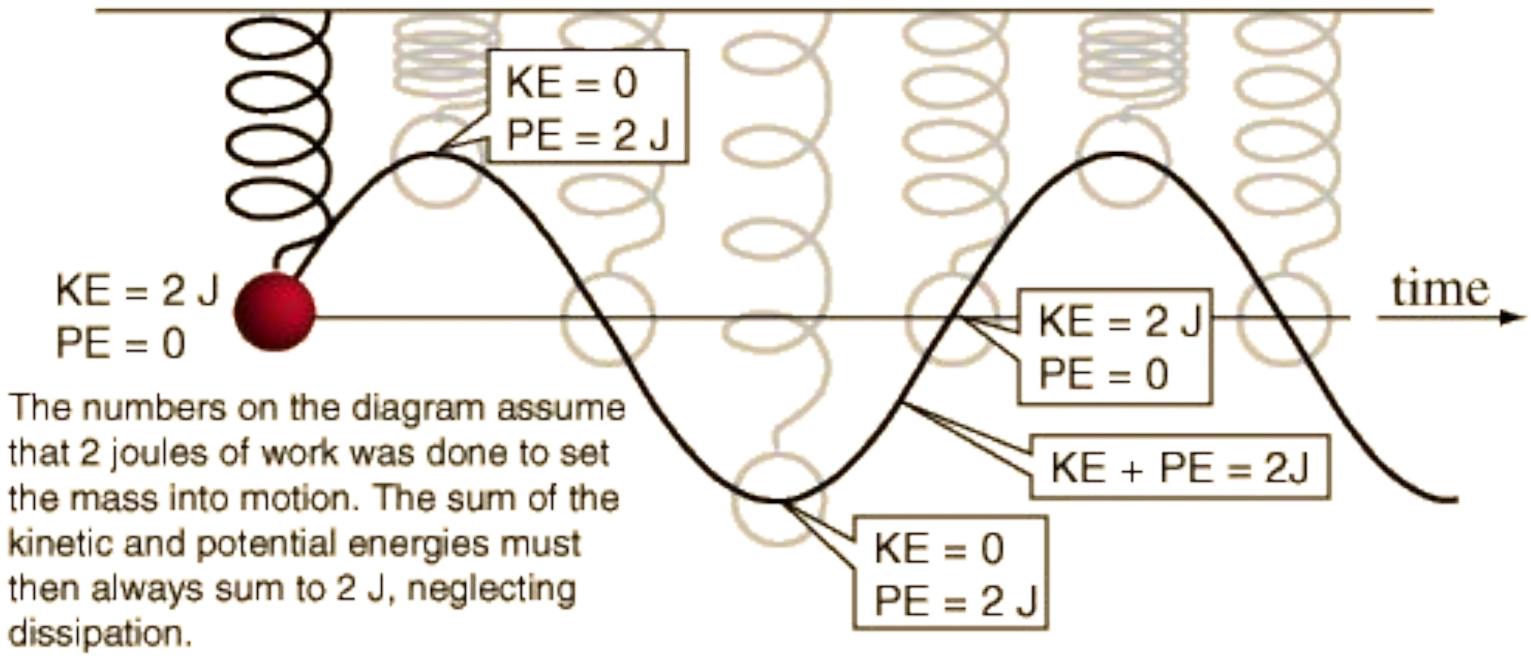
$$E_{total} = KE + U_{el} = \frac{1}{2}kA^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2}kA^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right)$$

$$= \frac{1}{2}kA^2$$

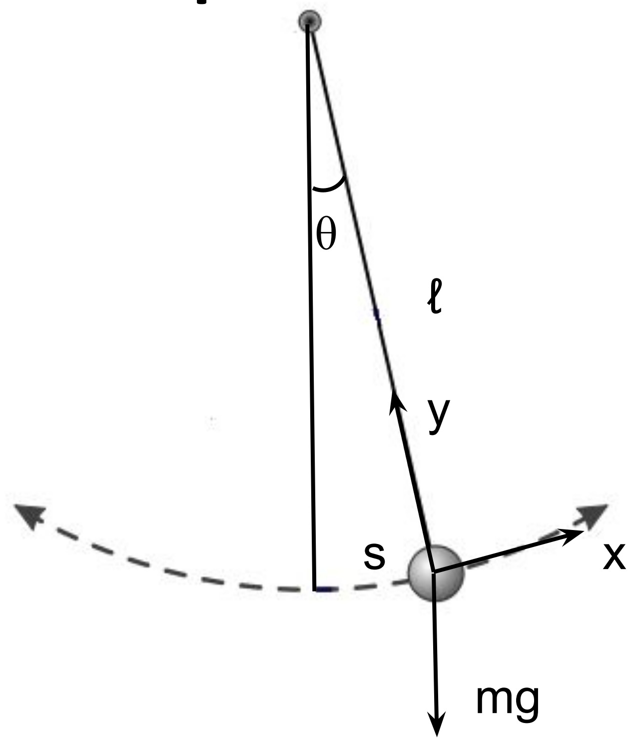
- But there is no time dependence here – conservation of energy!

Energy in Simple Harmonic Motion



No gravity…

The Simple Pendulum



$$F_x = -mg\sin\theta$$

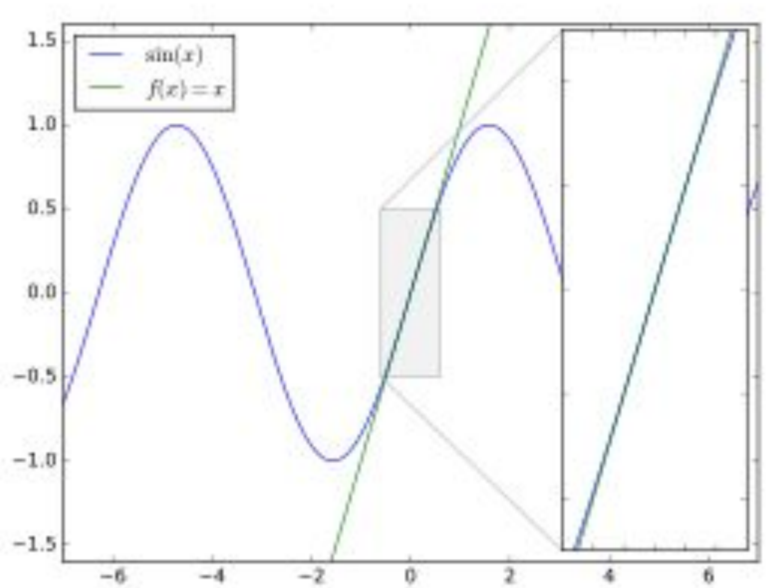
For small θ:

The sinθ small angle approximation

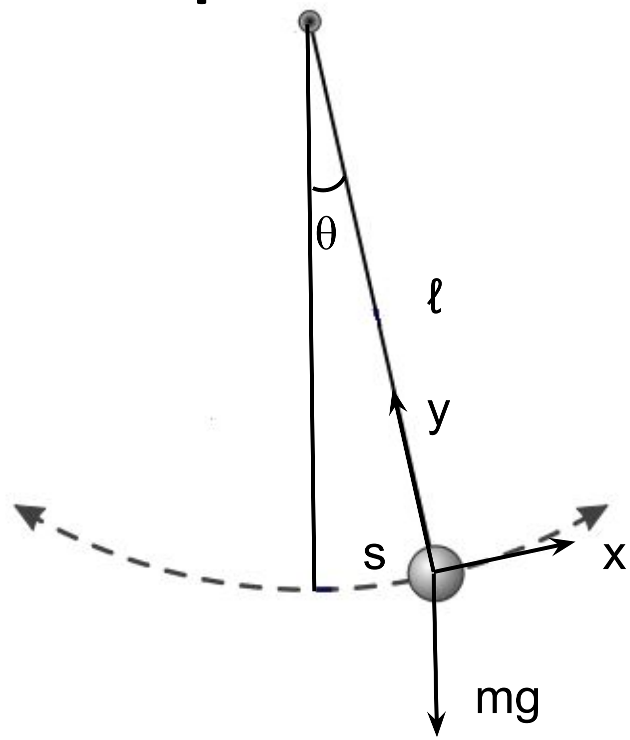
- Taylor expansion around θ=0:

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

- Or, just examine the graph near θ=0:



The Simple Pendulum



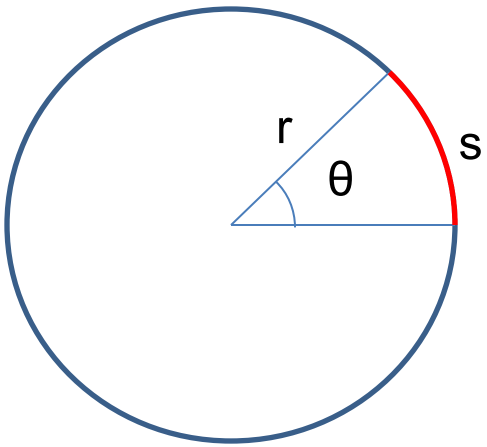
$$F_x = -mg\sin\theta$$

For small θ:

$$F_x = -mg\sin\theta = -mg\theta$$

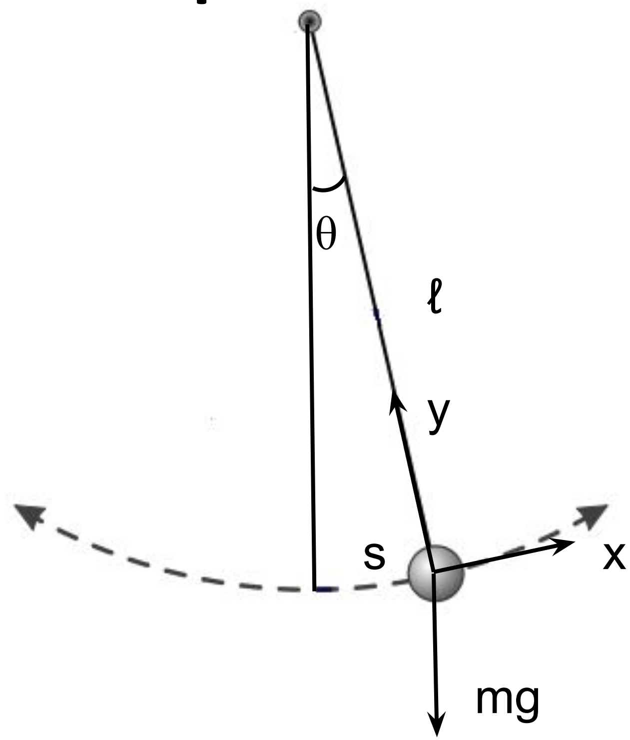
Arc length

- For a circle of radius r, the circumference is 2πr.
- Notice that the angle all the way around a circle is 2π angle.
- For an arc, subtending an angle θ, the arc length is just the same fraction of the circumference as the angle is to 2π:



$$s = 2\pi r \cdot \frac{\theta}{2\pi} = r\theta$$

The Simple Pendulum



$$F_x = -mg\sin\theta$$

For small θ:

$$F_x = -mg\sin\theta = -mg\theta = -mg\frac{s}{\ell}$$

$$F_x = -\frac{mg}{\ell}s = -ks, \quad k = \frac{mg}{\ell}$$

Same form as mass-spring!

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{mg}{\ell}}{m}} = \sqrt{\frac{g}{\ell}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$$

Torsion Pendulum

- Torsion spring applies a torque that is proportional to the angular displacement:

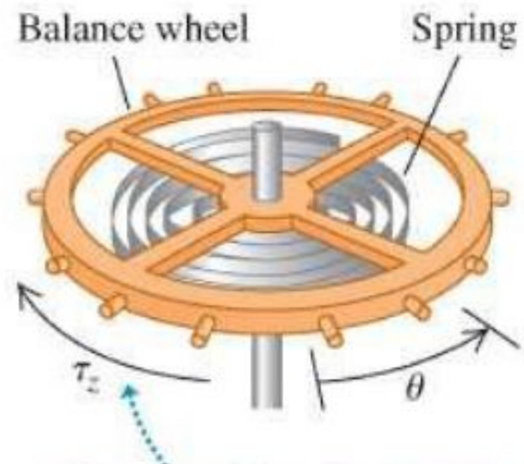
$$\tau \propto -\theta \Rightarrow \tau = -\kappa\theta$$

- From the rotational version of Newton's second law:

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} \Rightarrow$$

$$-\kappa\theta = I\frac{d^2\theta}{dt^2} \Rightarrow$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$



The spring torque τ_z opposes the angular displacement θ.

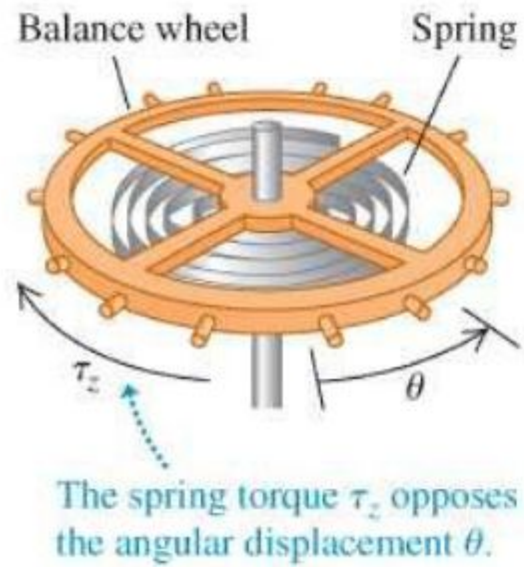
Torsion Pendulum

- But this is the same differential equation we had for a linear mass-spring system!

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \longleftrightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

- So, it has the same solutions:

$$\theta(t) = \Theta\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{\kappa}{I}}$$



Problem 13.36

13.36. A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (Fig. 13.32). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant of the fiber.