

# Lecture 27

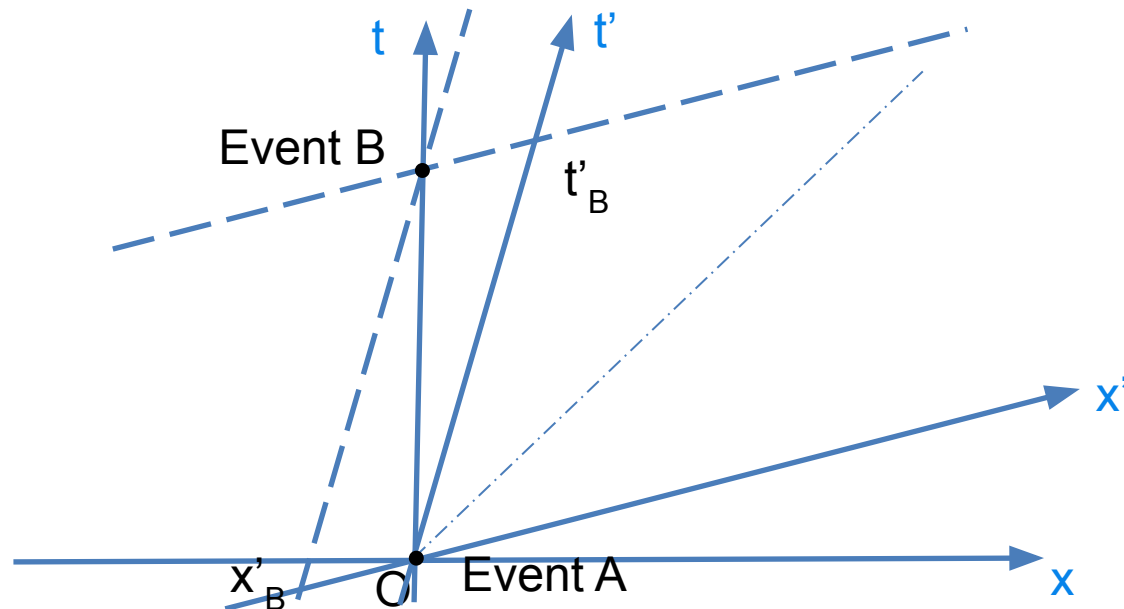
## (Special Relativity V – SR Metric & Time Dilation)

Physics 2310-01 Spring 2020

Douglas Fields

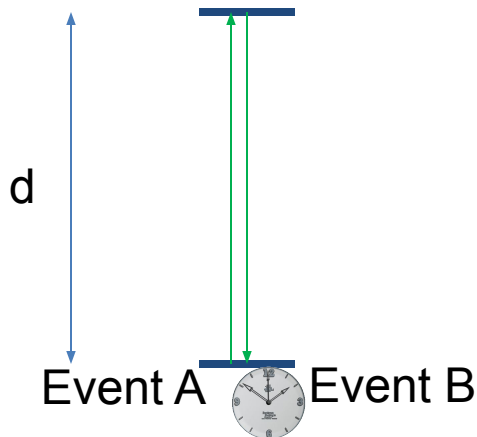
# Spacetime Interval

- So, if the spacetime interval is something real, how do we measure it in different reference frames?
- We will need our scale factors for our primed frame, and will need to figure out how exactly to determine the values of  $x'$  and  $t'$ ...



# Deriving the Metric Equation

- We have said that the Spacetime Interval should be the same in different frames, i.e., it should be invariant.
- But how do we determine the spacetime interval? Is there an equation for it?
  - Yes, it is called the spacetime metric equation.
- Here is how we'll derive it:
- Let's imagine a light clock in a certain inertial frame:
  - Light is emitted from the base, gets reflected from the top and then detected at the base.
  - For every detection, a certain time period has elapsed:
  - Let Event A be the light emission, and Event B the light detection.

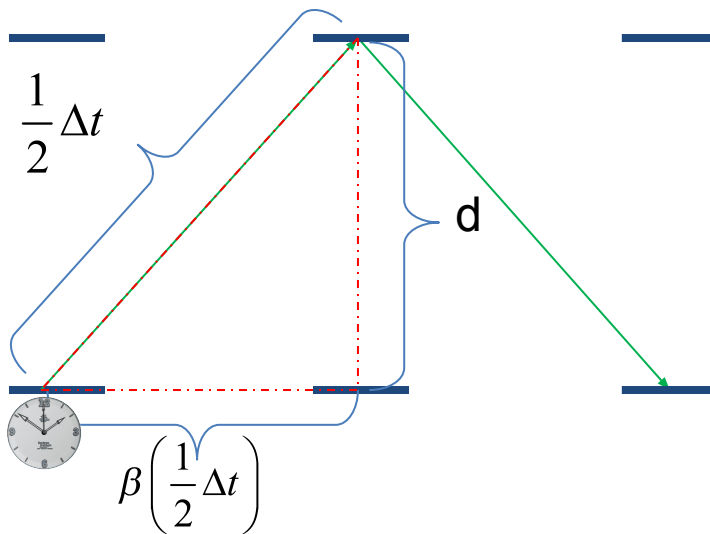


$$\Delta t = 2d$$

We will use SR units!  
Much of what you will find  
on the web, including  
Wikipedia, uses SI units.

# Deriving the Metric Equation

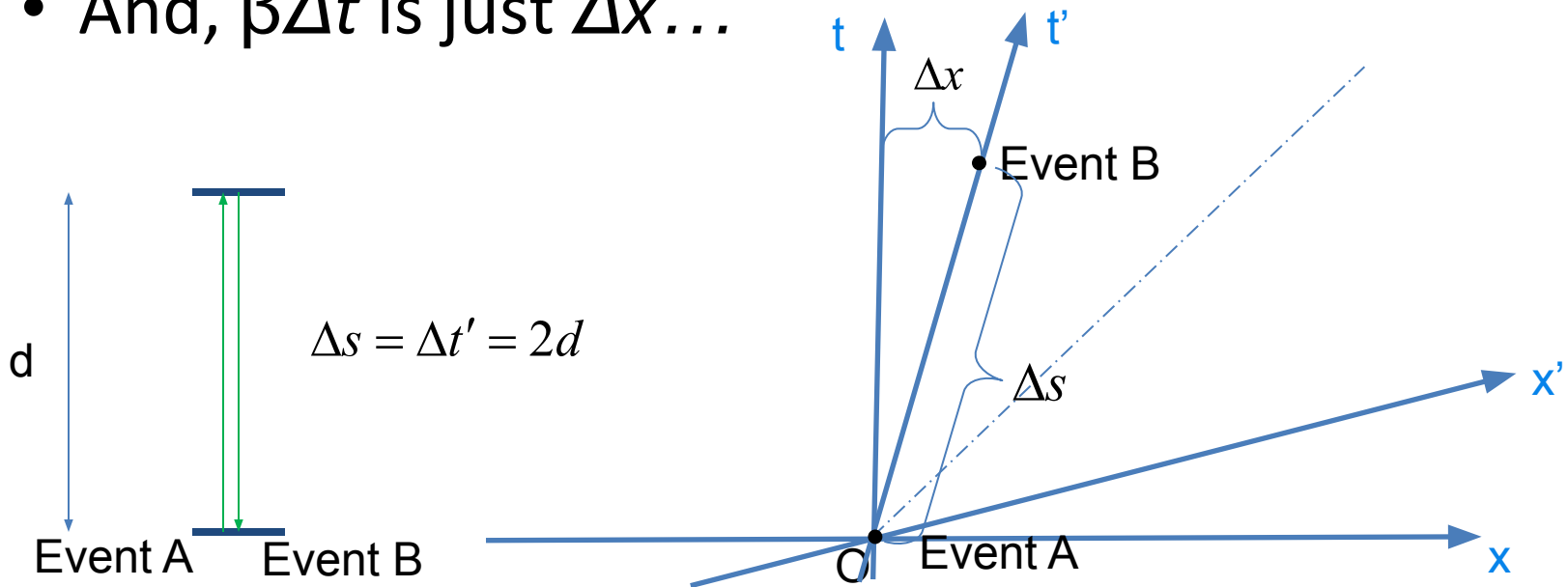
- Now, let's put this light clock on a moving train (moving very fast at velocity  $\beta$ ).
- What does the observer on the train see?
- The same thing, of course.  $\Delta t' = 2d$ 
  - The laws of physics are the same in any inertial frame!
- But what does an observer at the station see?



$$\left(\frac{1}{2}\Delta t\right)^2 = \left(\beta\frac{1}{2}\Delta t\right)^2 + d^2 \Rightarrow$$
$$\Delta t^2 = (\beta\Delta t)^2 + (2d)^2$$

# Deriving the Metric Equation

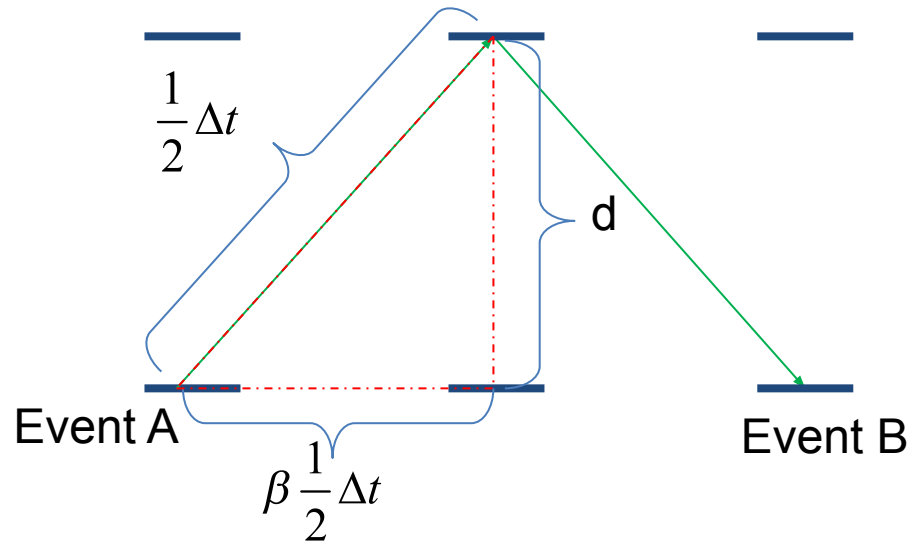
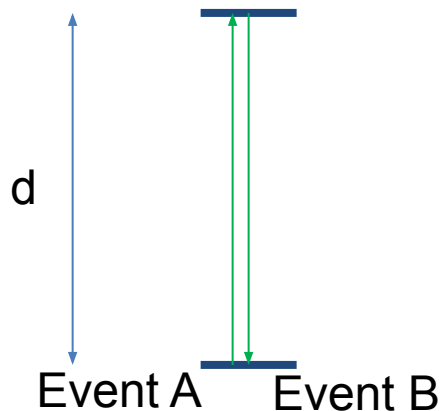
- Now, by looking at these events in a spacetime diagram, one can see that this time measurement is the Spacetime Interval,  $\Delta s$  – the proper time measured with an inertial clock.
- And,  $\beta\Delta t$  is just  $\Delta x$ ...



# Deriving the Metric Equation

- So,  $\Delta t^2 = (\beta\Delta t)^2 + (2d)^2$  becomes  $\Delta t^2 = \Delta x^2 + \Delta s^2$  .
- Or, solving for  $\Delta s$ ,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 \quad \text{Metric Equation}$$



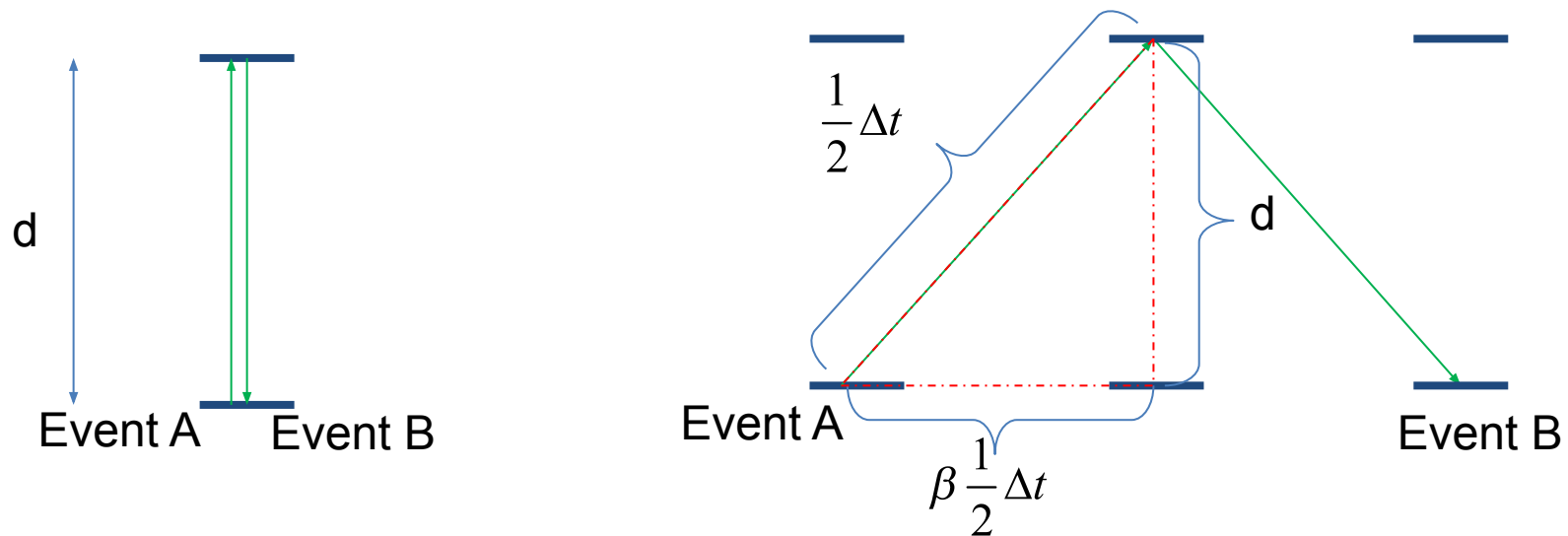
# Invariance of the Metric (Spacetime Interval)

- We have said that the Spacetime Interval is invariant, but only based on the fact that it can be measured in each frame by looking at an inertial clock at both events.
- But this derivation of the Spacetime Interval begs a few questions:
  - We assumed that the perpendicular distance,  $d$ , is the same in both frames. Is this true?
  - There is a geometry issue that really looks wrong...
- Let's go through both of these, as they will lead us to some important revelations about spacetime...

# Perpendicular (to relative motion)

## Distances

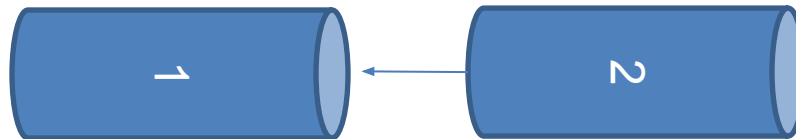
- Maybe, in this new world we are discovering, things are really weird and the distance  $d$ , which is measured perpendicular to the direction of relative motion between our two frames, is not the same in both frames...
- Okay, let try a thought experiment...





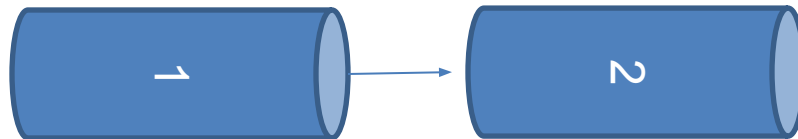
# Which pipe is smaller?

- We have two thin-walled pipes moving towards each other (relatively) along their mutual axes.
- In the frame of each pipe, an observer measures the diameter of the pipe in their own frame, and each measure 1m.
- Now, let's assume that perpendicular measurements do change in moving frames and see where this leads us.
- Perhaps the observer in pipe 1 says that pipe 2's diameter shrinks – then pipe 2 would pass on the inside of pipe 1.



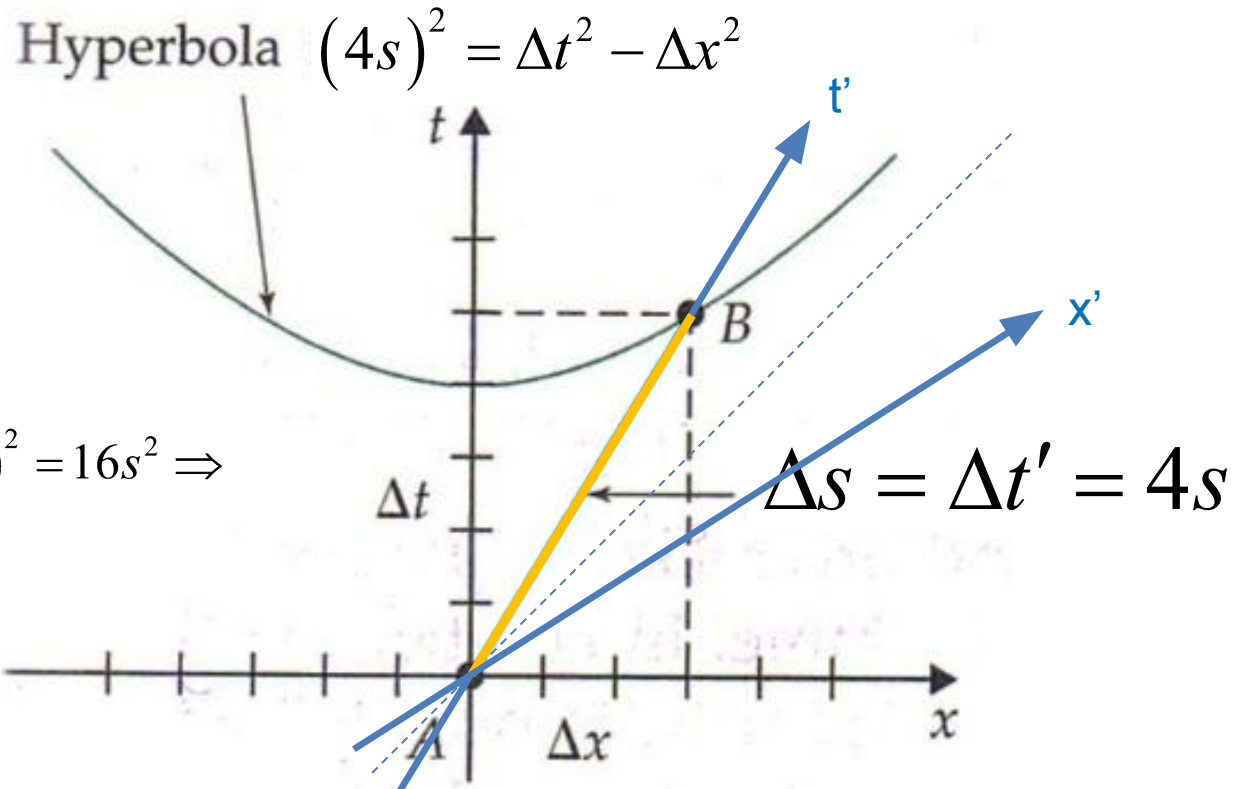
# Which pipe is smaller?

- But what would the observer in pipe 2 say?
- According to him, pipe 1 is moving and should pass inside his pipe!
- The only way for both observers to agree, is that the perpendicular distances don't change.



# The Geometry Issue

- Let's look at the geometry of the spacetime diagram.
- The metric equation describes a hyperbola.
- We see that  $\Delta t = 5s$  and  $\Delta x = 3s$  for Event B, and yet, the "hypotenuse" of the triangle formed is  $4s$ !
- This isn't even as bad as it gets...



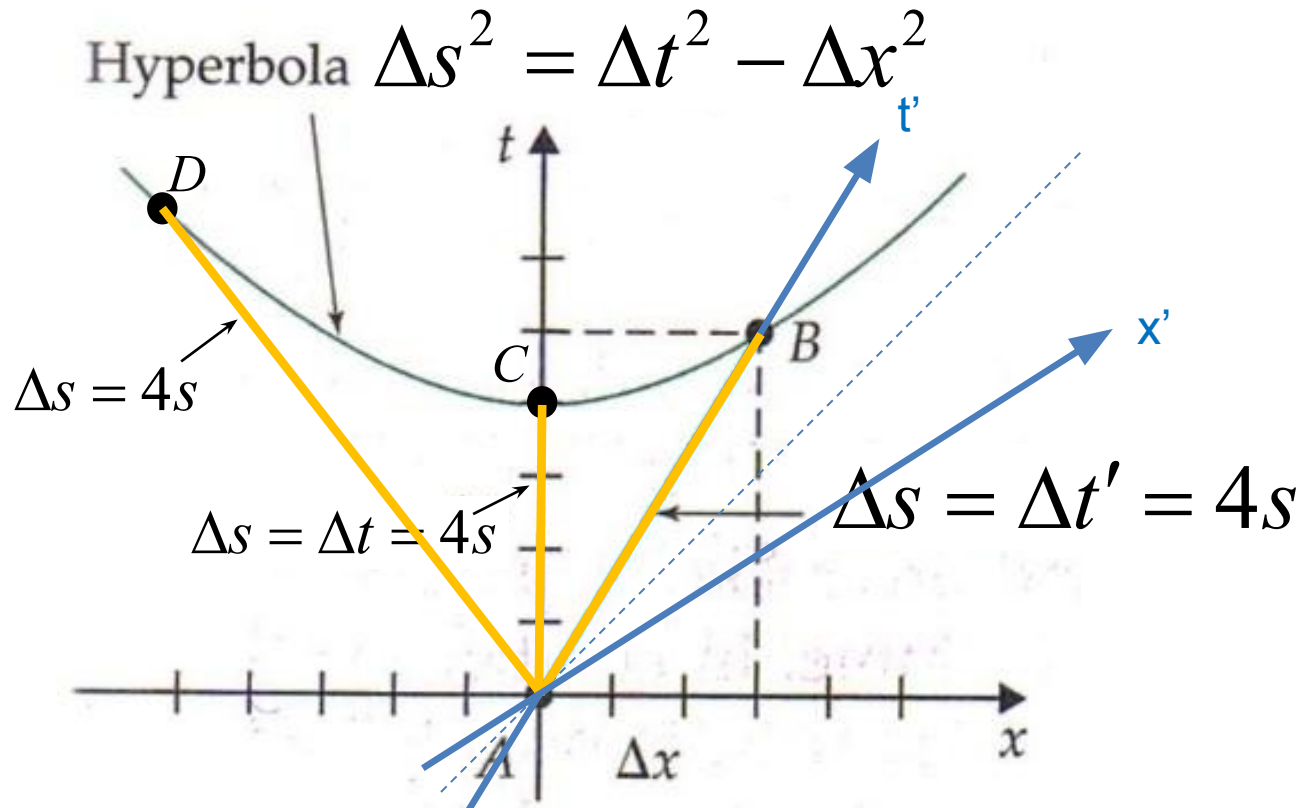
$$\Delta s^2 = \Delta t^2 - \Delta x^2$$

$$= (5s)^2 - (3s)^2 = 16s^2 \Rightarrow$$

$$\Delta s = 4s$$

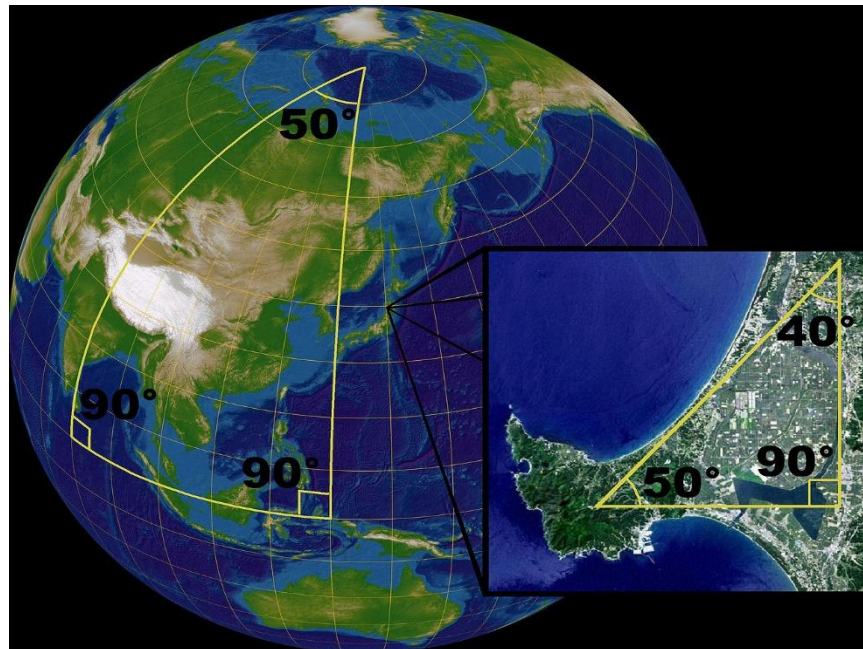
# The Geometry Issue

- Every point on the hyperbola has the same Spacetime Interval with respect to the origin.
- Remember that the Spacetime Interval is the time interval in Special Relativity that has meaning, in that it is invariant!
- But how can all these line segments be the same “length”?



# The Geometry Issue

- Well, how can a triangle have angles that add to more than  $180^\circ$ ?
- Spacetime is a Non-Euclidean geometry, just as the spherical geometry of the surface of the earth is non-Euclidean!

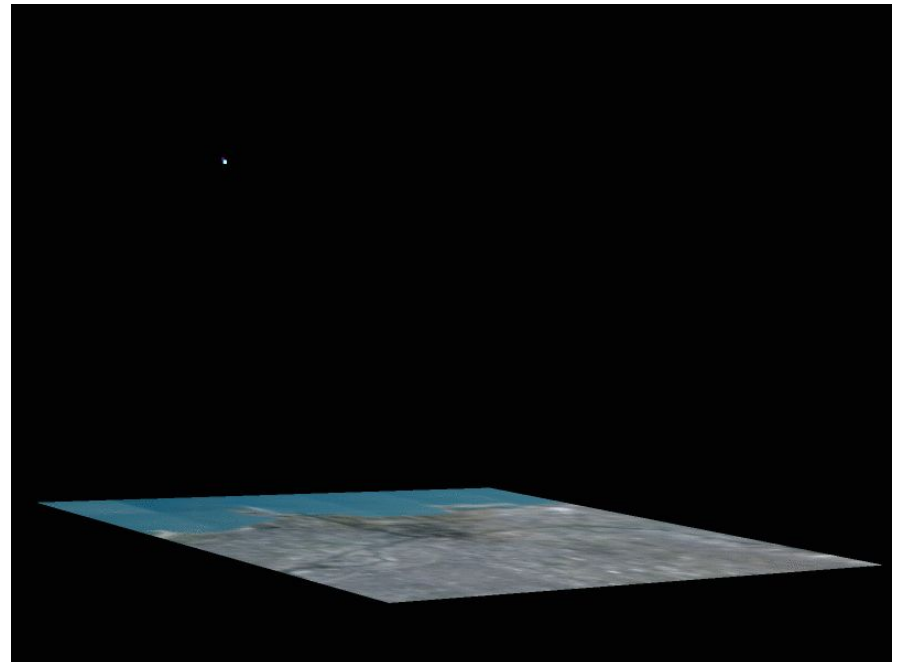
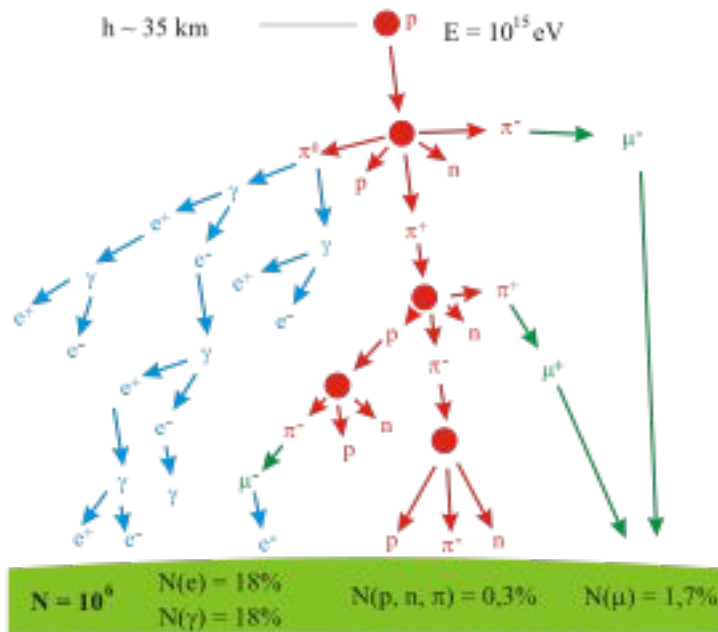


"Triangles (spherical geometry)" by Lars H. Rohwedder, Sarregouset - Own work from source files Image:OgaPeninsulaAkiJpLandsat.jpg (GFDL) and Image:Orthographic Projection Japan.jpg (GFDL and CC-BY-SA).. Licensed under CC BY-SA 3.0 via Commons - [https://commons.wikimedia.org/wiki/File:Triangles\\_\(spherical\\_geometry\).jpg#/media/File:Triangles\\_\(spherical\\_geometry\).jpg](https://commons.wikimedia.org/wiki/File:Triangles_(spherical_geometry).jpg#/media/File:Triangles_(spherical_geometry).jpg)

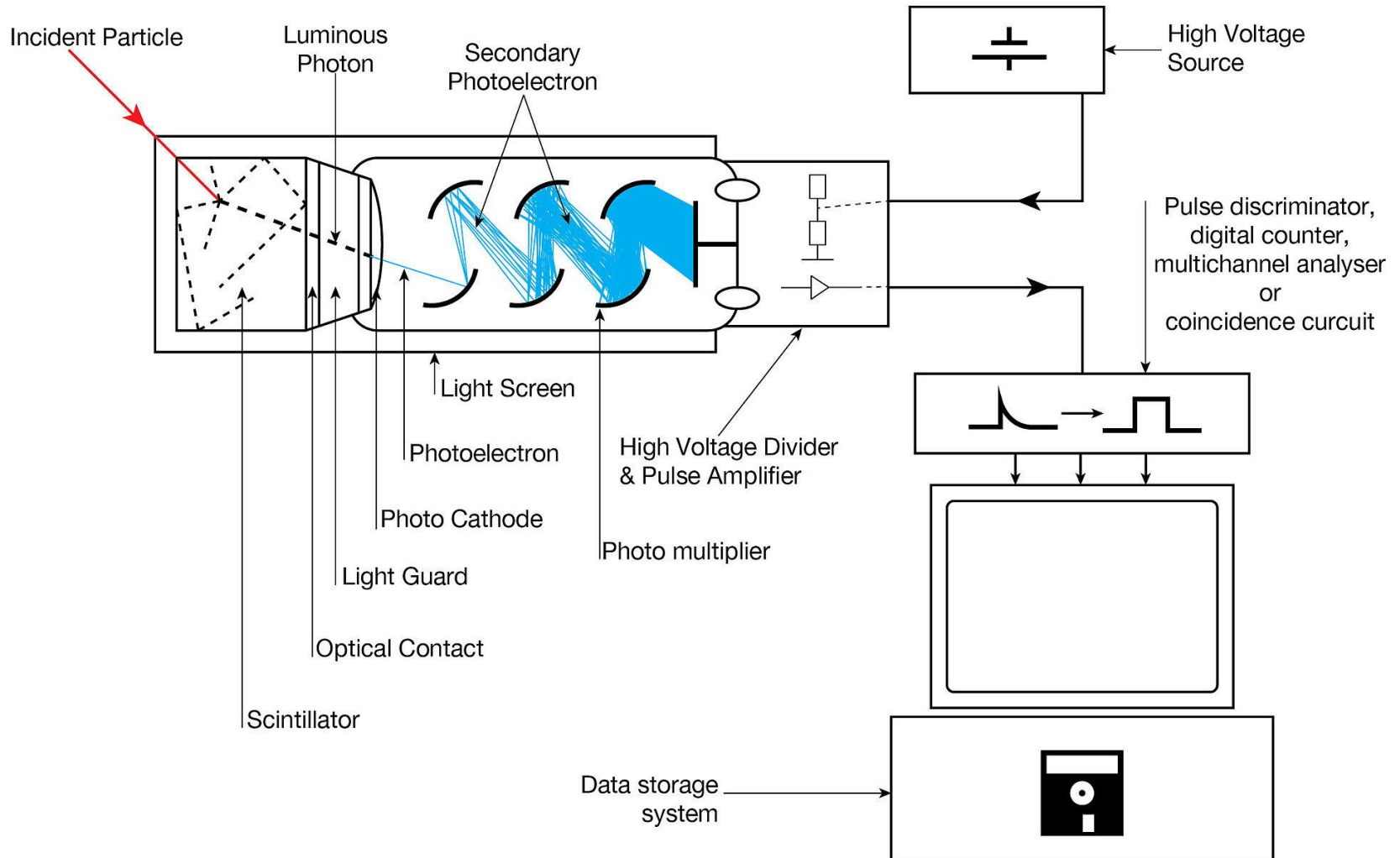


# Cosmic Ray Air Showers

- You may not have heard of muons before, but they are passing through you all the time.
- Primary cosmic rays (mostly protons from space) hit our atmosphere and cause a chain reaction.
- Particles are created, hit other particles, decay into new particles, etc.
- The end result of this is that some of these particles end up hitting the earth's surface.
- Since muons interact with normal matter (i.e., our atmosphere) less than other things like electrons and photons, many of the created muons strike the earth and can be detected as cosmic rays.



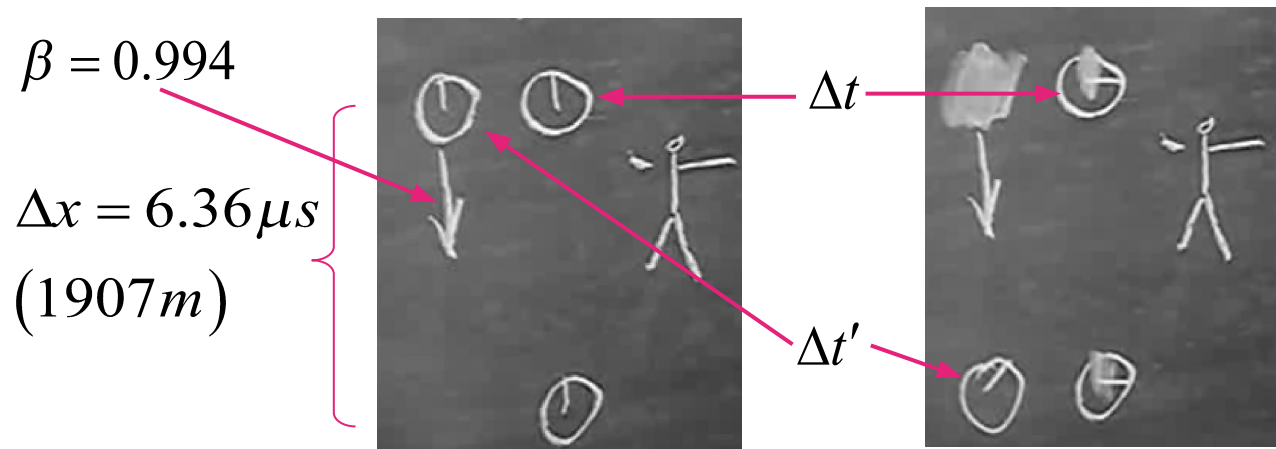
# Detecting Muons





# The Experiment

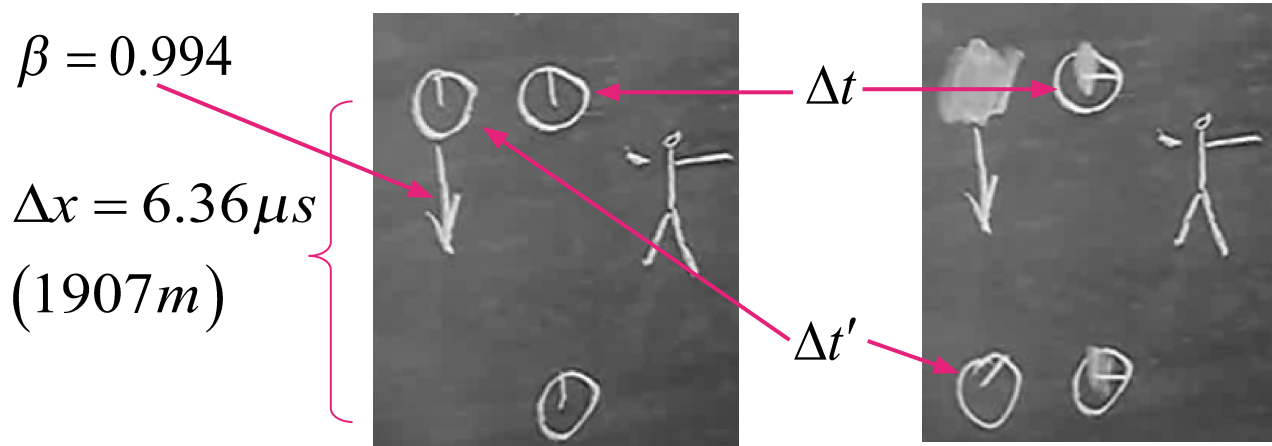
- Here's what we would like to do with the muons:
- We want to use them as clocks.
- We will have two detectors, one located 1907m above the other.
- Then, we will basically count the number of muons that pass through the top detector and still live to see the bottom detector.
- We will use a technique to just look at muons travelling at  $\beta = 0.994$ .
- Then, by knowing their lifetime we can predict how many should survive to see the bottom detector and compare to our measured results.
- This experiment was done in 1962 by David H. Frisch and James H. Smith.



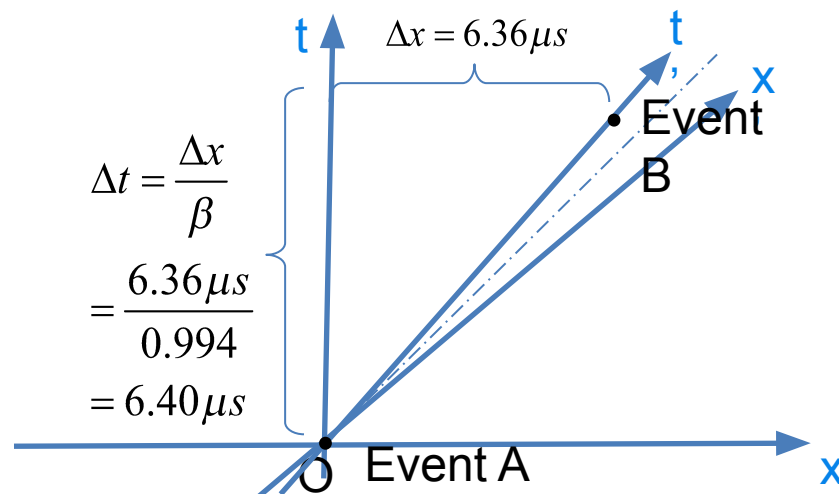
<http://www.scivee.tv/node/2415>

# The Experiment

- Let's also include a two-observer spacetime diagram.



<https://www.youtube.com/watch?v=tbsdrHILfVQ>

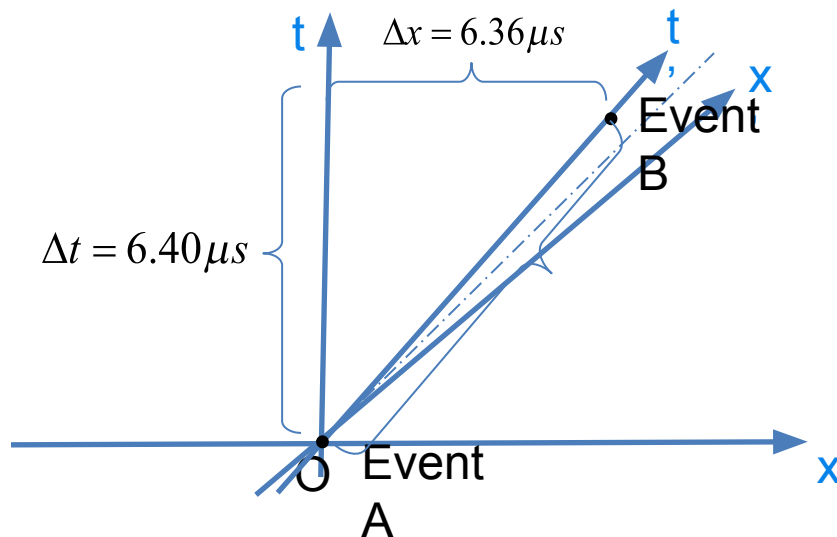


# The Experiment

- We can calculate the Spacetime Interval in the person's frame,

$$\begin{aligned}\Delta s^2 &= \Delta t^2 - \Delta x^2 \Rightarrow \\ &= (6.40\mu s)^2 - (6.36\mu s)^2 \Rightarrow \\ \Delta s &= 0.714\mu s\end{aligned}$$

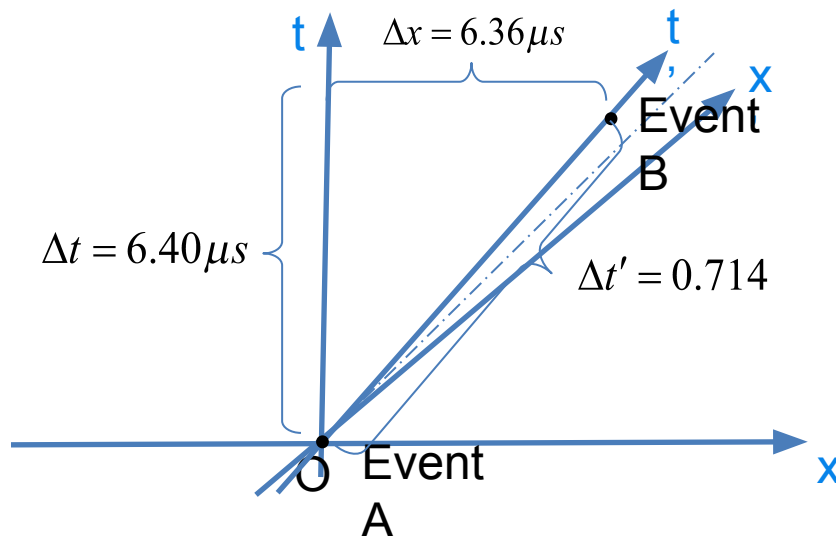
- And then, since it is invariant, we know the spacetime interval in the muon's frame.  $\Delta s' = 0.714\mu s$
- But, both events occur in the muon's rest frame at the same place (at the muon), so that the metric equation yields:



$$\begin{aligned}\Delta s' &= \sqrt{\Delta t'^2 - \cancel{\Delta x'^2}} = \Delta t' \\ \therefore \\ \Delta t' &= 0.714\mu s\end{aligned}$$

# The Results

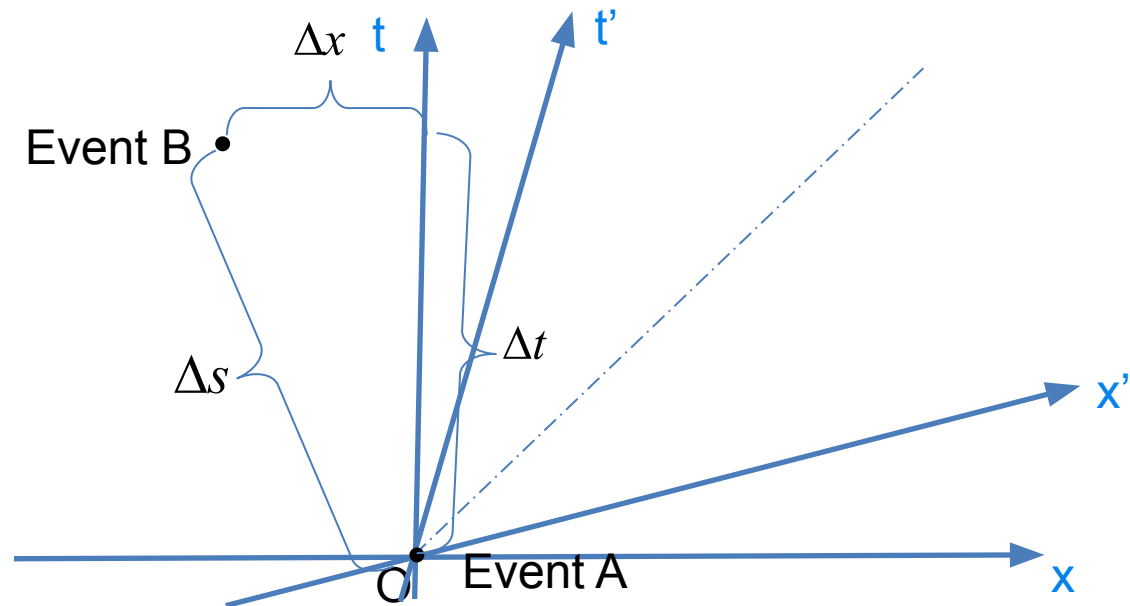
- So, the question is, does the “clock” in the muon’s frame (it’s half-life) read the same as the observer in the rest frame ( $6.40\mu\text{s}$ ) or what is given by the metric equation ( $0.714\mu\text{s}$ )?
- If it is  $6.40\mu\text{s}$ , then there should only be  $N \cdot (1/2)^{6.40\mu\text{s}/1.52\mu\text{s}} = N/18.5$  muons left at the lower detector.
- If it is  $0.714\mu\text{s}$ , then there should be  $N \cdot (1/2)^{0.714\mu\text{s}/1.52\mu\text{s}} = N/1.38$  muons left at the lower detector.
- The experimental results were clear – the number of muons that decayed were consistent with the fact that the “clocks” of the moving muons ran slow by a factor of  $6.40/0.714 \sim 9$ !



# Some Examples 1

- A firecracker explodes. A second Firecracker explodes 25ns away and 52ns later as measured in the Home Frame. In another inertial frame (the Other Frame), the two explosions are measured to occur 42ns apart in space. How long a time passes between the explosions in the Other Frame?

$$\begin{aligned}\Delta s^2 &= \Delta t^2 - \Delta x^2 \Rightarrow \\ &= (52ns)^2 - (25ns)^2 \Rightarrow \\ \Delta s &= 45.6ns\end{aligned}$$

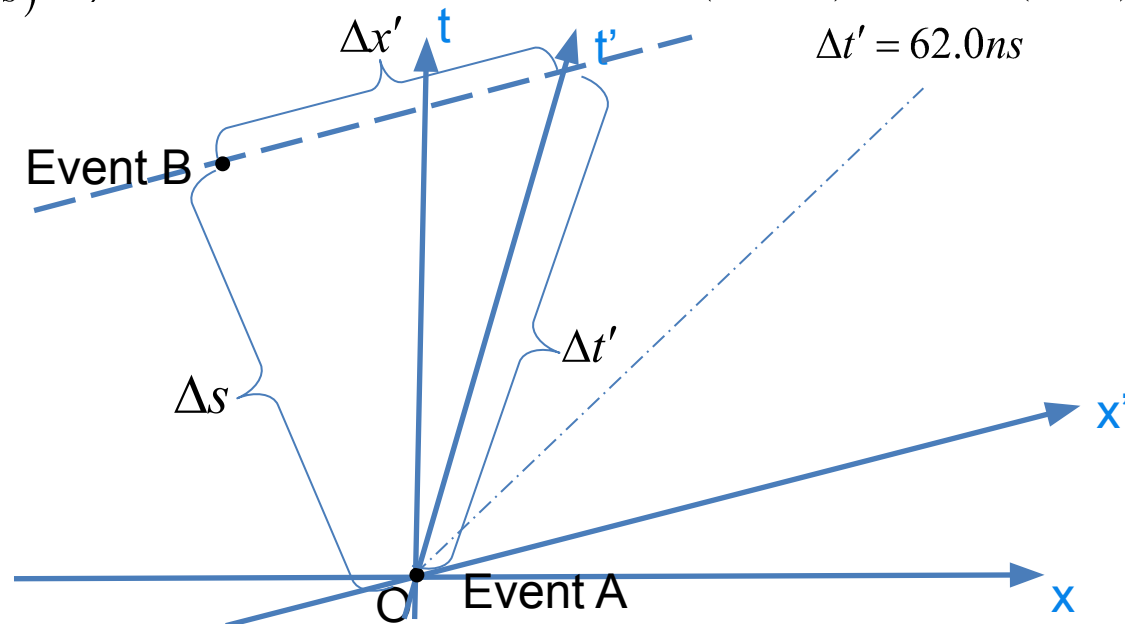


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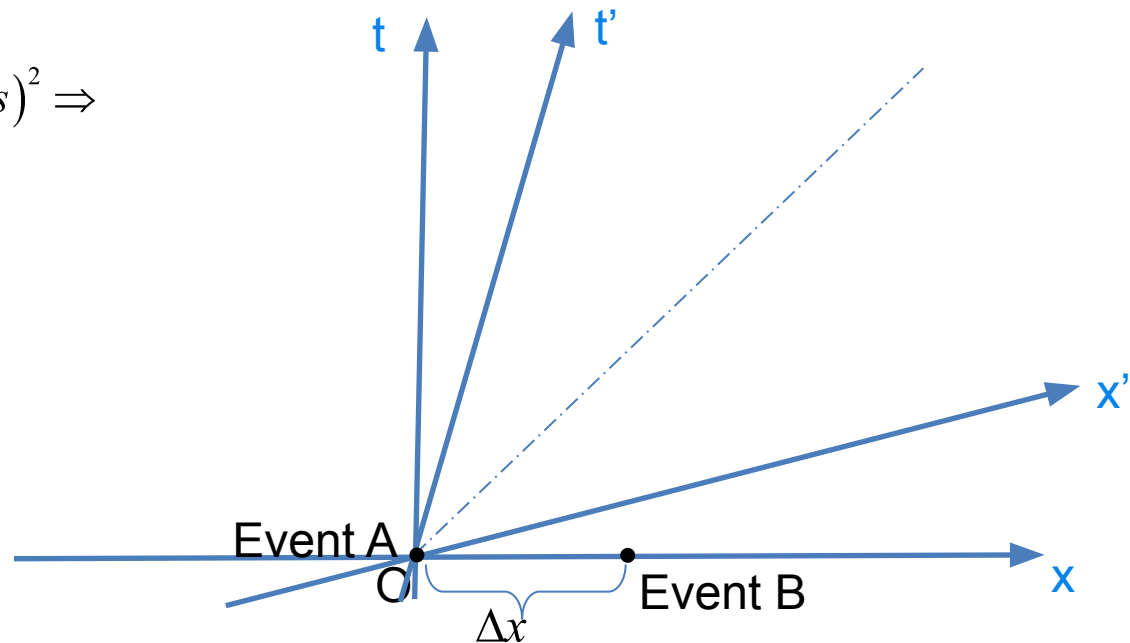
$$\begin{aligned}\Delta s^2 &= \Delta s'^2 = \Delta t'^2 - \Delta x'^2 \Rightarrow \\ (45.6ns)^2 &= \Delta t'^2 - (42ns)^2 \Rightarrow \\ \Delta t' &= 62.0ns\end{aligned}$$



# Some Examples 2

- Imagine that two events that are separated by 30ns of distance in the Home Frame are also simultaneous in that frame. If, in the Other Frame, the events are separated by 10ns of time, what is their spatial separation in that frame?

$$\begin{aligned}\Delta s^2 &= \Delta t^2 - \Delta x^2 \Rightarrow \\ &= (0ns)^2 - (30ns)^2 \Rightarrow \\ \Delta s^2 &= -(30ns)^2\end{aligned}$$

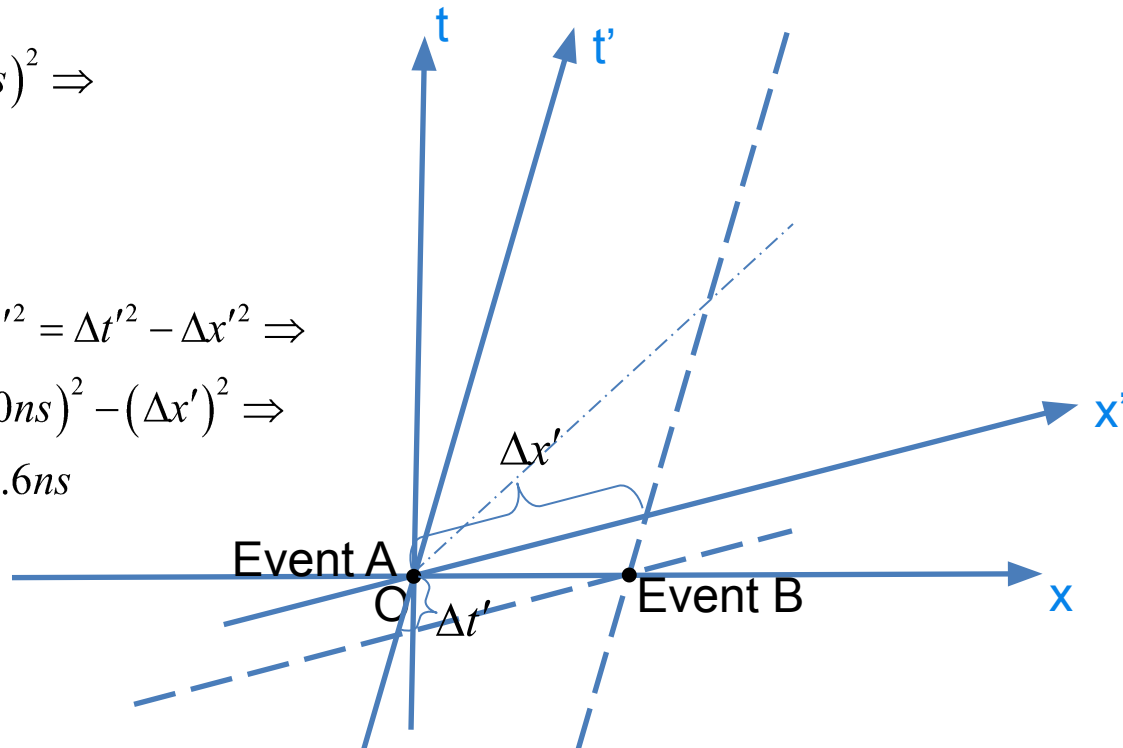


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$$\begin{aligned}\Delta s^2 &= \Delta s'^2 = \Delta t'^2 - \Delta x'^2 \Rightarrow \\ -(30ns)^2 &= (10ns)^2 - (\Delta x')^2 \Rightarrow \\ \Delta x' &= 31.6ns\end{aligned}$$



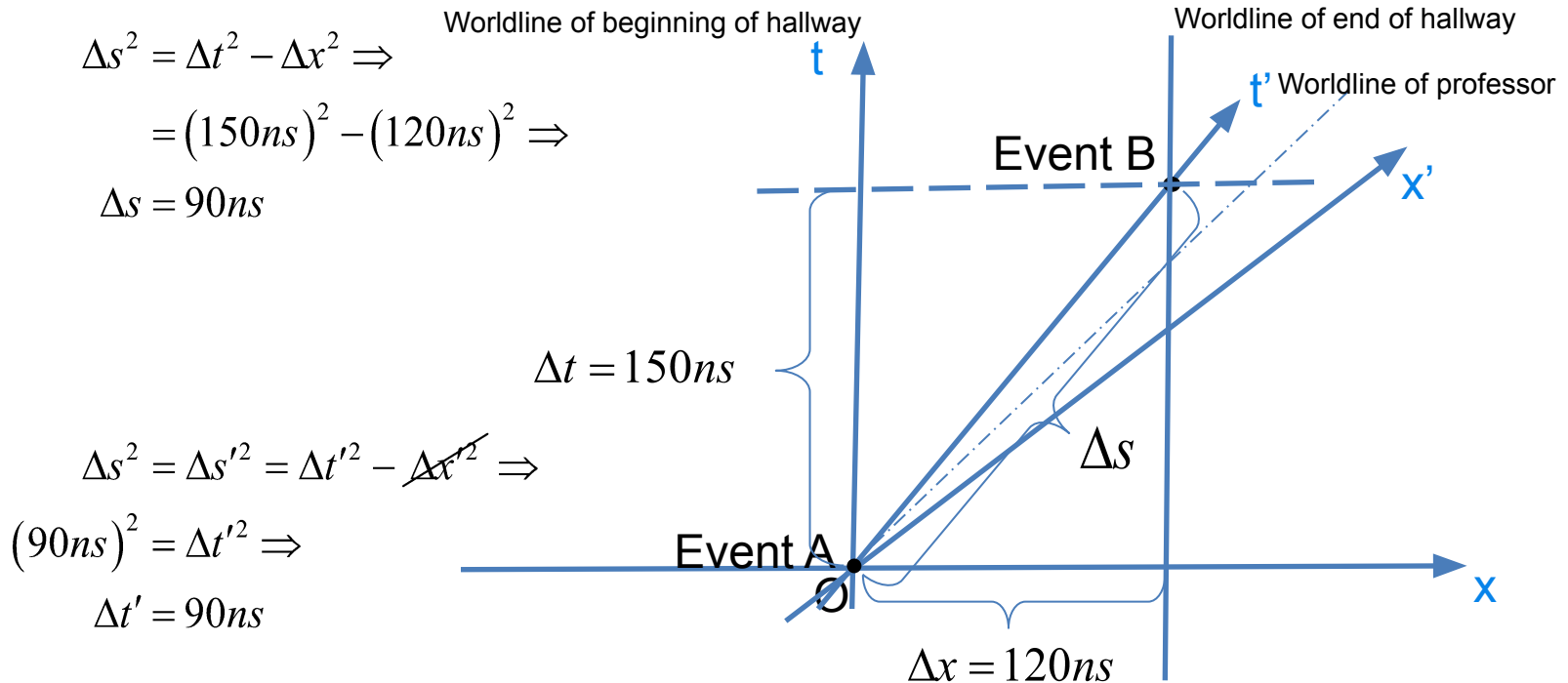


# Some Examples 3

- A certain physics professor fleeing the wrath of a set of irate students covers the length of the physics department hallway (a distance of 120ns) in a time of 150ns as measured in the frame of the earth. Assuming the professor moves at a constant velocity, how much time does the professor's watch measure during the trip from one end of the hallway to the other?

# Some Examples 3


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What is the speed of the Professor relative to earth in the previous problem?

- a) 1.25
- b) 0.5
- c) 0.25
- d) 0.5
- e) 0.8
- f) Not fast enough

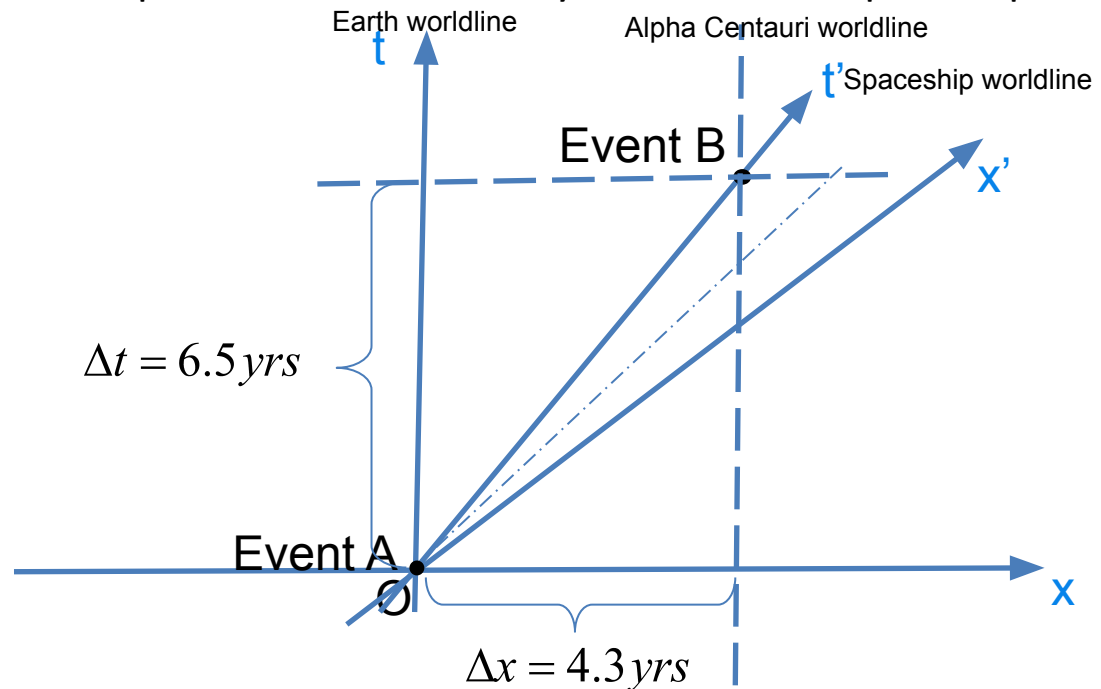
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# Some Examples 4

## Unrealistic Twin Paradox

- A spaceship departs our solar system and travels at constant speed to the star Alpha Centauri 4.3 light-years away, then instantaneously turns around (never-mind about the impossible accelerations involved) and returns to the solar system at the same constant speed. Assume that the trip takes 13 years as measured by clocks here on earth. How long does the trip take, as measured by clocks on the spaceship?



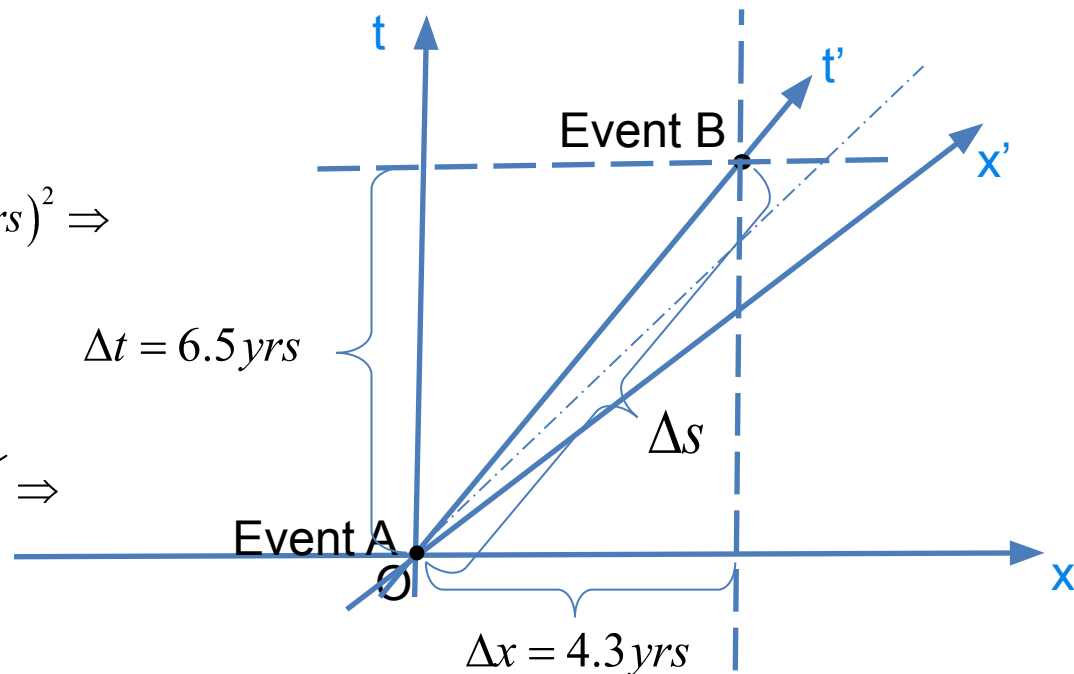
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$$\begin{aligned}\Delta s^2 &= \Delta t^2 - \Delta x^2 \Rightarrow \\ &= (6.5 \text{ yrs})^2 - (4.3 \text{ yrs})^2 \Rightarrow \\ \Delta s &= 4.9 \text{ yrs}\end{aligned}$$

$$\begin{aligned}\Delta s^2 &= \Delta s'^2 = \Delta t'^2 - \Delta x'^2 \Rightarrow \\ \Delta t' &= 4.9 \text{ yrs}\end{aligned}$$



# Why is it a paradox?

- Upon her return to earth, this says that the traveler to Alpha Centauri should say that only  $2 * 4.9\text{yrs} = 9.8\text{yrs}$  have past, but the twin back on earth would claim that  $2 * 6.5\text{yrs} = 13\text{yrs}$  have past.
- However, doesn't the "traveler" see herself as stationary and the twin on earth as the one who's clock is running slowly?
- Yes, and no...

# Why is it a paradox?

- During the constant velocity outbound and inbound trips, the traveler does, in fact see the earthbound twin's clocks running slowly, but...
- During the change from outgoing to incoming, the traveler sees the earthbound twin's clock make a huge jump:

