Lecture 3
(Coupled Harmonic Oscillators)

Physics 2310-01 Spring 2020

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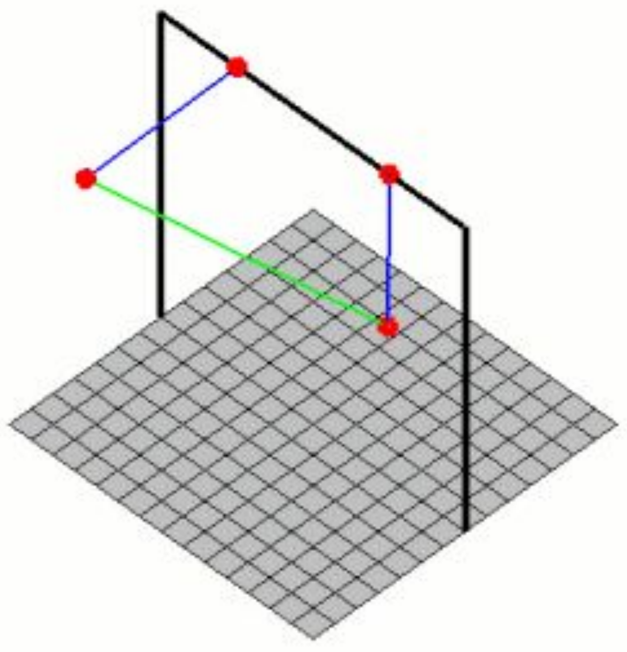
Coupled Harmonic Oscillators

- Let’s go back to our discussion of harmonic oscillators, specifically pendulums.
- When two SHO are put into contact some way, such that energy from one can be transferred to the other, we have coupled harmonic oscillators.



weakly coupled!

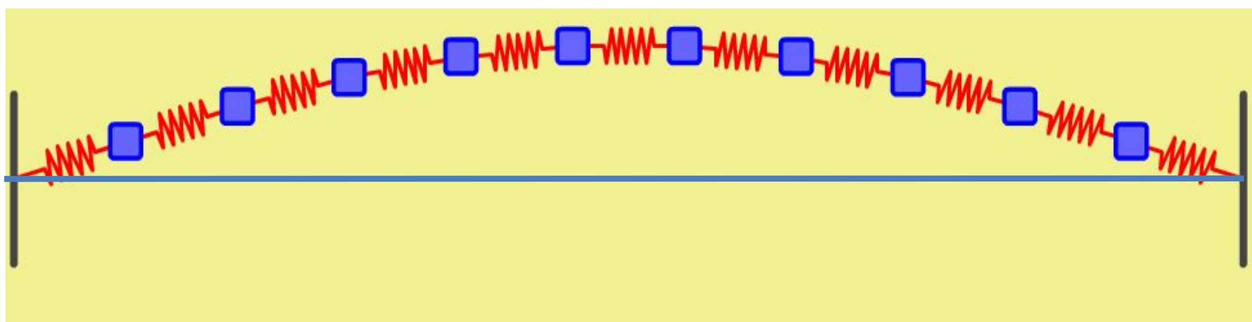
- https://phet.colorado.edu/sims/normal-modes/normal-modes_en.html
- http://www.falstad.com/loadedstring/

Normal modes?

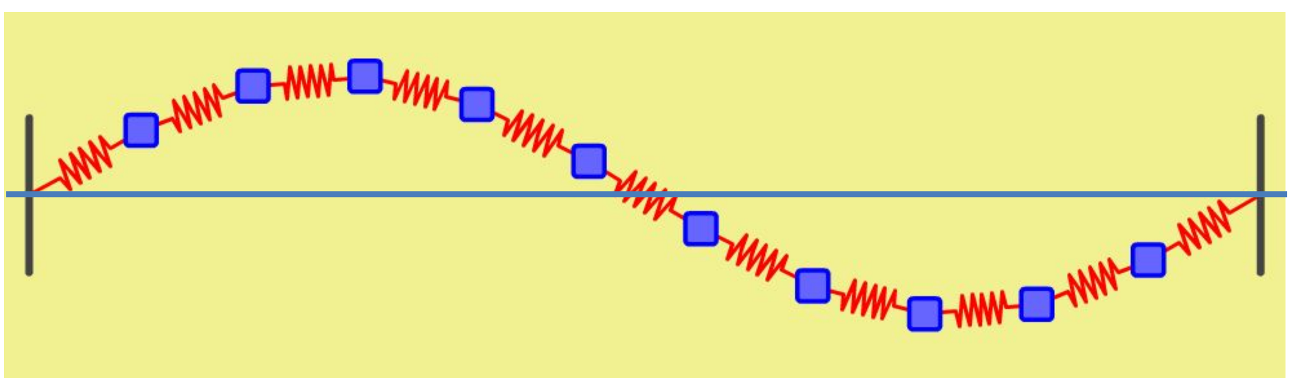
- **Notice** that:
  - For a particular normal mode, every mass has the same frequency of oscillation.
  - There is the same total number of normal modes as there are masses.
  - Normal modes alternate between being symmetric (with respect to the center) and anti-symmetric.
  - Each normal mode has a certain number of nodes and antinodes:
    - Nodes are points where the amplitude of oscillations is zero.
    - Antinodes are points where the amplitude of oscillations is maximal.

Normal modes?

- Notice that, for a given normal mode, the amplitude of oscillation for a particular mass does not change over time, but are different from mass to mass:



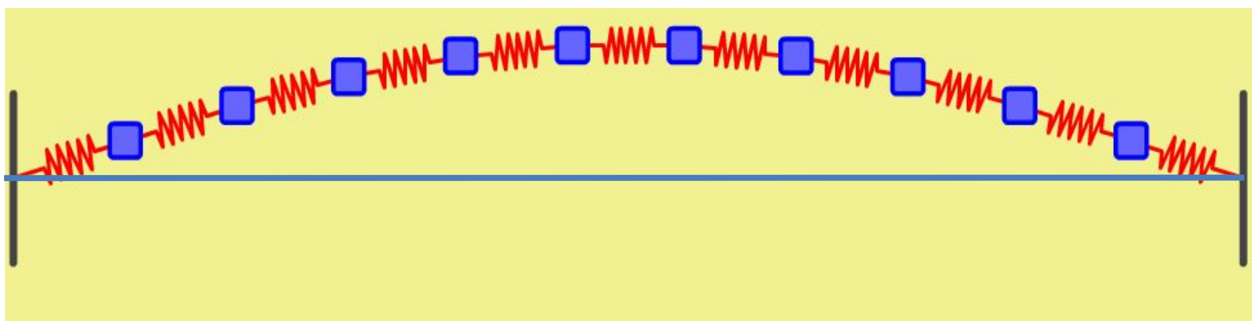
1st Normal mode; N = 1



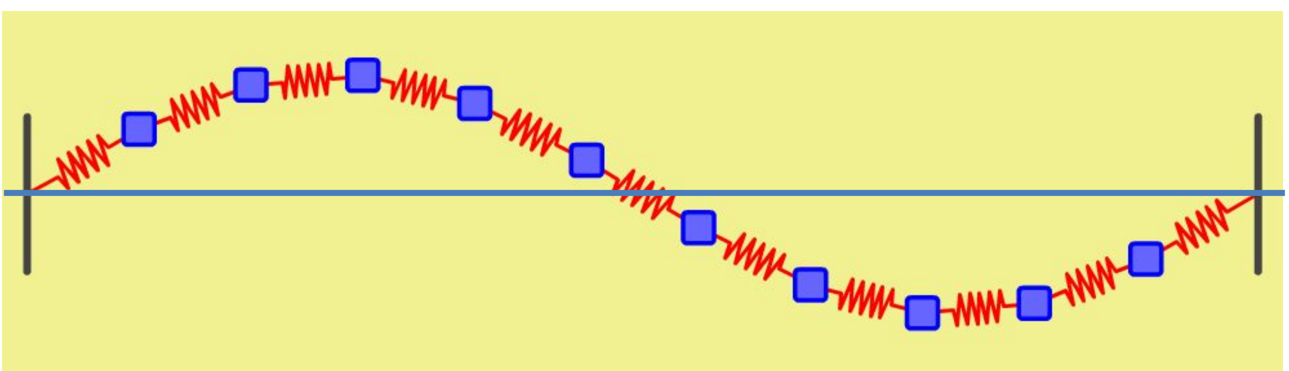
2nd Normal mode; N = 2

Normal modes?

- To try to make a mathematical description of the normal modes, we notice that the amplitudes vary as a sine function of the position:



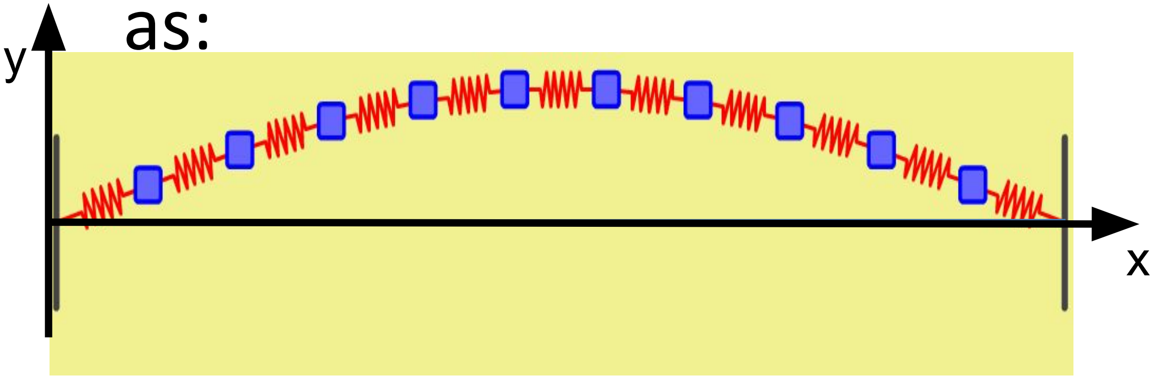
$$A_N(x_i) \propto \sin(k_N x_i)$$
$$= A \sin(k_N x_i)$$

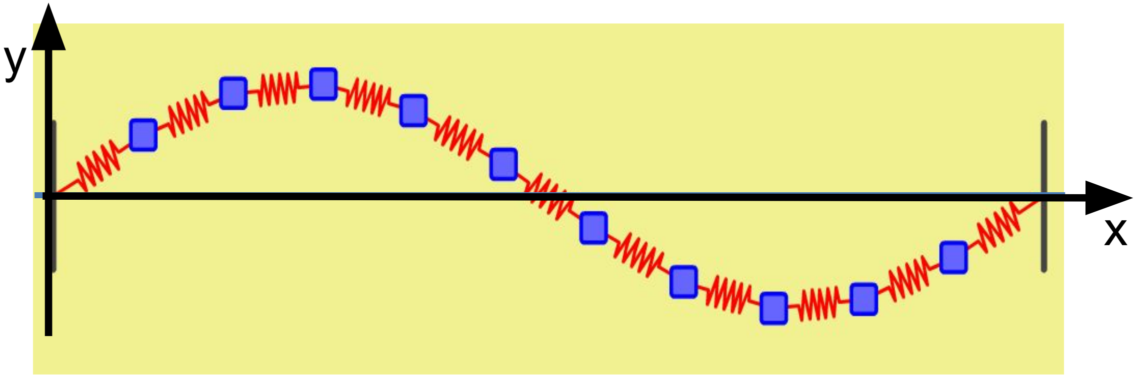


Read: the amplitude of the i^{th} mass in the N^{th} normal mode is proportional to the sin of the x-position of the i^{th} mass times a constant which depends on the mode number

Time Dependence

- Since, for the normal modes, all of the masses oscillate with the same frequency, their displacement from equilibrium can be described as:





$$y_N(x_i,t) = A_N \cos(\omega_N t)$$

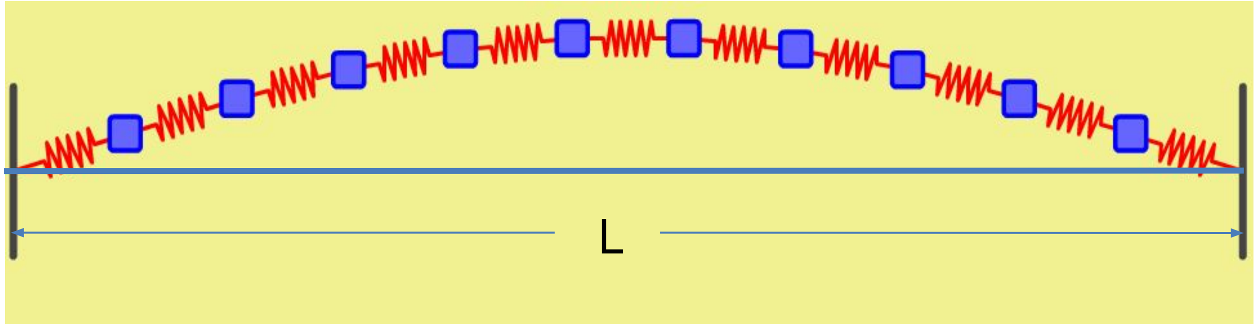
$$= \overbrace{A\sin(k_N x_i)}\cos(\omega_N t)$$

Questions

- What units do the k_N have?
- What units do the ω_N have?
- What is the value of k_2?

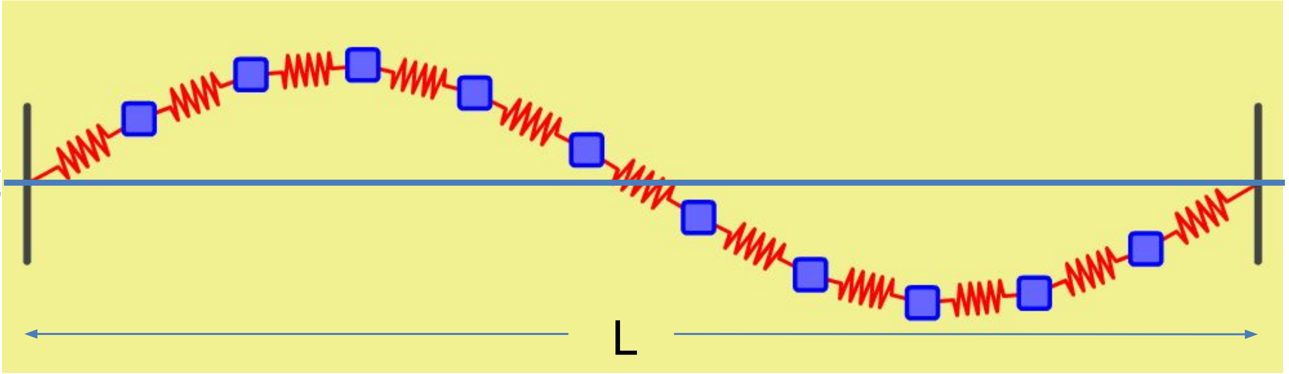
$$y_N(x_i,t) = A\sin(k_N x_i)\cos(\omega_N t)$$

N=1



$$A_1(x_i) \propto \sin(k_1 x_i)$$

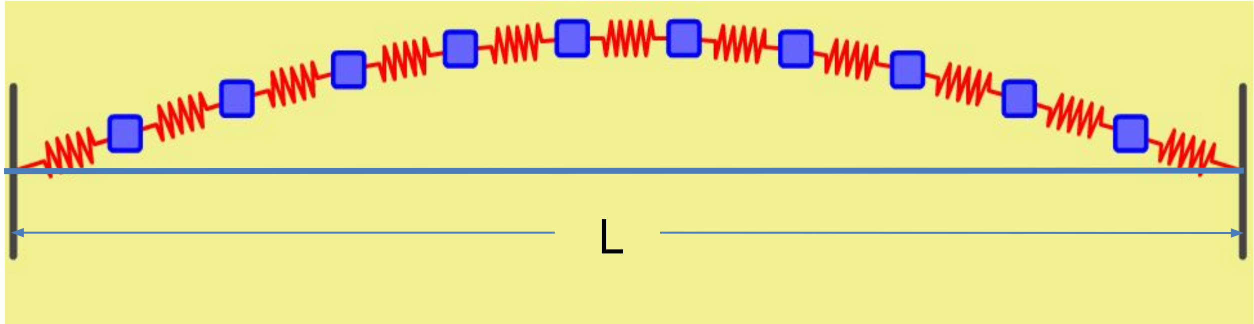
N=2



$$A_2(x_i) \propto \sin(k_2 x_i)$$

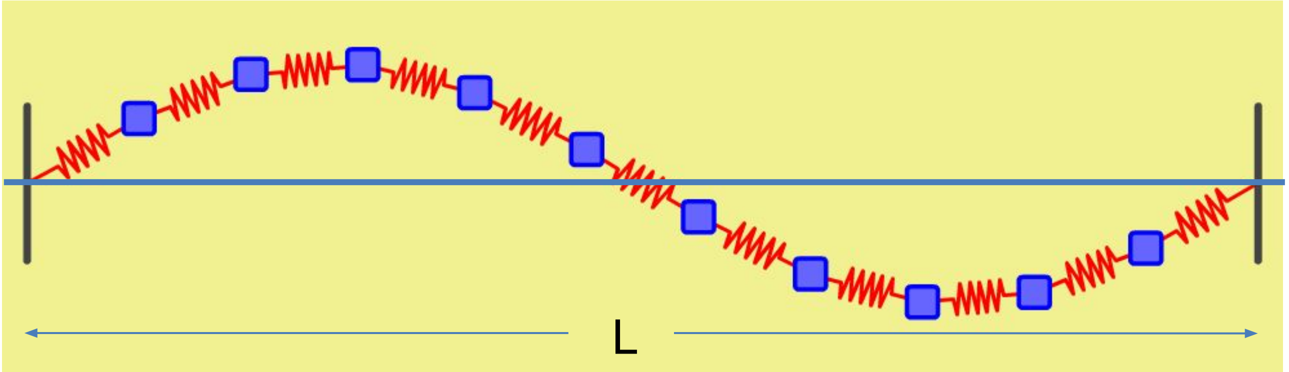
Answers

- The argument of a trigonometric function must be unitless, so k_N must have units of m^{-1}.
- For N=2, after the distance L, the sine function repeats, and since sine repeats every 2π, k_2 must be 2π/L.
- k_1 would then be 2 π/2L.



$$A_1(x_i) \propto \sin(k_1 x_i)$$

$$A_1(x_i) \propto \sin\left(\frac{\pi}{L} x_i\right)$$

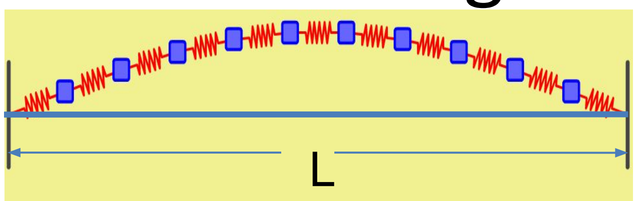


$$A_2(x_i) \propto \sin(k_2 x_i)$$

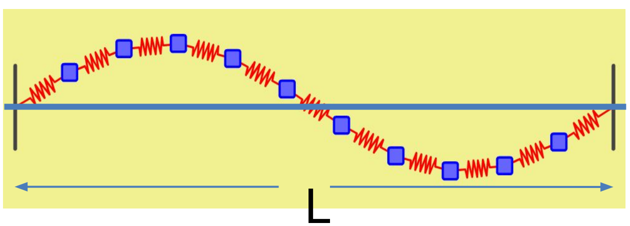
$$A_2(x_i) \propto \sin\left(\frac{2\pi}{L} x_i\right)$$

General k

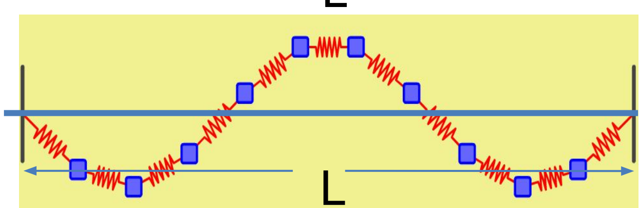
- By looking at a few more normal modes we can see the general form for k:



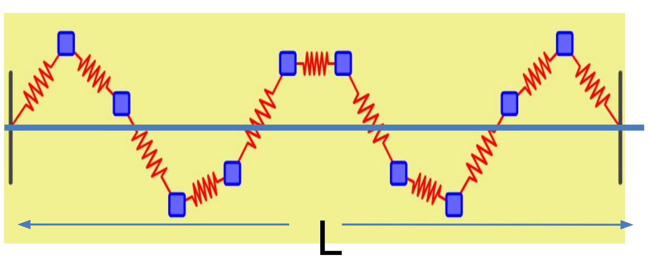
$$A_1(x_i) \propto \sin\left(\frac{\pi}{L}x_i\right)$$



$$A_2(x_i) \propto \sin\left(\frac{2\pi}{L}x_i\right)$$



$$A_3(x_i) \propto \sin\left(\frac{3\pi}{L}x_i\right)$$



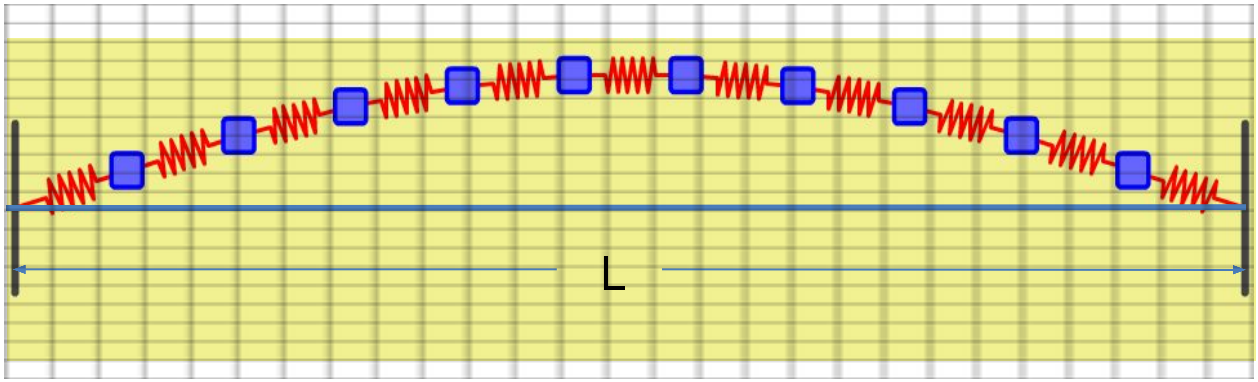
$$A_5(x_i) \propto \sin\left(\frac{5\pi}{L}x_i\right)$$

$$k_N = \frac{N\pi}{L}$$

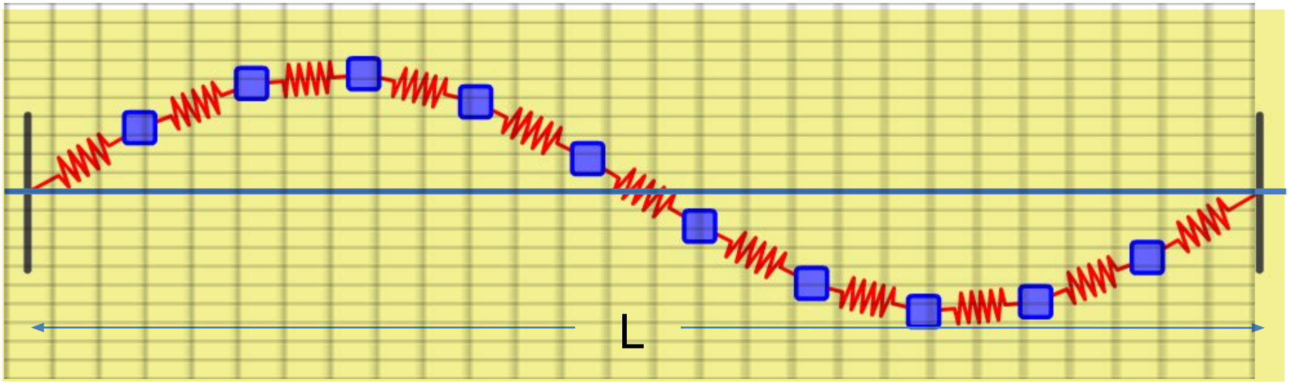
Why are they called **normal** modes?

- What is the answer to the following problem:

$$\sum_i A_1(x_i)\, A_2(x_i) = ?$$



$$A_1(x_i) \propto \sin\left(\frac{\pi}{L} x_i\right)$$

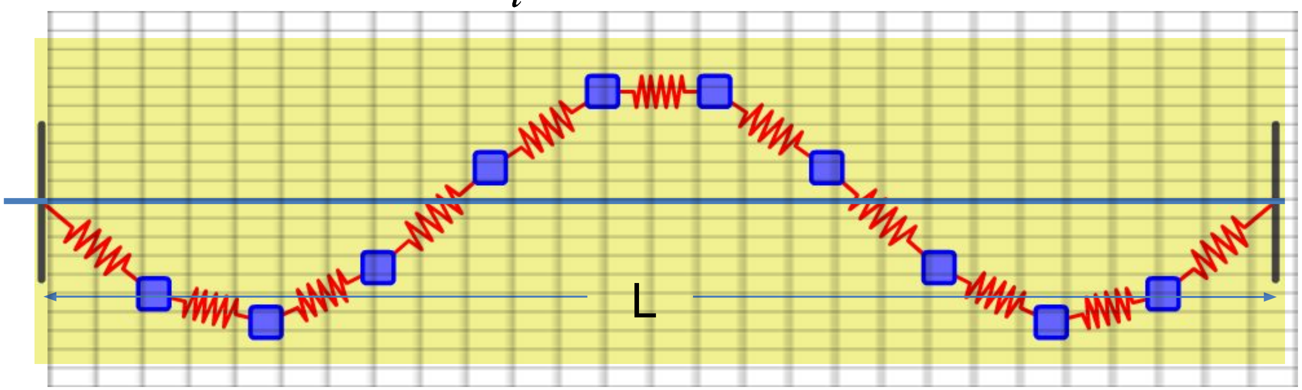


$$A_2(x_i) \propto \sin\left(\frac{2\pi}{L} x_i\right)$$

Why normal?

- What is the answer to the following problem:

$$\sum_i A_3(x_i)\,A_2(x_i) = ?$$



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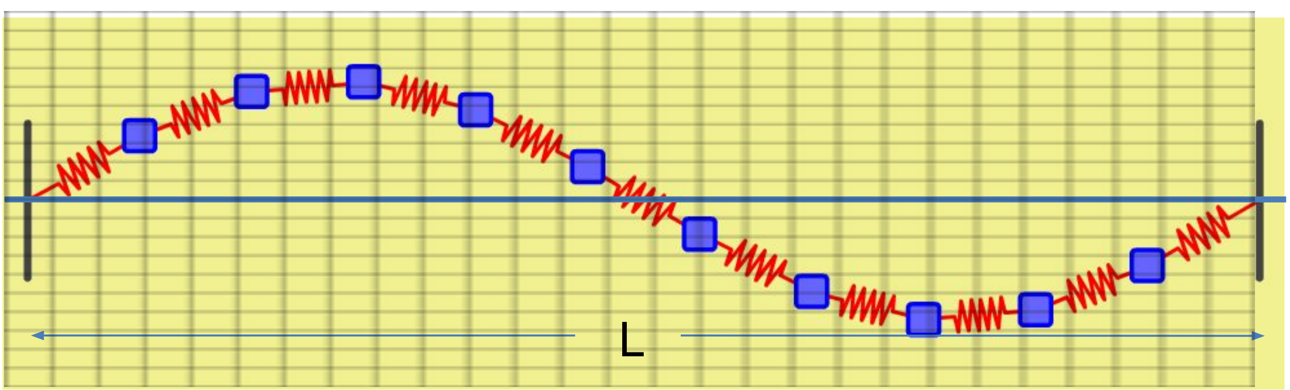
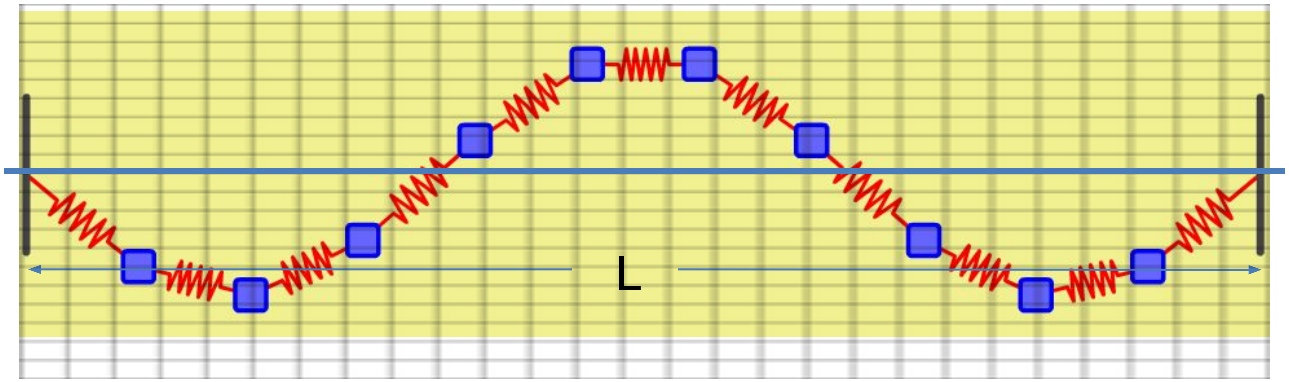


$$A_2(x_i) \propto \sin\left(\frac{2\pi}{L}x_i\right)$$

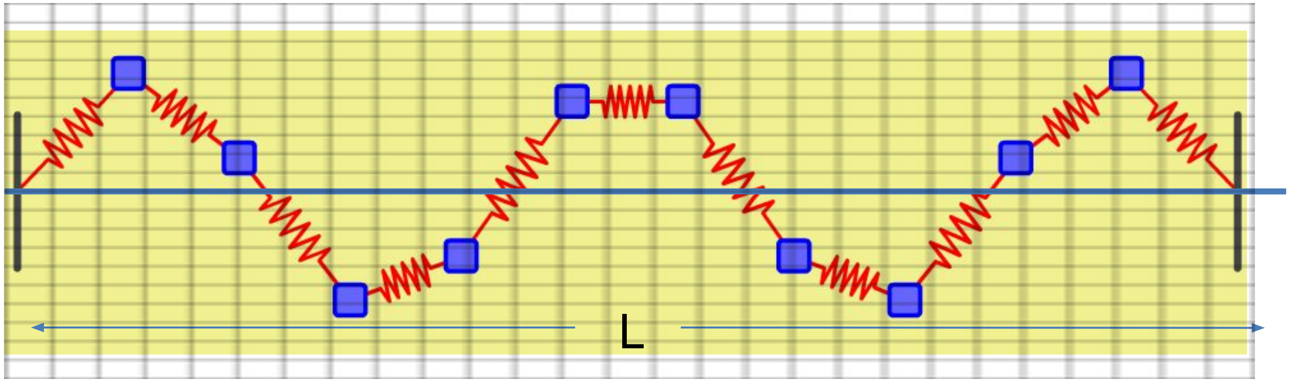
Why normal?

- What is the answer to the following problem:

$$\sum_i A_3(x_i)\,A_5(x_i) = ?$$



$$A_3(x_i) \propto \sin\left(\frac{3\pi}{L}x_i\right)$$



$$A_5(x_i) \propto \sin\left(\frac{5\pi}{L}x_i\right)$$

Why normal?

- They are called normal modes because:

$$\sum_i A_N(x_i)\,A_M(x_i) = \begin{cases} 0, & \text{if N} \neq \text{M} \\ \dfrac{L}{2}, & \text{if N} = \text{M} \end{cases}$$

- This is the condition for orthogonal functions.
- In fact, if we write these with a normalization constant of 2/L, they are orthonormal functions!

Are the only motions the normal modes?

- NO!
- But all motions can be described in terms of the normal modes:

$$y\left(x_i, t\right) = \sum_N c_N y_N\left(x_i, t\right)$$

- Where the constants c_N tell you how much of each normal mode to use.

Generalization

- What happens if we continue to increase the number of masses, while keeping the total length and mass of the “string” of masses constant?
  - As we approach infinitesimal masses, the number of normal modes approaches infinity.
  - Waves! We will discuss this next time.

- http://www.falstad.com/loadedstring/