Lecture 30
(Velocity Transformations & Relativistic Dynamics)

Physics 2310-01 Spring 2020

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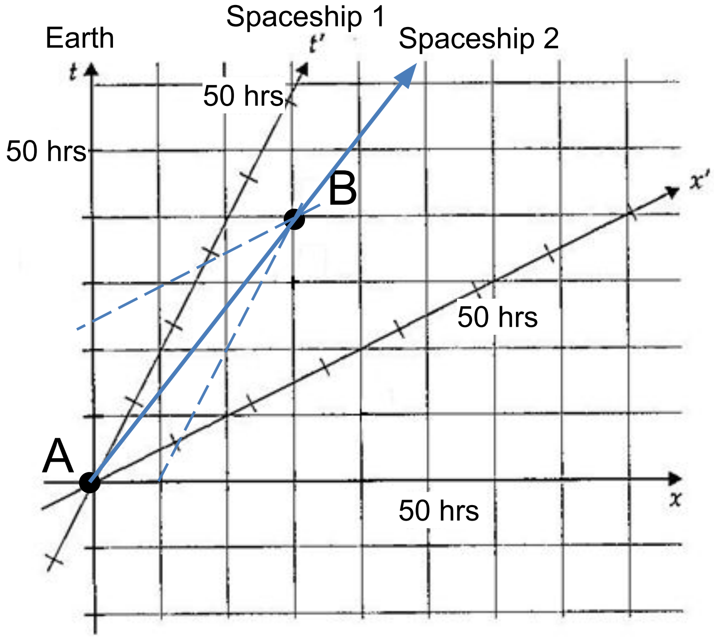
Velocity

- Okay, let's look at how velocities transform between reference frames.
- Let's say we launch from earth two spaceships at the same time, one which travels at β=0.5 (spaceship 1), and one that travels at v=0.75 (spaceship 2), both relative to the earth.
- With what speed does spaceship 2 have relative to spaceship 1?
- Well, we can look at two events along the worldline of spaceship 2, labeled as event A (the launch from earth) and event B, and just ask what is Δx'/Δt':

$$v' = \frac{\Delta x'}{\Delta t'}$$

$$\sim \frac{10.5hrs}{26.5hrs} = 0.40$$

Notice that it's not 0.25.



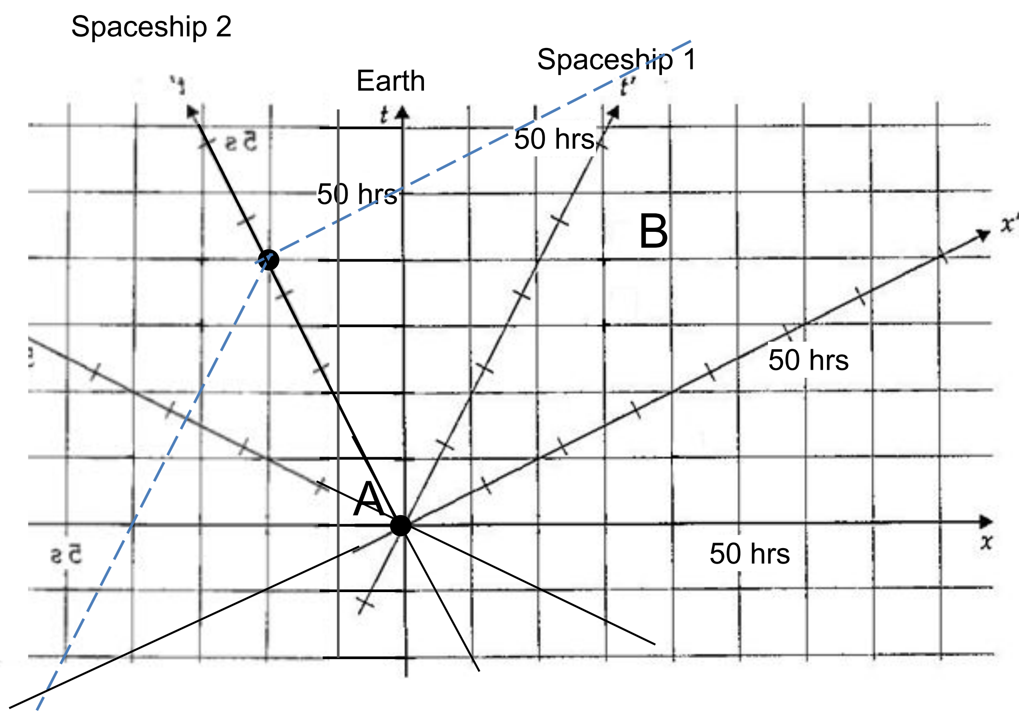
Velocity

- What if we have spaceship 2 travel in the opposite direction of spaceship 1, at v=-0.5?

$$v' = \frac{\Delta x'}{\Delta t'}$$

$$\sim \frac{-47 hrs}{59 hrs} = -0.80$$

Notice that it's not -1.



Velocity

- So, we can see that velocity transformations in Special Relativity are not trivial like they are in Newtonian Relativity. How would we express them mathematically?
- Let's use the Lorentz transformations:

$$v'_x = \frac{dx'}{dt'} = \frac{\gamma\left(dx - \beta dt\right)}{\gamma\left(dt - \beta dx\right)} = \frac{dx - \beta dt}{dt - \beta dx}$$

$$= \frac{\frac{dx}{dt} - \beta\frac{dt}{dt}}{\frac{dt}{dt} - \beta\frac{dx}{dt}} = \frac{v_x - \beta}{1 - \beta v_x}$$

Velocity

- So far we have only considered velocities in the same direction as the relative velocity between the two frames.
- Are velocities in the transverse directions the same ($v_y' = v_y$, $v_z' = v_z$?
- Let's use the Lorentz transformations:

$$v_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \beta dx\right)}$$

$$= \frac{\frac{dy}{dt}}{\gamma\left(\frac{dt}{dt} - \beta\frac{dx}{dt}\right)} = \frac{v_y\sqrt{1-\beta^2}}{1-\beta v_x}$$

And the same for velocities in the z-direction

Velocity Transformations

- So our velocity transformations are:

$$v'_x = \frac{v_x - \beta}{1 - \beta v_x}, \quad v'_y = \frac{v_y\sqrt{1-\beta^2}}{1 - \beta v_x}, \quad v'_z = \frac{v_z\sqrt{1-\beta^2}}{1 - \beta v_x}$$

- And the inverse transformations are just given by substituting -β for β:

$$v_x = \frac{v'_x + \beta}{1 + \beta v'_x}, \quad v_y = \frac{v'_y\sqrt{1-\beta^2}}{1 + \beta v'_x}, \quad v_z = \frac{v'_z\sqrt{1-\beta^2}}{1 + \beta v'_x}$$

Consequence of Velocity Transformations

- Let's ask: "What is the speed of light in the different frames?"
- If the home frame measures 1 for the speed of light, what does a moving frame measure?

$$v'_x = \frac{v_x - \beta}{1 - \beta v_x} = \frac{1 - \beta}{1 - \beta \cdot 1} = 1$$

- Regardless of the relative velocity between frames!

Consequence of Velocity Transformations

- Let's ask: "What is the speed of something if the frames are moving slowly relative to one another?"
- Then,

$$v'_x = \frac{v_x - \beta}{1 - \beta v_x} = \frac{(v_x - \beta)}{\left(1 - \underbrace{\beta v_x}_{\text{small compared to 1}}\right)} \approx v_x - \beta$$

$$v'_y = \frac{v_y\sqrt{1-\beta^2}}{1 - \beta v_x} \approx v_y, \qquad v'_z = \frac{v_z\sqrt{1-\beta^2}}{1 - \beta v_x} \approx v_z$$

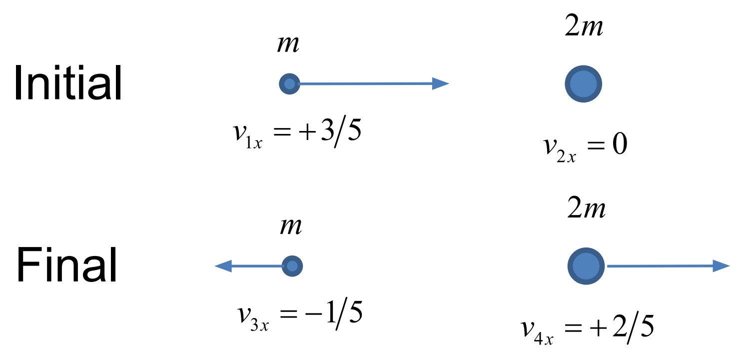
- Which is our Newtonian relativity!

Relativistic Dynamics

- So far, we have just explored the Special Relativistic analog of Chapter 1 in Physics 160.
- We just have our variables to measure distance and time, but we haven't yet seen how these translate to actions – how do objects behave when they act on each other.
- Newton's Laws, at their heart, deal with momentum, so let's first look at Newtonian momentum and see if it behaves well in our new paradigm.
- Newtonian momentum is just: $\vec{p} = m\vec{v}$
- Will this definition work in every inertial frame?

Conservation of Newtonian Momentum

- Let's look at the following example, where we look at a collision between two masses in the rest frame of one of them…



Total Momentum Before $$\vec{p}_{Tot_i} = mv_{1x} + 2mv_{2x} = m\left(+\frac{3}{5}\right) + 2m(0) = +\frac{3}{5}m$$

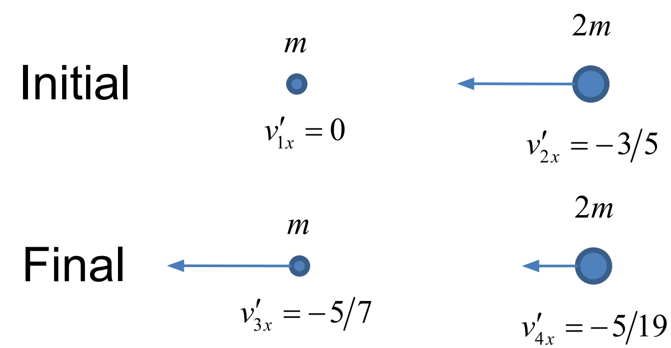
Total Momentum After $$\vec{p}_{Tot_f} = mv_{3x} + 2mv_{4x} = m\left(-\frac{1}{5}\right) + 2m\left(+\frac{2}{5}\right) = +\frac{3}{5}m$$

Conservation of Newtonian Momentum

- Now, let's just move to a frame that is moving along with the other particle.
- We will use the SR velocity transformation equations we just derived….

$$v'_{3x} = \frac{v_{3x} - \beta}{1 - \beta v_{3x}} = \frac{-\frac{1}{5} - \frac{3}{5}}{1 - \left(\frac{3}{5}\right)\left(-\frac{1}{5}\right)} = \frac{-\frac{4}{5}}{\frac{28}{25}} = -\frac{5}{7}$$

$$v'_{4x} = \frac{v_{4x} - \beta}{1 - \beta v_{4x}} = \frac{\frac{2}{5} - \frac{3}{5}}{1 - \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)} = \frac{-\frac{1}{5}}{\frac{19}{25}} = -\frac{5}{19}$$



Total Momentum Before $\quad \vec{p}'_{Tot_i} = mv'_{1x} + 2mv'_{2x} = m(0) + 2m\left(-\frac{3}{5}\right) = -\frac{6}{5}m$

Total Momentum After $\quad \vec{p}'_{Tot_f} = mv'_{3x} + 2mv'_{4x} = m\left(-\frac{5}{7}\right) + 2m\left(-\frac{5}{19}\right) = -\frac{65}{133}m$

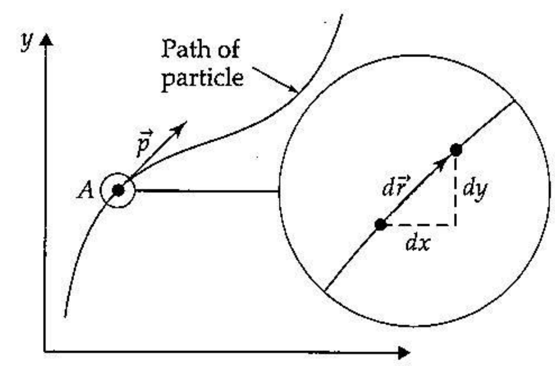
Conservation of Newtonian momentum is not frame independent!

Four Momentum

- Rather than giving up on the idea of momentum conservation, we need to look for a new definition of momentum that:
  - Gives us back Newtonian momentum at low speeds
  - Gives us momentum conservation when at relativistic speeds
  - It might even give us something more…
- There are many ways to derive the idea of relativistic momentum (see ***Special Relativity: A Modern Introduction*** by H.C. Ohanian for a more mathematically rigorous derivation).
- I will follow Moore's (***Six Ideas that Shaped Physics, Volume R***) derivation, which is more conceptual…
- Let's first look at the concept of Newtonian momentum:

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} = m\left[\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right]$$

Newtonian momentum is tangent to the particle's **path** and involves the **time** derivative of each of its **three** dimensions, times its mass.

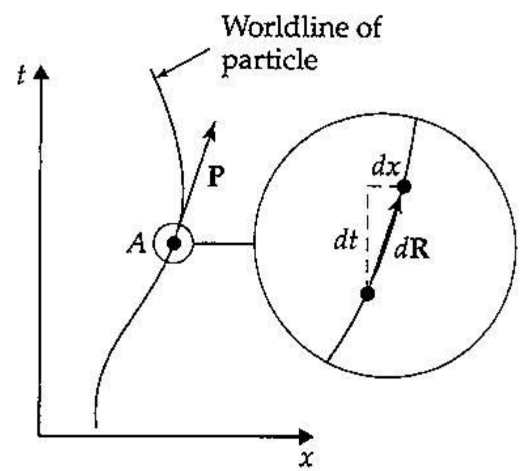


Four Momentum

- Now that we have discovered that time and space are tied together in spacetime, we want to give ***time*** an equal place with ***space*** in our momentum definition.
- Also, since time is not invariant by itself, rather than taking the time derivative, *dt* (which is different for different frames), we will take the derivative with respect to the proper time, *dτ*, which is frame independent.

$$\mathbf{P} = m\frac{d\mathbf{R}}{d\tau} = m\left[\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right]$$

Relativistic momentum is tangent to the particle's **worldline** and involves the **proper time** derivative of each of its **four** dimensions, times its mass.



Four Momentum

- Let's take a closer look at the components of the Four Momentum four-vector with the substitution: $d\tau = \sqrt{1-v^2}\,dt$

$$\mathbf{P} = m\frac{d\mathbf{R}}{d\tau} = m\left[\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right]$$

$$P_t = m\frac{dt}{d\tau} = m\frac{dt}{\sqrt{1-v^2}\,dt} = \frac{m}{\sqrt{1-v^2}}$$

$$P_x = m\frac{dx}{d\tau} = m\frac{dx}{\sqrt{1-v^2}\,dt} = \frac{mv_x}{\sqrt{1-v^2}}$$

$$P_y = m\frac{dy}{d\tau} = m\frac{dy}{\sqrt{1-v^2}\,dt} = \frac{mv_y}{\sqrt{1-v^2}}$$

$$P_z = m\frac{dz}{d\tau} = m\frac{dz}{\sqrt{1-v^2}\,dt} = \frac{mv_z}{\sqrt{1-v^2}}$$

- Finally, we get that:

$$\boxed{\mathbf{P} = \left[\frac{m}{\sqrt{1-v^2}},\frac{mv_x}{\sqrt{1-v^2}},\frac{mv_y}{\sqrt{1-v^2}},\frac{mv_z}{\sqrt{1-v^2}}\right]}$$

What is v???

Frame Transformations of the Four Momentum

- How does the Four Momentum four-vector transform between reference frames?

$$\mathbf{P} = m\frac{d\mathbf{R}}{d\tau} = m\left[\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right]$$

$$P_t' = m\frac{dt'}{d\tau} = m\frac{\gamma\left(dt - \beta dx\right)}{\sqrt{1-v^2}\,dt} = m\frac{\gamma\left(\frac{dt}{dt} - \beta\frac{dx}{dt}\right)}{\sqrt{1-v^2}\,\frac{dt}{dt}} = \gamma\left[\frac{m}{\sqrt{1-v^2}} - \frac{\beta m v_x}{\sqrt{1-v^2}}\right] = \gamma\left(P_t - \beta P_x\right)$$

$$P_x' = m\frac{dx'}{d\tau} = m\frac{\gamma\left(dx - \beta dt\right)}{\sqrt{1-v^2}\,dt} = \gamma\left[\frac{m v_x}{\sqrt{1-v^2}} - \frac{\beta m}{\sqrt{1-v^2}}\right] = \gamma\left(P_x - \beta P_t\right)$$

$$P_y' = m\frac{dy'}{d\tau} = m\frac{dy}{\sqrt{1-v^2}\,dt} = \frac{m v_y}{\sqrt{1-v^2}} = P_y$$

$$P_z' = m\frac{dz'}{d\tau} = m\frac{dz}{\sqrt{1-v^2}\,dt} = \frac{m v_z}{\sqrt{1-v^2}} = P_z$$

Frame Transformations of the Four Momentum

- So, the Four Momentum transforms exactly as the spacetime coordinates.
- Just use the Lorentz transformations:

$$\mathbf{R} = \left[dt, dx, dy, dz\right]$$
$$dt' = \gamma\left(dt - \beta dx\right)$$
$$dx' = \gamma\left(dx - \beta dt\right)$$
$$dy' = dy$$
$$dz' = dz$$

$$\mathbf{P} = m\frac{d\mathbf{R}}{d\tau} = m\left[\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right]$$
$$P_t' = \gamma\left(P_t - \beta P_x\right)$$
$$P_x' = \gamma\left(P_x - \beta P_t\right)$$
$$P_y' = P_y$$
$$P_z' = P_z$$