Lecture 31
(Conservation of Four Momentum)

Physics 2310-01 Spring 2020

Douglas Fields

Frame Transformations of the Four Momentum

- So, the Four Momentum transforms exactly as the spacetime coordinates.
- Just use the Lorentz transformations:

$$\mathbf{R} = \left[dt, dx, dy, dz\right]$$

$$dt' = \gamma\left(dt - \beta dx\right)$$

$$dx' = \gamma\left(dx - \beta dt\right)$$

$$dy' = dy$$

$$dz' = dz$$

$$\mathbf{P} = m\frac{d\mathbf{R}}{d\tau} = m\left[\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right]$$

$$P_t' = \gamma\left(P_t - \beta P_x\right)$$

$$P_x' = \gamma\left(P_x - \beta P_t\right)$$

$$P_y' = P_y$$

$$P_z' = P_z$$

Lorentz Invariant Quantities

- So, each component of the spacetime coordinates and the four momentum depend on the reference frame…
- But we have already seen that the spacetime interval does not.
- Is there an analogous quantity for the four momentum that is frame independent?

$$\mathbf{R} = \left[dt, dx, dy, dz\right]$$

$$dt' = \gamma\left(dt - \beta dx\right)$$

$$dx' = \gamma\left(dx - \beta dt\right)$$

$$dy' = dy$$

$$dz' = dz$$

$$\Delta s^2 = \Delta t^2 - \Delta d^2 = \Delta s'^2$$

$$\mathbf{P} = m\frac{d\mathbf{R}}{d\tau} = m\left[\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right]$$

$$\mathbf{P}^2 = \mathbf{P}\cdot\mathbf{P} = \left(m\frac{dt}{d\tau}\right)^2 - \left[\left(m\frac{dx}{d\tau}\right)^2 + \left(m\frac{dy}{d\tau}\right)^2 + \left(m\frac{dz}{d\tau}\right)^2\right]$$

$$= m^2\left[\left(\frac{dt}{d\tau}\right)^2 - \left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right]\right]$$

$$= m^2\frac{dt^2 - \left[dx^2 + dy^2 + dz^2\right]}{d\tau^2} = m^2\left(\frac{ds}{d\tau}\right)^2$$

$$= m^2$$

Conservation of Four Momentum

- Let’s look at a two-body collision, and see if we can conserve four-momentum in different reference frames:

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4 \Rightarrow$$

$$\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_3 - \mathbf{P}_4 = 0 \Rightarrow$$

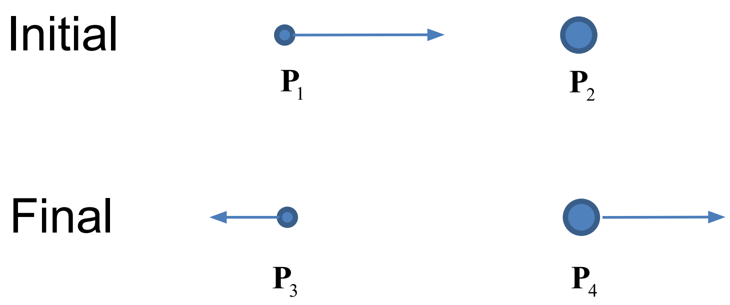
$$\begin{bmatrix} P_{1t} \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} P_{2t} \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} - \begin{bmatrix} P_{3t} \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix} - \begin{bmatrix} P_{4t} \\ P_{4x} \\ P_{4y} \\ P_{4z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$P_{1t} + P_{2t} - P_{3t} - P_{4t} = 0$$

$$P_{1x} + P_{2x} - P_{3x} - P_{4x} = 0$$

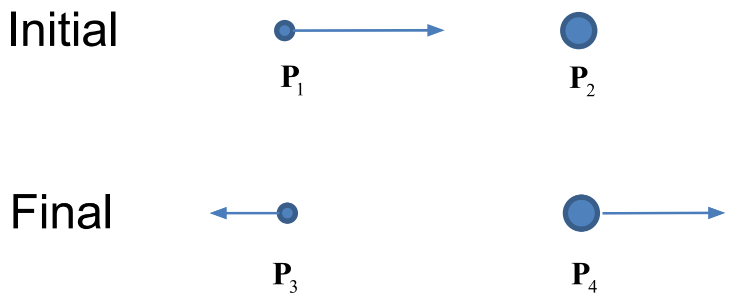
$$P_{1y} + P_{2y} - P_{3y} - P_{4y} = 0$$

$$P_{1z} + P_{2z} - P_{3z} - P_{4z} = 0$$



Conservation of Four Momentum

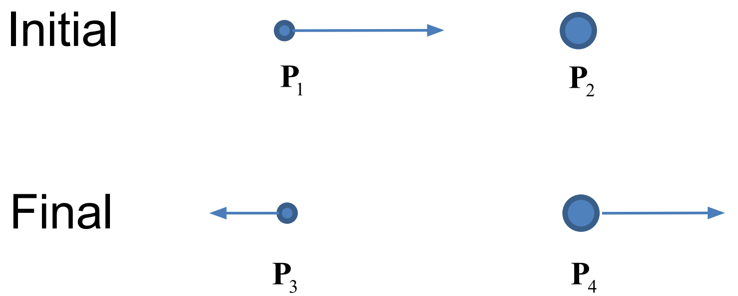
- OK, if in one frame these four constraints are true, is it also true in another frame?
- Let's try for the x-components:



$$P'_{1x} + P'_{2x} - P'_{3x} - P'_{4x} = 0? \Rightarrow$$
$$\gamma(P_{1x} - \beta P_{1t}) + \gamma(P_{2x} - \beta P_{2t}) - \gamma(P_{3x} - \beta P_{3t}) - \gamma(P_{4x} - \beta P_{4t}) = 0? \Rightarrow$$
$$\gamma[P_{1x} + P_{2x} - P_{3x} - P_{4x}] - \gamma\beta[P_{1t} + P_{2t} - P_{3t} - P_{4t}] = 0!$$

Conservation of Four Momentum

- And, of course, it can be showed for the other components as well.
- So, if four-momentum is conserved in one frame, it is conserved in any inertial frame!



$$P'_{1x} + P'_{2x} - P'_{3x} - P'_{4x} = 0? \Rightarrow$$
$$\gamma(P_{1x} - \beta P_{1t}) + \gamma(P_{2x} - \beta P_{2t}) - \gamma(P_{3x} - \beta P_{3t}) - \gamma(P_{4x} - \beta P_{4t}) = 0? \Rightarrow$$
$$\gamma[P_{1x} + P_{2x} - P_{3x} - P_{4x}] - \gamma\beta[P_{1t} + P_{2t} - P_{3t} - P_{4t}] = 0!$$

An Interesting Thing About the Time Component…

- Let's examine the nature of the time component of the four-momentum:

$$\mathbf{P} = \left[\frac{m}{\sqrt{1-v^2}}, \frac{mv_x}{\sqrt{1-v^2}}, \frac{mv_y}{\sqrt{1-v^2}}, \frac{mv_z}{\sqrt{1-v^2}}\right]$$

$$P_t = \frac{m}{\sqrt{1-v^2}}$$

for $v << 1$, can Taylor expand around $\mathrm{v} = 0$,

$$P_t = m\left(1-v^2\right)^{-1/2} = m\left[1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots\right]$$

$$\approx m\left(1 + \frac{1}{2}v^2\right)$$

$$P_t = m + \frac{1}{2}mv^2$$

- So, at low velocity, we see that the time component is the sum of the mass and the kinetic energy…

Relativistic Energy

- So, taking the hint, we define the total relativistic energy as:

$$E \equiv P_t = \frac{m}{\sqrt{1-v^2}}$$

- And, taking the hint from the low-velocity limit, we define the relativistic kinetic energy as:

$$KE \equiv E - m = \frac{m}{\sqrt{1-v^2}} - m$$

- Which, leaves us with another interesting result, that is that even at zero velocity, there is still energy:

$$\boxed{E_{rest} = m}$$

Relativistic Momentum

- With the definition of the four momentum as

$$\mathbf{P} = \left[\frac{m}{\sqrt{1-v^2}}, \frac{mv_x}{\sqrt{1-v^2}}, \frac{mv_y}{\sqrt{1-v^2}}, \frac{mv_z}{\sqrt{1-v^2}}\right] = \left[E, \frac{mv_x}{\sqrt{1-v^2}}, \frac{mv_y}{\sqrt{1-v^2}}, \frac{mv_z}{\sqrt{1-v^2}}\right]$$

- We can identify the relativistic momentum as the last three components of the four- momentum

$$\overrightarrow{p} = \left[\frac{mv_x}{\sqrt{1-v^2}}, \frac{mv_y}{\sqrt{1-v^2}}, \frac{mv_z}{\sqrt{1-v^2}}\right]$$

- So that we can compress our definition of the four-momentum as the four vector with the first component as the total relativistic energy, and the remaining three components being the relativistic momentum

$$\mathbf{P} = \left[E, \overrightarrow{p}\right]$$

This puts energy and momentum into the same relationship as time and space in special relativity!

Relativistic Energy and Momentum

- Then,

$$\mathbf{P}^2 = m^2 = P_t^2 - \left(P_x^2 + P_y^2 + P_z^2\right)$$

$$m^2 = E^2 - p^2$$

- And

$$\frac{\vec{p}}{E} = \frac{m\vec{v} \big/ \sqrt{1-v^2}}{m \big/ \sqrt{1-v^2}} = \vec{v}$$

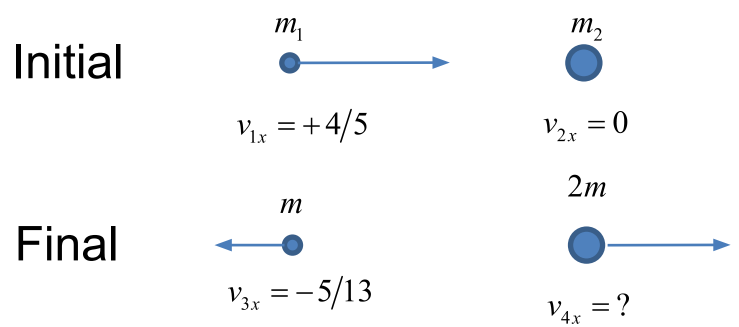
- And

$$\mathbf{P} = \left[E, E\vec{v}\right]$$

- So we can think of relativistic momentum as the rate at which relativistic energy is transmitted through space…

Elastically Colliding Rocks

- Imagine a rock (mass, m_1 = 12kg) moving with v_{1x} = +4/5 in some inertial frame. This rock then strikes another rock (mass, m_2 = 28kg) at rest in that frame. Pretend that the collision is elastic (not likely) and that after the collision, the lighter rock is seen to have a velocity v_{3x} = -5/13. What is the velocity of the more massive rock, v_{4x} ?



Total Four-Momentum Before = Total Four-Momentum After

$$\mathbf{P} = \left[\frac{m}{\sqrt{1-v^2}}, \frac{mv_x}{\sqrt{1-v^2}}, \frac{mv_y}{\sqrt{1-v^2}}, \frac{mv_z}{\sqrt{1-v^2}}\right] = \left[E, P_x, P_y, P_z\right]$$

$$\begin{bmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} = \begin{bmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix} + \begin{bmatrix} E_4 \\ P_{4x} \\ P_{4y} \\ P_{4z} \end{bmatrix}$$

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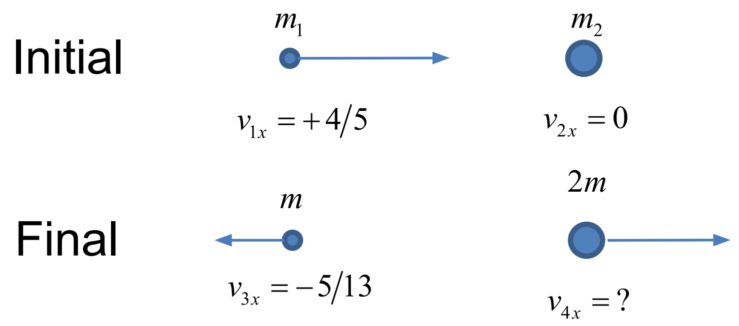

Total Four-Momentum Before = Total Four-Momentum After

$$E_1 = \frac{m_1}{\sqrt{1-v_1^2}} = \frac{12kg}{\sqrt{1-(4/5)^2}} = 20kg$$

$$P_{1x} = \frac{m_1 v_{1x}}{\sqrt{1-v_1^2}} = \frac{(12kg)(4/5)}{\sqrt{1-(4/5)^2}} = +16kg$$

$$E_2 = \frac{m_2}{\sqrt{1-v_2^2}} = \frac{28kg}{\sqrt{1-(0)^2}} = 28kg$$

$$P_{2x} = \frac{m_2 v_{2x}}{\sqrt{1-v_2^2}} = \frac{(28kg)(0)}{\sqrt{1-(0)^2}} = 0kg$$

$$E_3 = \frac{m_3}{\sqrt{1-v_3^2}} = \frac{12kg}{\sqrt{1-(-5/13)^2}} = 13kg$$

$$P_{3x} = \frac{m_3 v_{3x}}{\sqrt{1-v_3^2}} = \frac{(12kg)(-5/13)}{\sqrt{1-(-5/13)^2}} = -5kg$$

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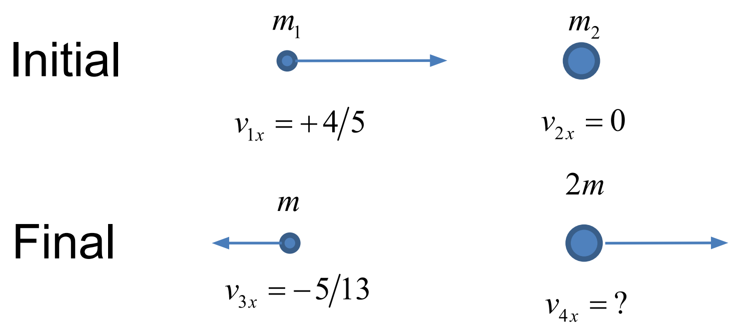

Total Four-Momentum Before = Total Four-Momentum After

$$\begin{bmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} = \begin{bmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix} + \begin{bmatrix} E_4 \\ P_{4x} \\ P_{4y} \\ P_{4z} \end{bmatrix} \Rightarrow \begin{bmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} - \begin{bmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix} = \begin{bmatrix} E_4 \\ P_{4x} \\ P_{4y} \\ P_{4z} \end{bmatrix} \Rightarrow \begin{bmatrix} E_4 = E_1 + E_2 - E_3 = 20kg + 28kg - 13kg = 35kg \\ P_{4x} = P_{1x} + P_{2x} - P_{3x} = 16kg + 0kg + 5kg = 21kg \\ P_{4y} = P_{1y} + P_{2y} - P_{3y} = 0kg + 0kg - 0kg = 0kg \\ P_{4z} = P_{1z} + P_{2z} - P_{3z} = 0kg + 0kg - 0kg = 0kg \end{bmatrix}$$

Elastically Colliding Rocks

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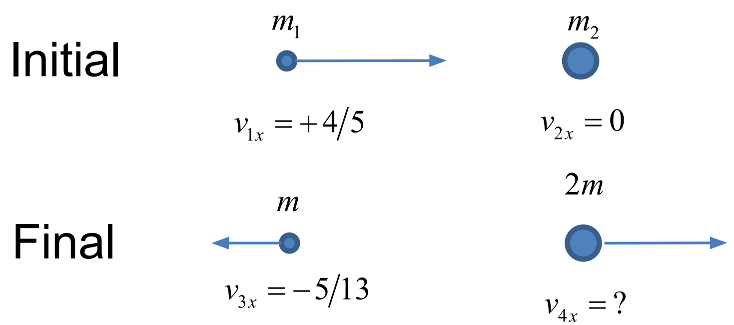


Total Four-Momentum Before = Total Four-Momentum After

$$\mathbf{P}_4 = \begin{bmatrix} E_4 = 35kg \\ P_{4x} = 21kg \\ P_{4y} = 0kg \\ P_{4z} = 0kg \end{bmatrix}$$

$$v_{4x} = \frac{p_{4x}}{E_4} = \frac{21kg}{35kg} = \frac{3}{5}$$

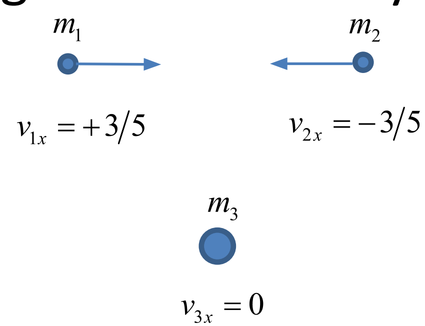
$$m_4 = \sqrt{(E_4)^2 - (P_{4x})^2 - (P_{4y})^2 - (P_{4z})^2}$$
$$= \sqrt{(35kg)^2 - (21kg)^2 - (0kg)^2 - (0kg)^2}$$
$$= 28kg$$

Mass of a System of Particles

- Consider the collision of two identical balls of putty (mass, $m_1 = m_2 = 4$kg) which, in some inertial frame, are moving towards each other with identical speeds of v = 3/5. After the collision, they stick together (inelastic) and are at rest in this frame. What is the mass of the two balls together after they are stuck together?

Initial

Final



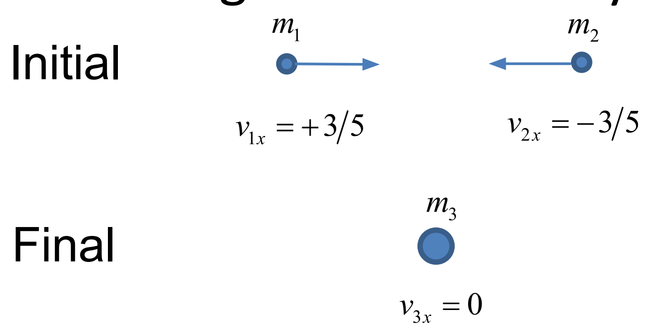
Total Four-Momentum Before = Total Four-Momentum After

$$\mathbf{P} = \left[\frac{m}{\sqrt{1-v^2}}, \frac{mv_x}{\sqrt{1-v^2}}, \frac{mv_y}{\sqrt{1-v^2}}, \frac{mv_z}{\sqrt{1-v^2}}\right]$$
$$= \left[E, P_x, P_y, P_z\right]$$

$$\begin{bmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} = \begin{bmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix}$$

Mass of a System of Particles

- Consider the collision of two identical balls of putty (mass, $m_1 = m_2 = 4kg$) which, in some inertial frame, are moving towards each other with identical speeds of v = 3/5. After the collision, they stick together (inelastic) and are at rest in this frame. What is the mass of the two balls together after they are stuck together?



Total Four-Momentum Before = Total Four-Momentum After

$$E_1 = \frac{m_1}{\sqrt{1-v_1^2}} = \frac{4kg}{\sqrt{1-(3/5)^2}} = 5kg \qquad E_2 = \frac{m_2}{\sqrt{1-v_2^2}} = \frac{4kg}{\sqrt{1-(-3/5)^2}} = 5kg$$

$$P_{1x} = \frac{m_1 v_{1x}}{\sqrt{1-v_1^2}} = \frac{(4kg)(3/5)}{\sqrt{1-(3/5)^2}} = 3kg \qquad P_{2x} = \frac{m_2 v_{2x}}{\sqrt{1-v_2^2}} = \frac{(4kg)(-3/5)}{\sqrt{1-(-3/5)^2}} = -3kg$$

Mass of a System of Particles

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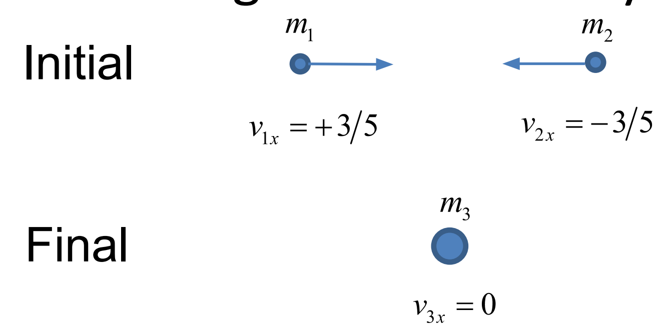
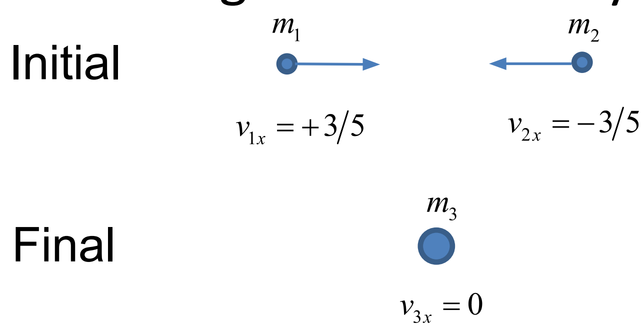


Total Four-Momentum Before = Total Four-Momentum After

$$\begin{bmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} = \begin{bmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix} \Rightarrow \begin{bmatrix} E_3 = E_1 + E_2 = 5kg + 5kg = 10kg \\ P_{3x} = P_{1x} + P_{2x} = 3kg - 3kg = 0kg \\ P_{3y} = P_{1y} + P_{2y} = 0kg + 0kg = 0kg \\ P_{3z} = P_{1z} + P_{2z} = 0kg + 0kg = 0kg \end{bmatrix}$$

Mass of a System of Particles

- Consider the collision of two identical balls of putty (mass, $m_1 = m_2 = 4kg$) which, in some inertial frame, are moving towards each other with identical speeds of v = 3/5. After the collision, they stick together (inelastic) and are at rest in this frame. What is the mass of the two balls together after they are stuck together?



Total Four-Momentum Before = Total Four-Momentum After

$$\mathbf{P}_3 = \begin{bmatrix} E_3 = 10kg \\ P_{3x} = 0kg \\ P_{3y} = 0kg \\ P_{3z} = 0kg \end{bmatrix}$$

$$v_{3x} = \frac{p_{3x}}{E_3} = \frac{0kg}{10kg} = 0$$

$$m_3 = \sqrt{(E_3)^2 - (P_{3x})^2 - (P_{3y})^2 - (P_{3z})^2}$$
$$= \sqrt{(10kg)^2 - (0kg)^2 - (0kg)^2 - (0kg)^2}$$
$$= 10kg$$

Notice that this is MORE than the masses of the initial balls!

Four-Momentum of Light

- So, what about light?

$$v_{light} \equiv 1 = \frac{p_{light}}{E_{light}} \rightarrow$$

$$p = E \quad \text{(for light)}$$

- And so:

$$\begin{aligned} m_{\text{rest}} &= \sqrt{(E)^2 - (P_{3x})^2 - (P_{3y})^2 - (P_{3z})^2} \\ &= \sqrt{(E)^2 - (P)^2} \\ &= 0 \end{aligned}$$

Doppler Shift of Light

- Using the Lorentz transform of the four-momentum:

$$P_t' = \gamma\left(P_t - \beta P_x\right), \quad but \quad E = P_t = P_x$$

$$E' = \gamma\left(E - \beta E\right)$$

$$E' = \gamma\left(1-\beta\right)E = \frac{1}{\sqrt{1-\beta^2}}\left(1-\beta\right)E = \frac{\left(1-\beta\right)}{\sqrt{\left(1-\beta\right)\left(1+\beta\right)}}E$$

$$E' = \sqrt{\frac{\left(1-\beta\right)}{\left(1+\beta\right)}}E$$

- Which is exactly the transformation that the book derives for the doppler shift of the frequency for light. Is the energy of light related to its frequency…?

Problem

An object of mass m sits at rest in a particular frame. A light flash moving in the +x direction with a total energy of 2m hits this object and is totally absorbed. What are the mass and x-velocity of the object after absorbing the light?

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An object of mass m sits at rest in a particular frame. A light flash moving in the +x direction with a total energy of 2m hits this object and is totally absorbed. What are the mass and x-velocity of the object after absorbing the light?

$$E_1 = \frac{m_1}{\sqrt{1-v_1^2}} = \frac{m}{\sqrt{1-(0)^2}} = m$$

$$P_{1x} = \frac{m_1 v_{1x}}{\sqrt{1-v_1^2}} = \frac{(m)(0)}{\sqrt{1-(0)^2}} = 0kg$$

$$E_2 = 2m$$

$$P_{2x} = E_2$$

$$\begin{bmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix} + \begin{bmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} = \begin{bmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{bmatrix}$$

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$$v_{3x} = \frac{p_{3x}}{E_3} = \frac{2m}{3m} = \frac{2}{3}$$

$$\begin{aligned} m_3 &= \sqrt{(E_3)^2 - (P_{3x})^2 - (P_{3y})^2 - (P_{3z})^2} \\ &= \sqrt{(3m)^2 - (2m)^2 - (0kg)^2 - (0kg)^2} \\ &= \sqrt{5}m \end{aligned}$$