Lecture 5
(Wave Equation and Wave Speed)

Physics 2310-01 Spring 2020

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Wave Equation

- Let's look at our displacement solutions, and some of its derivatives:

$$y(x,t) = A\cos(kx - \omega t)$$

$$\frac{\partial y}{\partial t} = -\omega A\left[-\sin(kx - \omega t)\right] = \omega A \sin(kx - \omega t)$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y}$$

Wave Equation

- Let's look at our displacement solutions, and some of its derivatives:

$$y(x,t) = A\cos(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t)$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y}$$

Wave Equation

- And we can put these together:

$$y(x,t) = A\cos(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y \qquad \text{and} \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

$so,$

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

$but, \quad \omega = vk \Rightarrow \omega^2 = v^2 k^2 \Rightarrow$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

WAVE EQUATION – very important

General Solution

- Although we have only looked at periodic solutions to the wave equation so far, the general solution is:

$$y(x,t) = a\,\mathrm{F}(x+vt) + b\,\mathrm{G}(x-vt)$$

- In other words, ANY function that moves at the wave speed in either the plus or minus x-direction (one dimensional case).

General Solution

- Proof: $$y(x,t) = a\,\mathrm{F}(x+vt) + b\,\mathrm{G}(x-vt)$$

Let: $\xi \equiv x + vt$

$$\frac{\partial\,\mathrm{F}(\xi)}{\partial x} = \frac{\partial\,\mathrm{F}(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x} = \mathrm{F}'(\xi) \qquad \frac{\partial\,\mathrm{F}(\xi)}{\partial t} = \frac{\partial\,\mathrm{F}(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} = \mathrm{F}'(\xi)\,v$$

$$\frac{\partial^2\,\mathrm{F}(\xi)}{\partial x^2} = \mathrm{F}''(\xi) \qquad \frac{\partial^2\,\mathrm{F}(\xi)}{\partial t^2} = \mathrm{F}''(\xi)\,v^2$$

which obeys the wave equation:

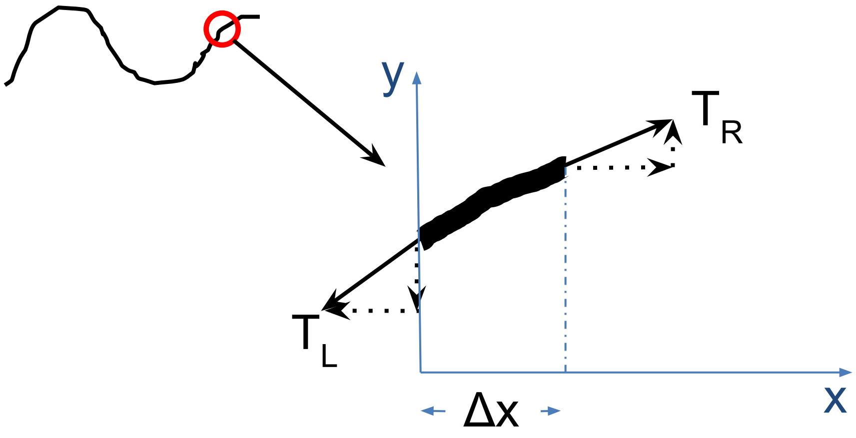
$$\frac{\partial^2\,\mathrm{F}(\xi)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\,\mathrm{F}(\xi)}{\partial t^2}$$

Wave Equation Derivation

- In the previous slides, we started with our wave functions, and just “accidentally” happened upon a differential equation whose solutions were the functions.
- Can we start with what we know about the physics of a string and derive the wave equation?
- Let’s start with a deformed string of unknown shape, and mass density (mass/length) of μ.

Wave Equation Derivation

- Examine the vertical forces on a small length (Δx) of the string:



For small slopes, $T_x \approx T$, so

$$T_{Ly} = -T_x \left.\frac{\partial y}{\partial x}\right|_{x_L} \Rightarrow$$

$$T_{Ly} = -T \left.\frac{\partial y}{\partial x}\right|_{x_L}$$

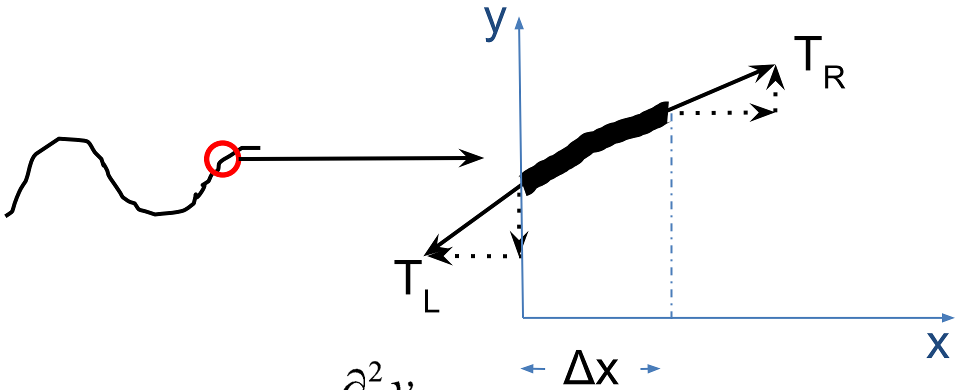
and

$$T_{Ry} = T \left.\frac{\partial y}{\partial x}\right|_{x_L + \Delta x = x_R}$$

$$\sum F_y = T_{Ry} + T_{Ly} = T\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right] = ma_y = (\Delta x \cdot \mu)\frac{\partial^2 y}{\partial t^2} \Rightarrow$$

Wave Equation Derivation

- Simple derivation:



$$\sum F_y = T_{Ry} + T_{Ly} = T\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right] = ma_y = \left(\Delta x \cdot \mu\right)\frac{\partial^2 y}{\partial t^2} \Rightarrow$$

$$\frac{\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right]}{\Delta x} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

$$in \quad \lim \Delta x \to 0$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} \leftrightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \Rightarrow$$

$$v = \sqrt{\frac{T}{\mu}}$$

Experiments with Strings

- Let's compare some wave speeds with different density strings and different tensions:

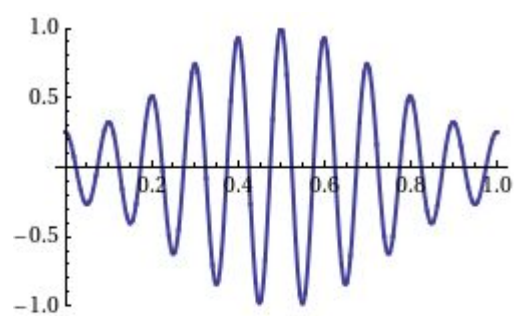
$$v = \sqrt{\frac{T}{\mu}}$$

Wave Speed(s)

- Let’s look at a wave and you tell me what it’s speed is:

By Kraaiennest - Own work, GFDL, https://commons.wikimedia.org/w/index.php?curid=3651297

- How about this one:



By Geek3 - Own work;This mathematical image was created with Mathematica, CC BY 3.0, https://commons.wikimedia.org/w/index.php?curid=14772688

Phase Velocity vs Group Velocity

- The phase velocity is just given as

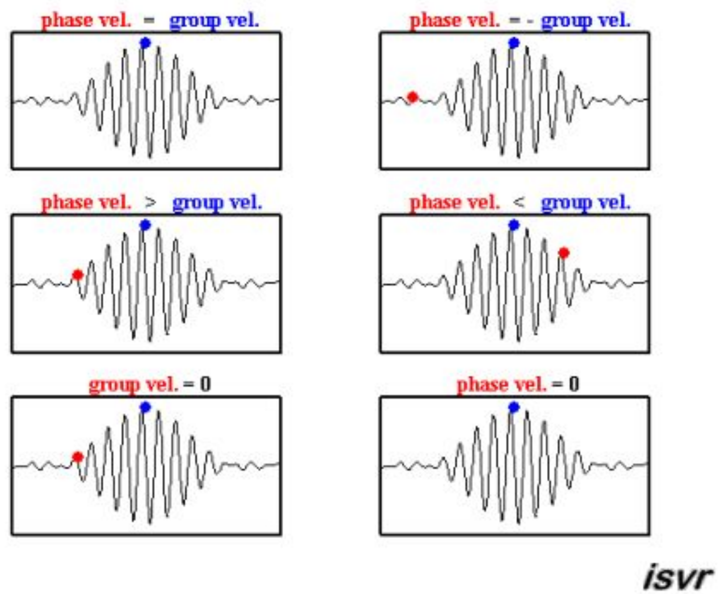
$$v_p = \frac{\omega}{k}$$

- The group velocity is

$$v_g = \frac{\partial \omega}{\partial k}$$

Phase Velocity vs Group Velocity

- If ω is directly proportional to k, then the phase and group velocities are the same:



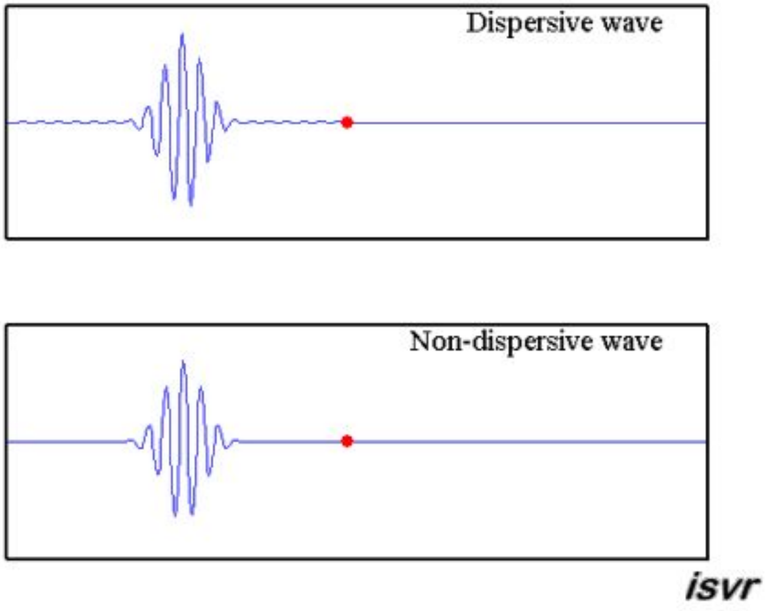
$$v_p = \frac{\omega}{k}$$

$$v_g = \frac{\partial \omega}{\partial k}$$

- The function ω(k), which gives ω as a function of k, is known as the dispersion relation.

Dispersion

- Because of the dependence of the phase velocity on frequency, the group envelope will get distorted with time:



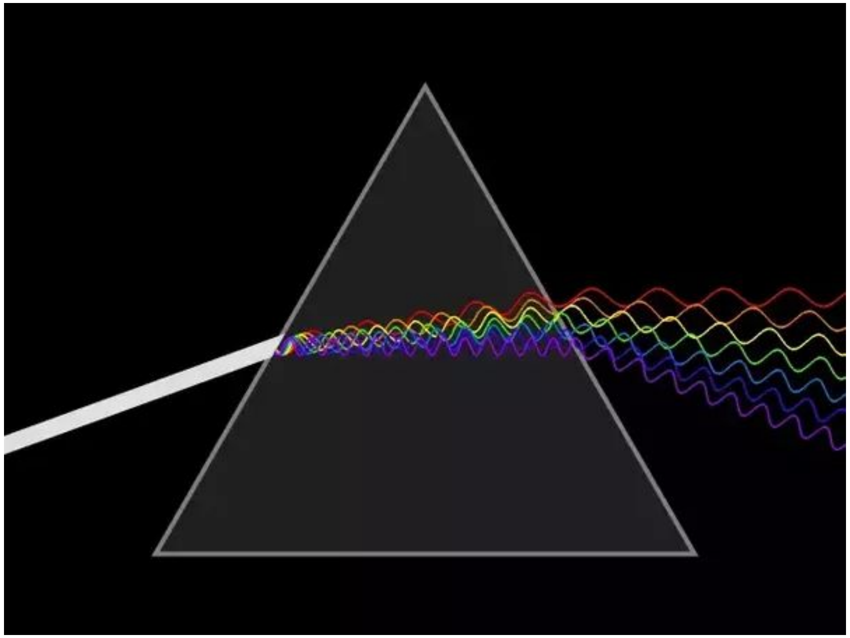
Chromatic Dispersion

- For instance, if the wave velocity depends on the wavelength such that *v = c/n,* and n depends on the wave number *,* then:

$$\omega = vk = \frac{c}{n}k$$

$$v_g = \frac{\partial \omega}{\partial k} = \frac{c}{n} - \frac{ck}{n^2} \cdot \frac{\mathrm{d}n}{\mathrm{d}k}$$

$$v_p = \frac{\omega}{k} = \frac{c}{n}$$



Constants

A 1.80-m string of weight 0.0121 N is tied to the ceiling at its upper end, and the lower end supports a weight W. Neglect the very small variation in tension along the length of the string that is produced by the weight of the string. When you pluck the string slightly, the waves traveling up the string obey the equation $y(x,t) = (8.50 \text{ mm})\cos(172 \text{ rad}\cdot\text{m}^{-1}\,x - 2730 \text{ rad}\cdot\text{ s}^{-1}\,t)$ Assume that the tension of the string is constant and equal to W.

Part A

How much time does it take a pulse to travel the full length of the string?

t = s

Submit Request Answer

Part B

What is the weight W?

W = N

Submit Request Answer