Lecture 7
(Normal Modes)

Physics 2310-01 Spring 2020

Douglas Fields

A version of this idea will be on the exam…

MP#6

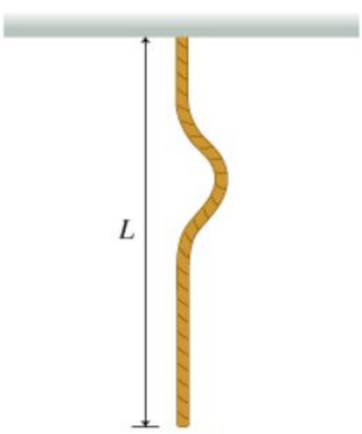
Wave in a Dangling Rope

Constants

A uniform rope of length L and negligible stiffness hangs from a solid fixture in the ceiling (Figure 1).

Figure

1 of 1



Part A

The free lower end of the rope is struck sharply at time $t = 0$. What is the time t it takes the resulting wave on the rope to travel to the ceiling, be reflected, and return to the lower end of the rope?

Express your answer in terms of L and constants such as g (the magnitude of the acceleration due to gravity), π, etc.

View Available Hint(s)

t =

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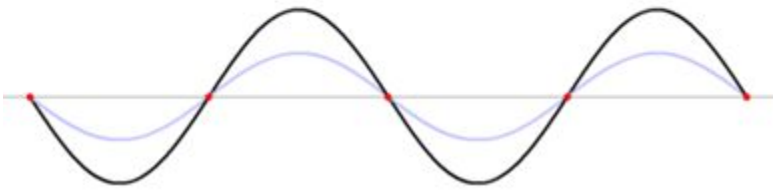
Next

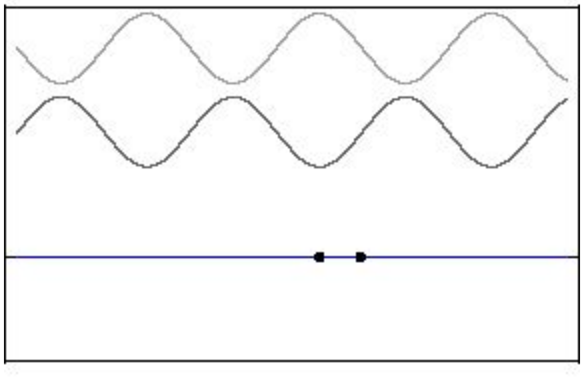
Standing Waves

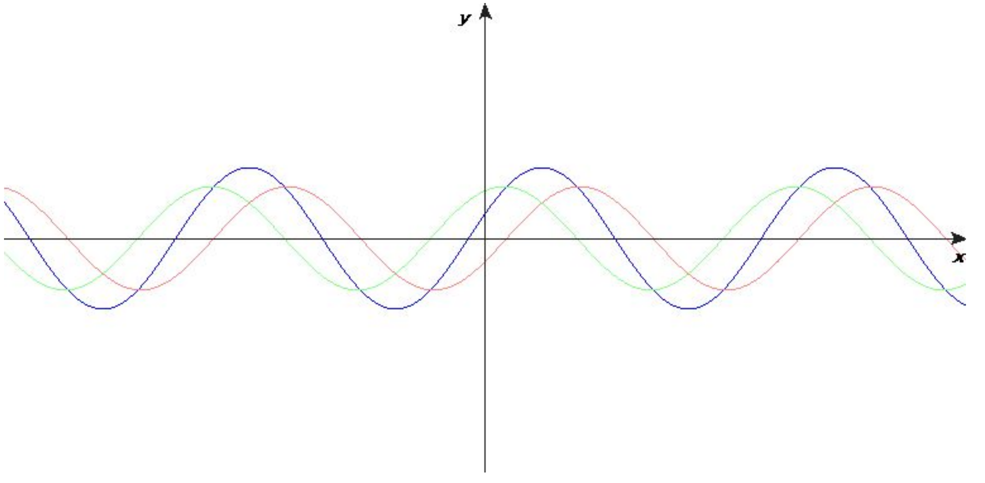
- Examine this solution:



$$y_1(x,t)+y_2(x,t)=\left[2A\cos(\omega t)\right]\sin(kx)$$

- In position, there is a sin dependence whose amplitude oscillates harmonically with time.





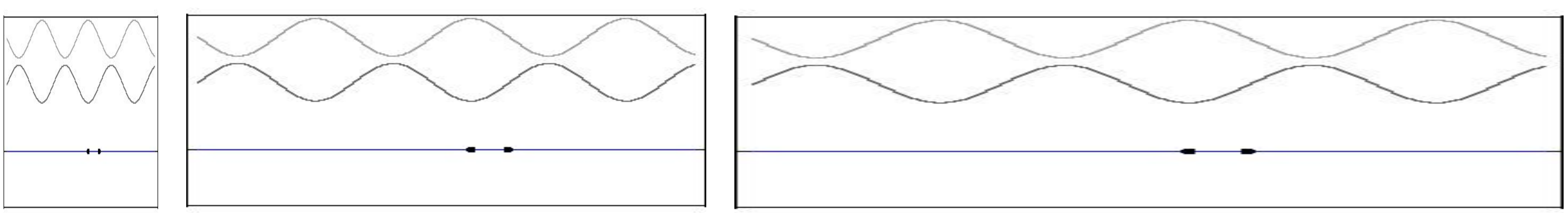
No boundaries…

- Notice that this solution is for two waves travelling in opposite directions with the same wavelength and frequency:

$$y_1(x,t) + y_2(x,t) = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

$$= 2A\sin(kx)\cos(\omega t)$$

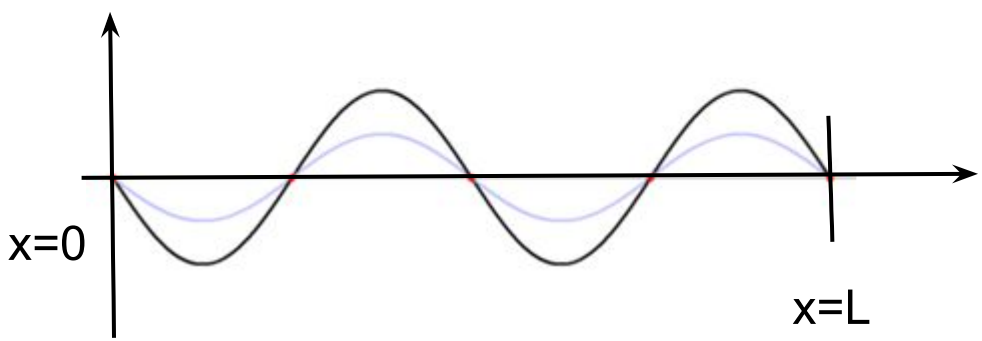
- But there is no restrictions on the wavelength, it can be anything – you always get a standing wave.



Standing Waves with Boundaries

- Now, if there are boundaries, then there are conditions that must be met for there to be standing waves:
  - For instance, if there are fixed ends, then there must be nodes at both ends.

$$y(x,t) = y_1(x,t) + y_2(x,t) = 2A\sin(kx)\cos(\omega t)$$



Normal Modes

- So, we have:

$$y(0,t) = 2A\sin(0)\cos(\omega t) = 0$$

$$and$$

$$y(L,t) = 2A\sin(kL)\cos(\omega t) = 0 \Rightarrow$$

$$kL = n\pi \Rightarrow$$

$$\frac{2\pi L}{\lambda} = n\pi \Rightarrow$$

$$\lambda_n = \frac{2L}{n}, \quad n = 1,2,3,4\cdots$$

$$v = f\lambda \Rightarrow$$

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}$$

Normal Modes

- For n=1,

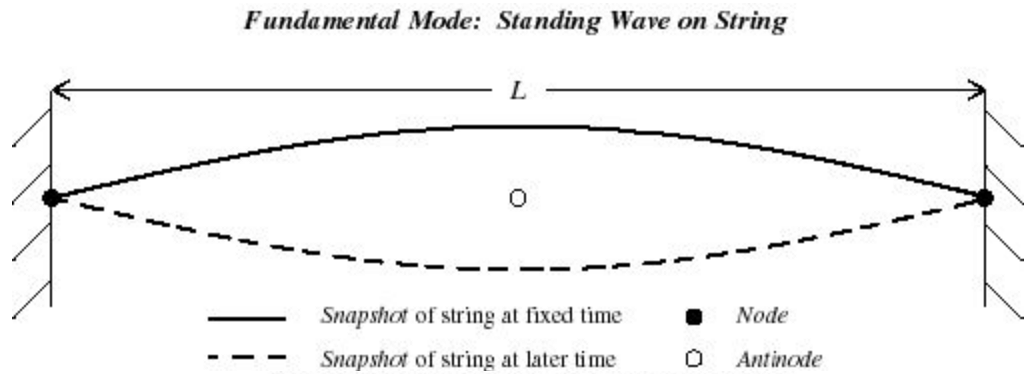
$$\lambda_1 = \frac{2L}{1} = 2L$$

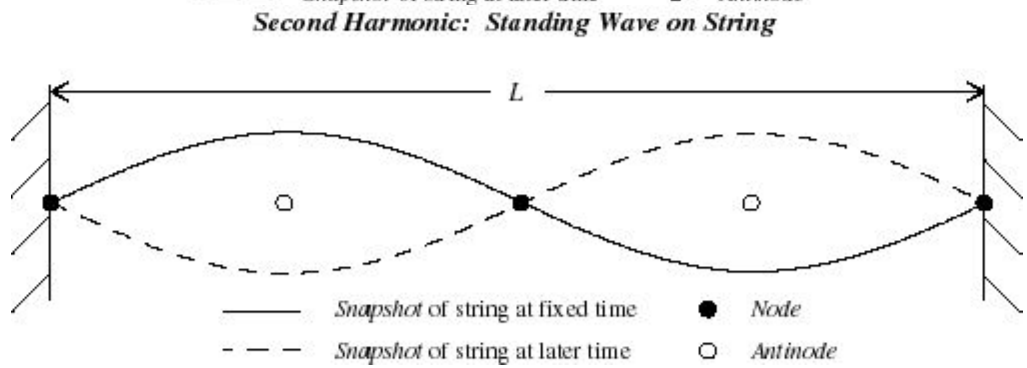
- For n=2,

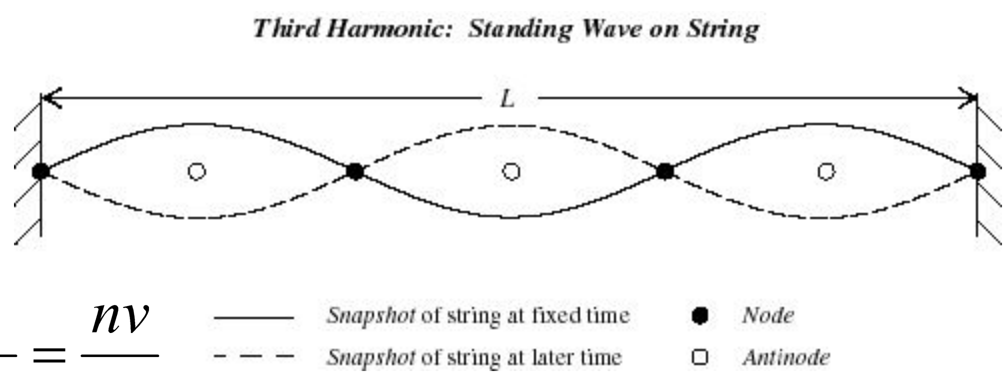
$$\lambda_2 = \frac{2L}{2} = L$$

- For n=3,

$$\lambda_3 = \frac{2L}{3} = \frac{2}{3}L$$







$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}$$

Boundary Conditions

- Does this mean that I cannot have any other wavelengths (frequencies) on the string?
- Of course you can, but they will not create standing waves, and will not have large amplitude oscillations.
- Only the frequencies associated with the normal modes are “natural frequencies” or “resonant frequencies”.

Boundary Conditions

- One of the most profound results from applying boundary conditions is that the natural frequencies are no longer continuous – they become *quantized*:

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} = n\frac{v}{2L} = nf_1$$

Problem 15.46

15.46. (a) A horizontal string tied at both ends is vibrating in its fundamental mode. The traveling waves have speed v, frequency f, amplitude A, and wavelength λ. Calculate the maximum transverse velocity and maximum transverse acceleration of points located at (i) $x = \lambda/2$, (ii) $x = \lambda/4$, and (iii) $x = \lambda/8$ from the left-hand end of the string. (b) At each of the points in part (a), what is the amplitude of the motion? (c) At each of the points in part (a), how much time does it take the string to go from its largest upward displacement to its largest downward displacement?

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

$$\frac{\partial y(x,t)}{\partial t} = -2A\omega\sin(kx)\sin(\omega t)$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -2A\omega^2\sin(kx)\cos(\omega t)$$

Problem 15.70

15.70. A guitar string is vibrating in its fundamental mode, with nodes at each end. The length of the segment of the string that is free to vibrate is 0.386 m. The maximum transverse acceleration of a point at the middle of the segment is $8.40 \times 10^3\ \text{m/s}^2$ and the maximum transverse velocity is 3.80 m/s. (a) What is the amplitude of this standing wave? (b) What is the wave speed for the transverse traveling waves on this string?

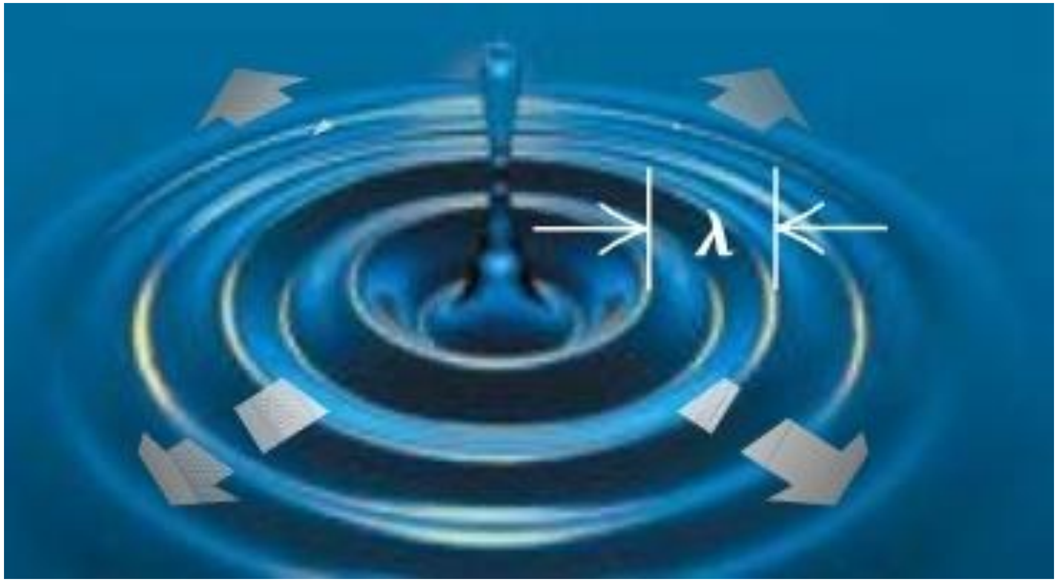
$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

$$\frac{\partial y(x,t)}{\partial t} = -2A\omega\sin(kx)\sin(\omega t)$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -2A\omega^2\sin(kx)\cos(\omega t)$$

2-Dimensional Waves

- Waves can, of course move in more than one dimension:

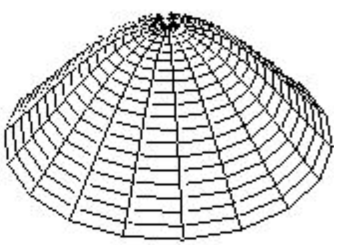


2-D

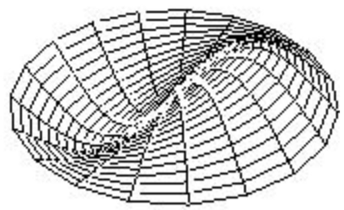
- https://phet.colorado.edu/sims/normal-modes/normal-modes_en.html

Normal Modes in 2-D

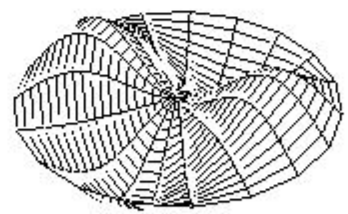
- We can also have boundaries in 2-dimensions, for instance, a drum head:



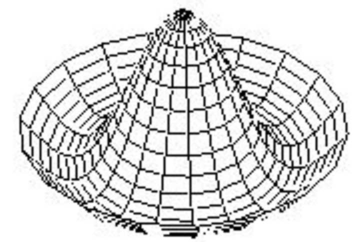
(0,1)



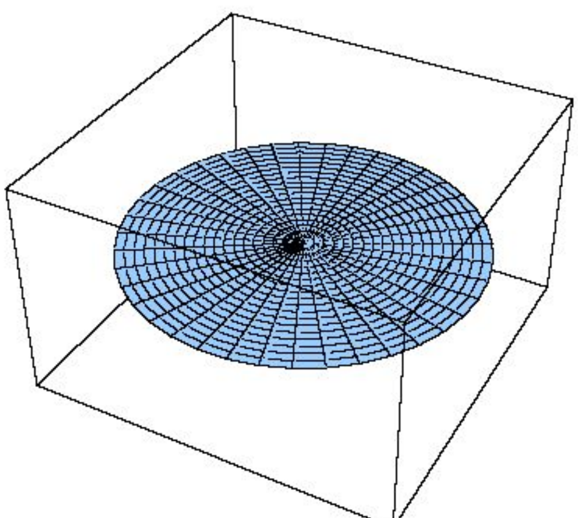
(1,1)



(2,1)



(0,2)



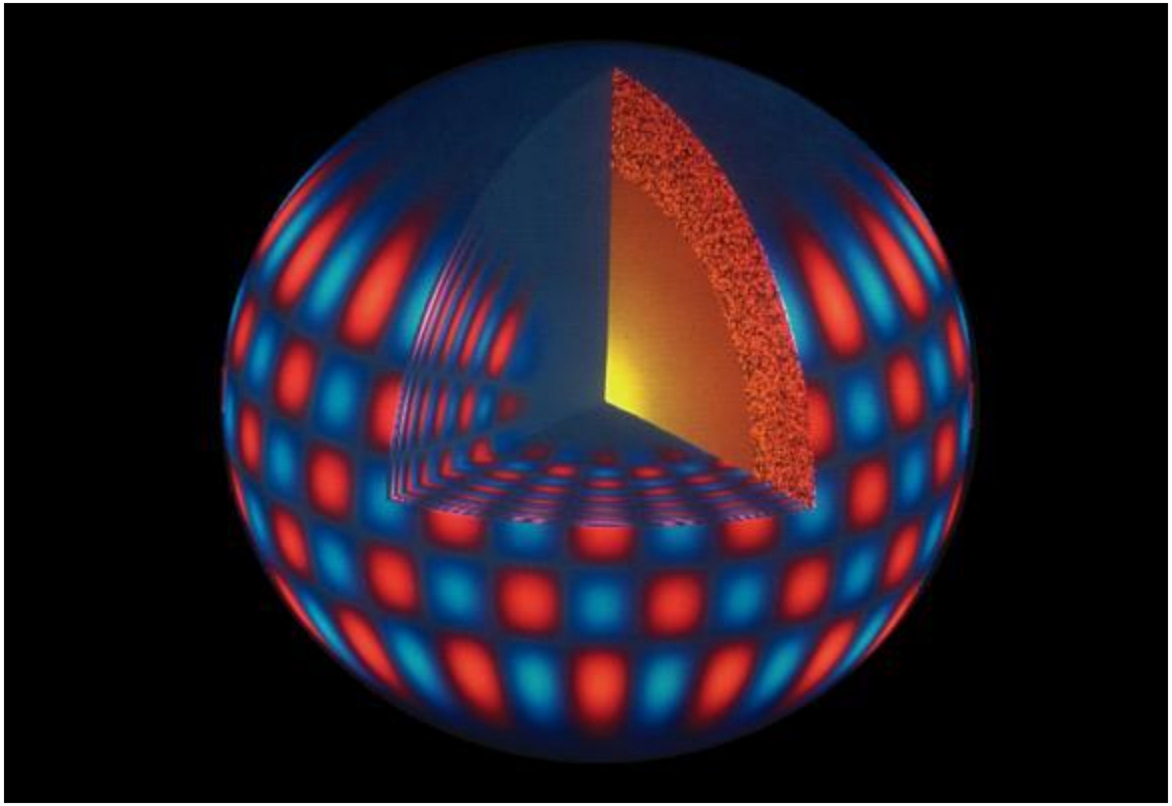
(2,2)

Notice, that in two dimensions, there are two indices for the normal modes corresponding to the two dimensions. This is just a result of solving the two-dimensional differential equation with boundary conditions.

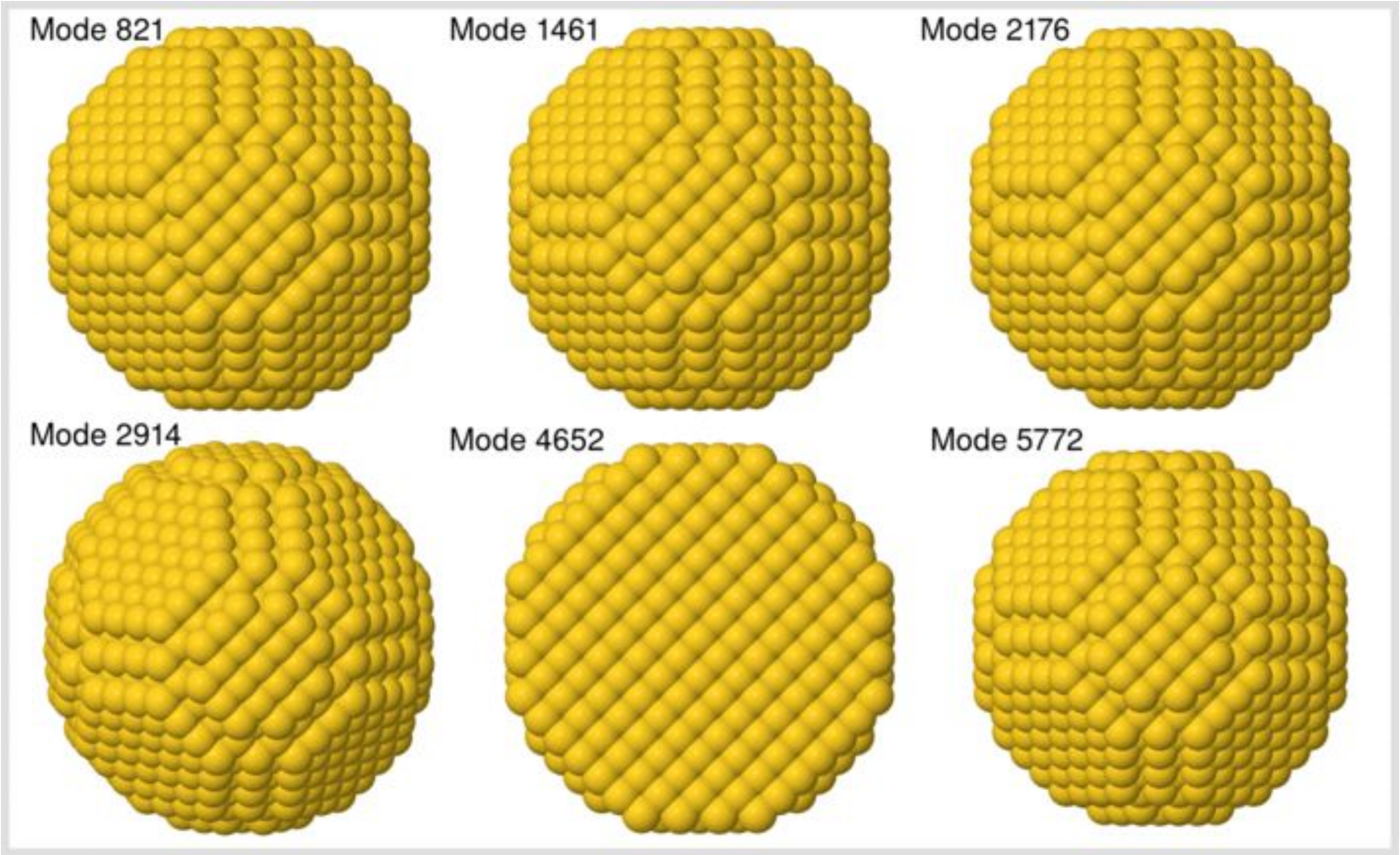
Normal Modes in 3-D

Normal Modes in 3-D

- For instance, the sun:



Gold Nanoparticles



Size and Shape Dependence of the Vibrational Spectrum and Low-Temperature Specific Heat of Au Nanoparticles

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Take-aways

- Normal modes correspond to resonance frequencies of a medium.
- In an unbounded medium, normal modes are continuous.
- In a bounded medium, normal modes are quantized!
- The number of quantum numbers of a system corresponds to the number of degrees of freedom.
- Any motion of the system can be expressed as a linear combination of the normal modes.