Lecture 8
(Wave Pulses and Fourier Transforms)

Physics 2310-01 Spring 2020

Douglas Fields

Wave Pulses

- So, if our solution to the wave equation is:

$$y(x,t) = A\cos(kx - \omega t)$$

- How do we get a wave pulse???

$$y_1(x,t) = \sin(kx - \omega t)$$

$$y_2(x,t) = \sin\big((k + dk)x - (\omega + d\omega)t\big)$$

- Using sinA + sinB = 2cos[(A-B)/2]sin[(A+B)/2]

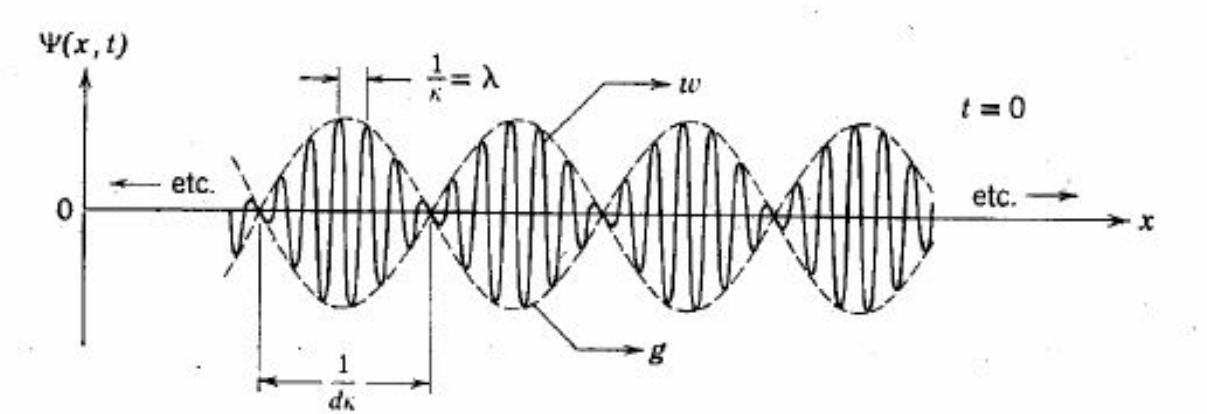
$$y(x,t) = 2\cos\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right)\sin\left(\frac{2k + dk}{2}x - \frac{2\omega + d\omega}{2}t\right)$$

$$= 2\cos\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right)\sin(kx - \omega t)$$

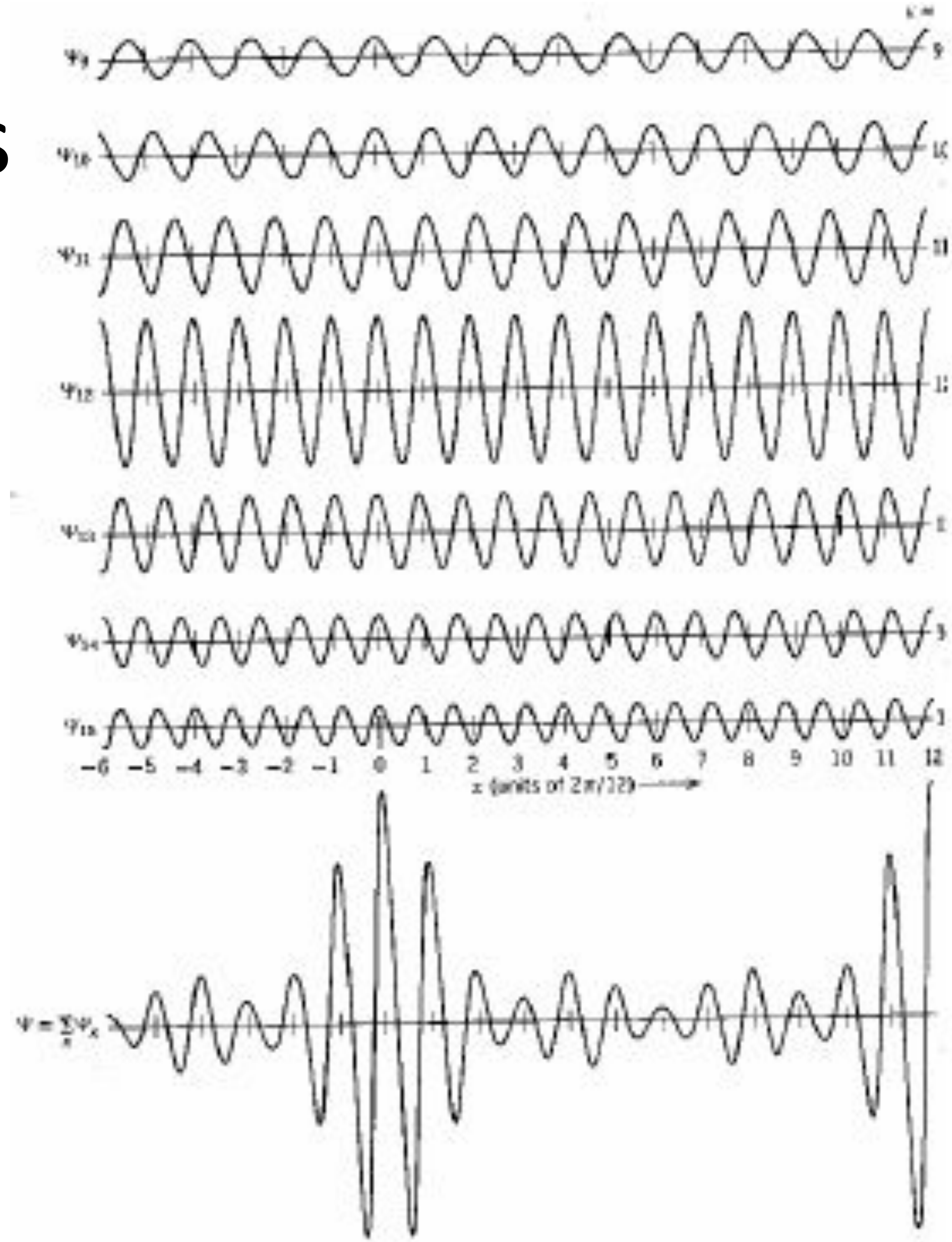
Since $dk \ll k$ and $d\omega \ll \omega$

Wave Pulses (Beats)

- $$y(x,t) = 2\cos\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right)\sin(kx - \omega t)$$



Wave Pulses



Bounded Problem

- For a bounded problem, there are discrete (quantized) normal modes. Let's say we have some mix of those normal modes:

$$\psi(x,t) = a_1\psi_1(x,t) + a_2\psi_2(x,t) + a_3\psi_3(x,t) + \ldots$$

- How do we find the values a_1, a_2, …?

Bounded Problem

- Let's multiply both sides by one of the normal mode wave functions:

$$\psi(x,t)\psi_n(x,t) = a_1\psi_1(x,t)\psi_n(x,t) + a_2\psi_2(x,t)\psi_n(x,t) + a_3\psi_3(x,t)\psi_n(x,t) + \ldots$$

- Now, let's integrate over all space (or time) on both sides of the equation:

$$\int_{B1}^{B2} \psi(x,t)\psi_n(x,t)\,dx = \int_{B1}^{B2} a_1\psi_1(x,t)\psi_n(x,t)\,dx + \int_{B1}^{B2} a_2\psi_2(x,t)\psi_n(x,t)\,dx + \ldots$$

$$= a_1\int_{B1}^{B2} \psi_1(x,t)\psi_n(x,t)\,dx + a_2\int_{B1}^{B2} \psi_2(x,t)\psi_n(x,t)\,dx + \ldots$$

- But the integrals on the right are either 0 (if n not equal to the index), or 1 (if it is). So,

$$a_n = \int_{-\infty}^{\infty} \psi(x,t)\psi_n(x,t)\,dx$$

http://www.falstad.com/loadedstring

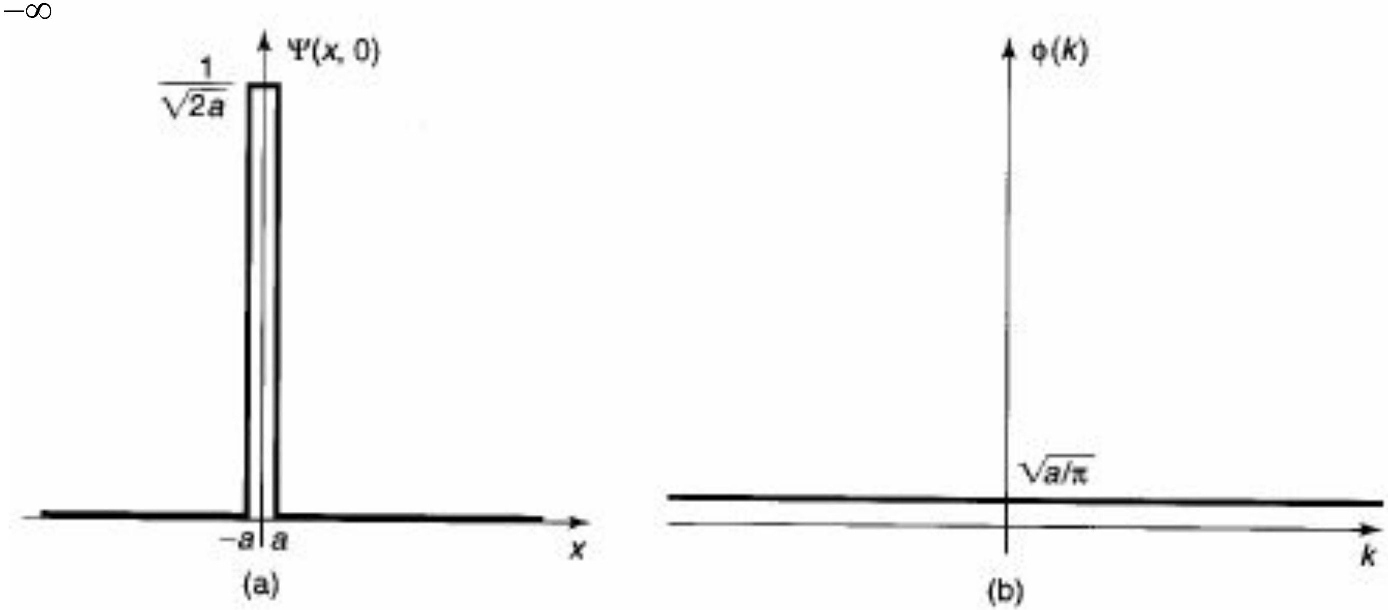
Wave Pulses

In the case of an **unbounded** problem, with continuous values of wave numbers:

$$\Psi(x,t) = \int_{-\infty}^{\infty} \varphi(k)\sin(kx - \omega t)\,dk$$

$$\varphi(k) = \int_{-\infty}^{\infty} \Psi(x,t)\sin(kx - \omega t)\,dx$$

k and x are conjugate variables

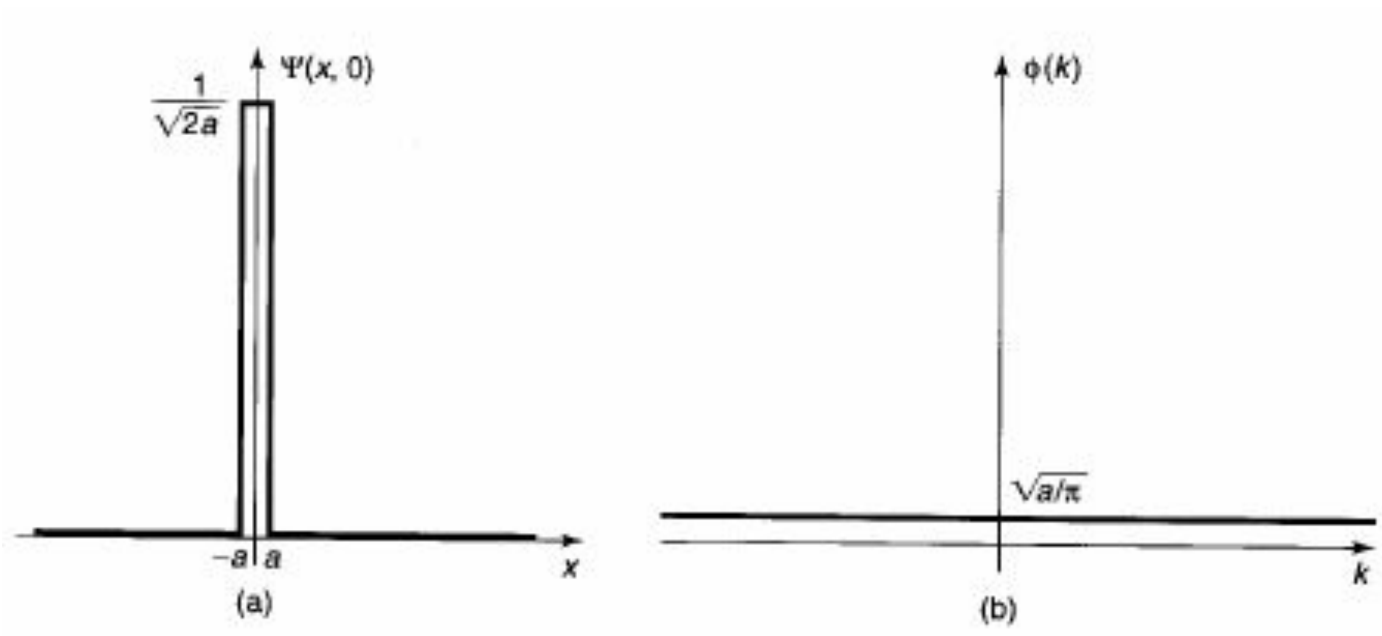


Wave Pulses

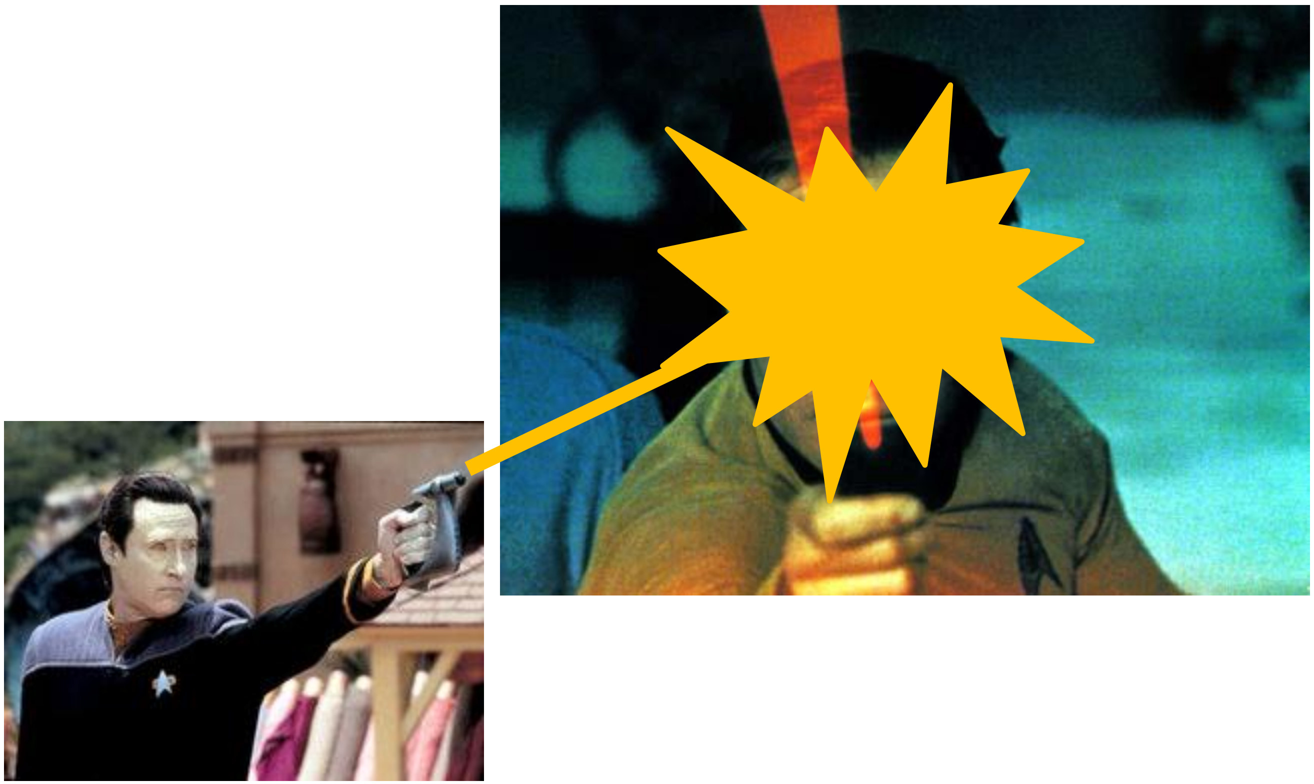
ω and t are conjugate variables

$$\Psi(x,t)=\int_{-\infty}^{\infty}\varphi(\omega)\sin(kx-\omega t)\,d\omega$$

$$\varphi(\omega)=\int_{-\infty}^{\infty}\Psi(x,t)\sin(kx-\omega t)\,dt$$

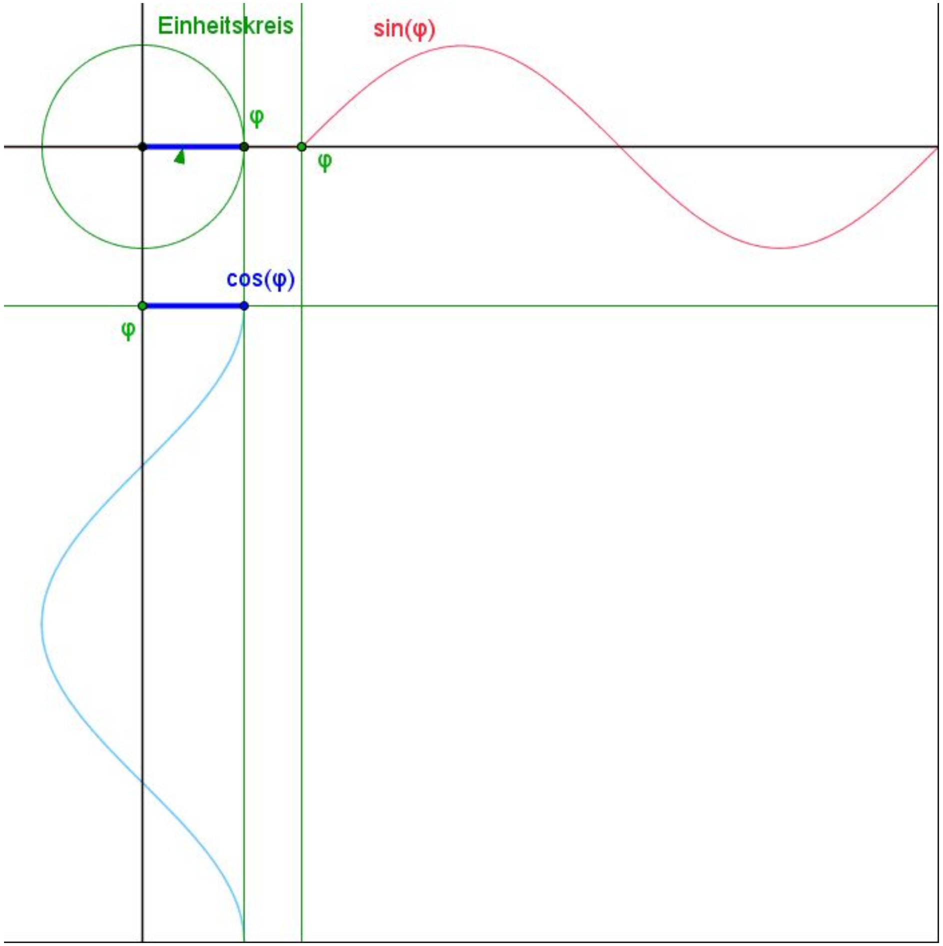


Phasors

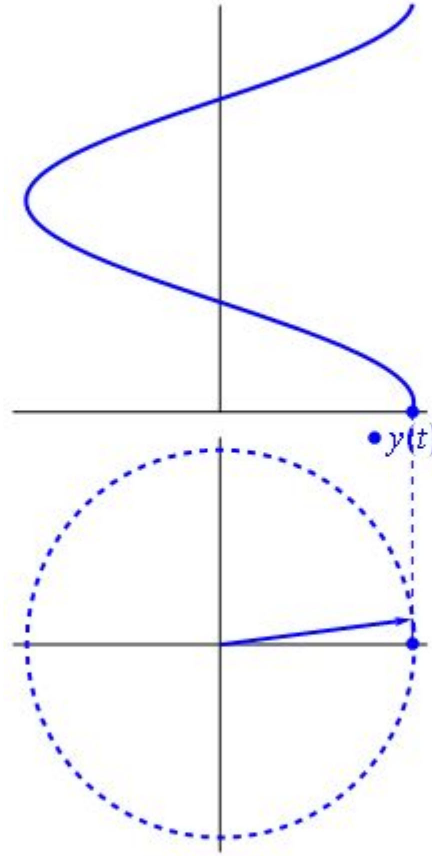


Phasors

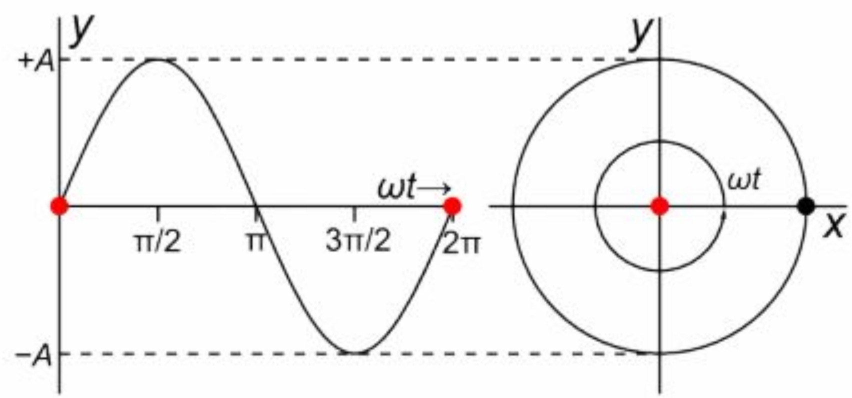
- One can represent sin(ωt) or cos(ωt) as a x- or y-projection of a phasor which rotates at an angular velocity ω.



Phasors

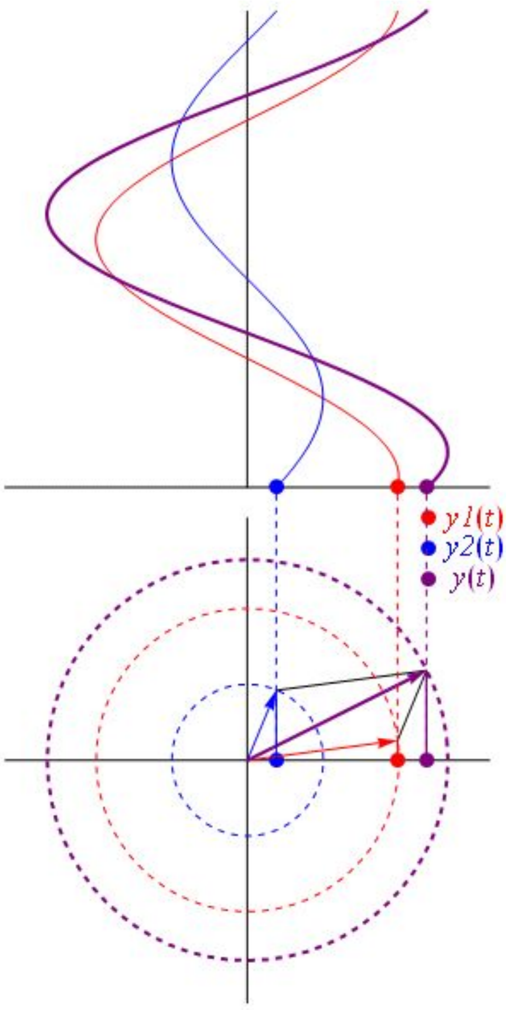


- Let's just focus on cos(ωt).
- The phasor is rotating with angular velocity ω, so the angle w.r.t. the x-axis is just ωt.
- Then, the x-component of the phasor is just cos(ωt).
- Or, the y-component of the phasor is just sin(ωt).
- Why is this useful?



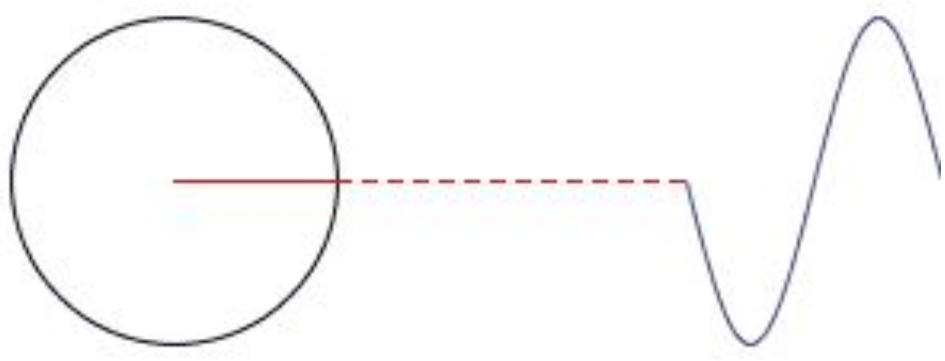
Phasors

- Let's add two waves, in this case, two waves with the same frequency, but with different relative phases.
- Note that the each wave can be represented by a phasor, and that the sum of the waves are represented by the vector sum of the phasors!

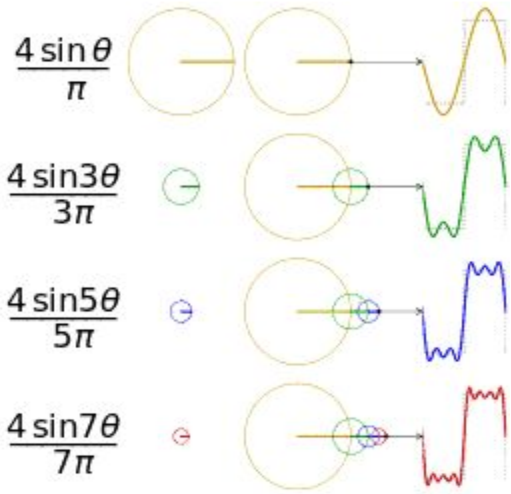


Phasors

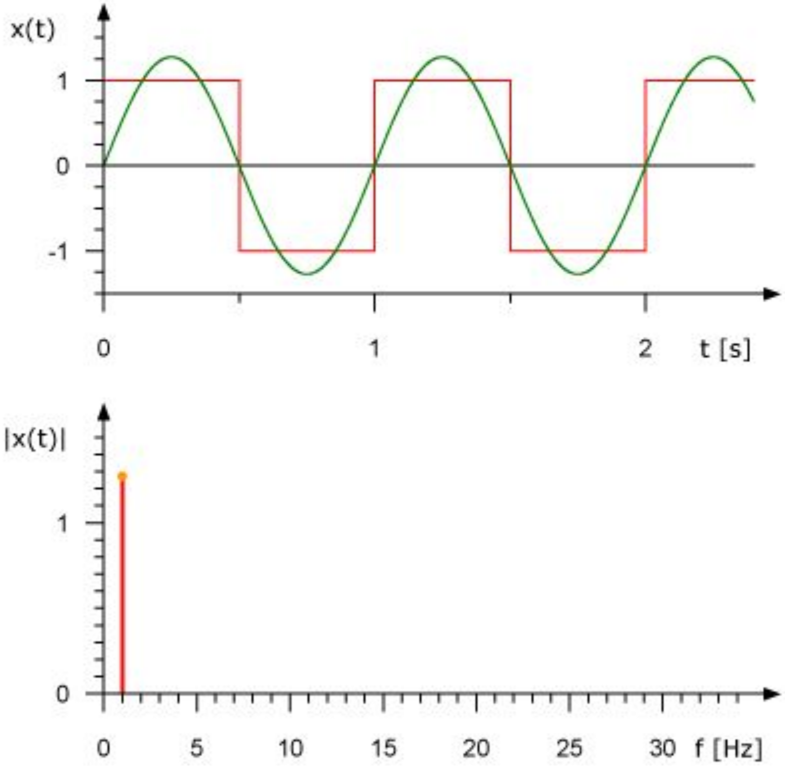
- But this still doesn't seem too useful…
- What about if the phasors rotate at different angular velocities?



Square Wave

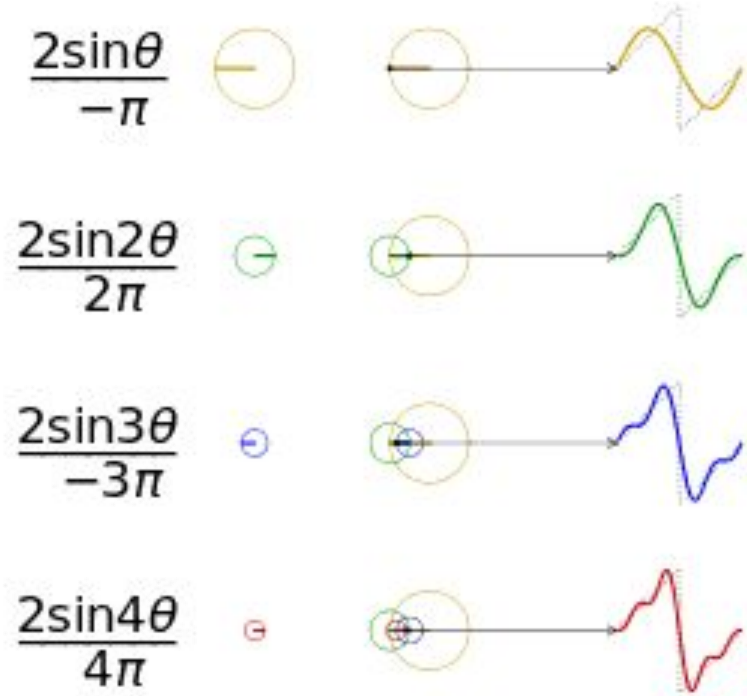


$\frac{4\sin\theta}{\pi}$
$\frac{4\sin 3\theta}{3\pi}$
$\frac{4\sin 5\theta}{5\pi}$
$\frac{4\sin 7\theta}{7\pi}$



x(t)
1
0
-1
0
1
2
t [s]
|x(t)|
1
0
0
5
10
15
20
25
30
f [Hz]

Saw-tooth Wave



- https://youtu.be/QVuU2YCwHjw
- https://www.youtube.com/watch?v=qS4H6PEcCCA
- And continue down the rabbit hole.