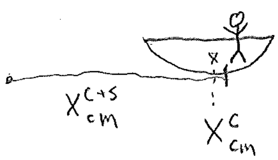
Physics 303 Fall 2022 Exam 2 NAME: Solutions
SHOW ALL WORK

1) A 70 kg Physics 303 student is standing in a 100 kg canoe, facing ~~away from~~ towards the shore, one meter farther from the shore than the canoe's center of mass. If the student now walks in the canoe towards the shore such that they are now standing one meter on the other side of the canoe's center of mass, how much closer are they to the shore? Assume that there is no resistance between the canoe and the water.



$$X^{C+P}_{cm\,\text{before}} = \frac{m_p x_p + m_c x_c}{m_p + m_c} = \frac{m_p x_p + m_c(x_p - 1)}{m_p + m_c} = \frac{(m_p + m_c)x_p - m_c}{m_p + m_c}$$

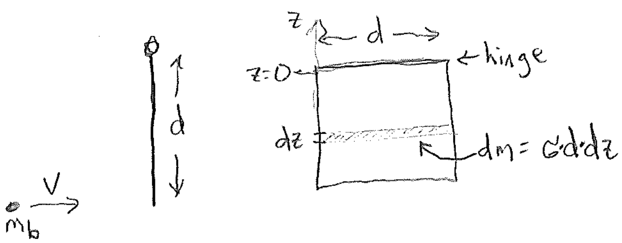
$$\parallel$$

$$X^{C+P}_{cm\,\text{after}} = \frac{m_p x_p' + m_c(x_p' + 1)}{m_p + m_c} = \frac{(m_p + m_c)x_p' + m_c}{m_p + m_c}$$

$$\therefore\ x_p - \frac{m_c}{m_p + m_c} = x_p' + \frac{m_c}{m_p + m_c}$$

$$\therefore\ \Delta x_p = x_p' - x_p = \frac{-2m_c}{m_p + m_c} = \frac{-2\cdot 100\,kg}{170\,kg} = -1.18\,m$$

2) A ball of putty, mass m_b, is shot horizontally with velocity v at a rectangular plate of uniform density and total mass m_p, with sides of length d. The plate is hinged at the top and hangs vertically initially with its flat surface perpendicular to the ball's velocity. The ball strikes the plate at the bottom edge and sticks to it. Through what angle will the plate rotate before swinging back down?



$$\Rightarrow I = \int z^2\,dm = \int_0^d z^2 \sigma \cdot d \cdot dz = \sigma \cdot d \cdot \frac{z^3}{3}\Big|_0^d = \sigma\frac{d^4}{3}$$

$$\text{but } \sigma = \frac{M_p}{A} = \frac{m_p}{d^2} \quad \therefore\ I = m_p\frac{d^2}{3}$$

Since angular momentum about the axis is conserved (but not linear momentum!),

$$L_i = m_b v d = L_f = m_b d^2\omega + I\omega,$$ ω = angular velocity of plate immediately after collision

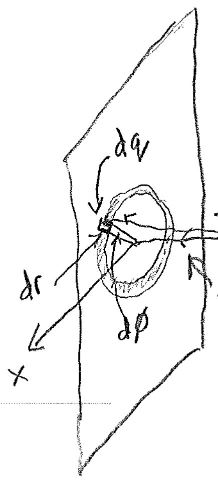
$$\therefore\ \omega = \frac{m_b v d}{m_b d^2 + I_p} = \frac{m_b v d}{I_{tot}}$$

After the collision, energy is conserved:

$$E_i = \tfrac{1}{2} I_t\,\omega^2 = E_f = g m_b d(1-\cos\theta) + m_p g \tfrac{d}{2}(1-\cos\theta) \Rightarrow$$

$$\therefore\ \theta = \cos^{-1}\left[1 - \frac{\frac{1}{2}(m_b v d)^2}{\left(m_b d^2 + m_p\frac{d^2}{3}\right)\left(dg\left(m_b + \frac{m_p}{2}\right)\right)}\right]$$

4) An infinite plane of charge with areal density σ is situated in the x-y plane. Use Coulomb's Law to find the force on a charge q located a distance z from the plane, and determine the potential energy of the configuration.



$$dq = \sigma r\,d\phi\,dr \Rightarrow dF = \frac{kq\,dq}{(r^2+z^2)}$$

But, from symmetry, we only have the z-component;

$$dF_z = \frac{kq\,dq}{(r^2+z^2)}\cos\theta = \frac{kq\sigma r\,d\phi\,dr}{(r^2+z^2)}\frac{z}{(r^2+z^2)^{1/2}}$$

$$= kq\sigma z\frac{r\,d\phi\,dr}{(r^2+z^2)^{3/2}}$$

For the shaded ring, we integrate around ϕ (from 0 to 2π)

to get: $dF_z = 2\pi kq\sigma z\dfrac{r\,dr}{(r^2+z^2)^{3/2}}$

and now must integrate from $r=0$ to $r=\infty$;

$$F_z = 2\pi kq\sigma z\int_0^\infty \frac{r\,dr}{(r^2+z^2)^{3/2}} = -2\pi kq\sigma z\left.\frac{1}{(r^2+z^2)^{1/2}}\right|_0^\infty$$

$$= -2\pi kq\sigma z\left(0-\frac{1}{z}\right) = 2kq\pi\sigma$$

The potential is the negative of the work done moving from one point to another;

$$\Delta U = -\int_{z_i}^{z_f} 2kq\pi\sigma\,dz = -2\pi kq\sigma\,\Delta z$$

3) For the force $\mathbf{F} = k(y^3, x^2y, xyz)$, determine if it is conservative.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & x^2y & xyz \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(x^2y)\right) + \hat{j}\left(\frac{\partial}{\partial z}(y^3) - \frac{\partial}{\partial x}(xyz)\right)$$

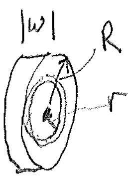
$$+ \hat{k}\left(\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(y^3)\right)$$

$$= \hat{i}(xz - 0) + \hat{j}(0 - yz) + \hat{k}(2xy - 3y^2)$$

$$\neq 0$$

$\therefore$ not conservative

5) Consider an Atwood's machine, with masses m_1 and m_2 connected by a string which hangs without slipping around a frictionless, cylindrical pulley of mass m_p with radius R. The masses are held stationary, and then released at time t=0. Using energy considerations, what is the angular velocity of the pulley at a later time, t?

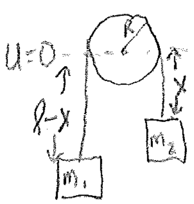


First, we must calculate the moment of inertia of the pulley:

the density of the pulley is $\frac{m_p}{V_p} = \frac{m_p}{\pi R^2 w} = \rho$. For a cylindrical shell of radius r, $dm = \rho 2\pi r w dr$ and,

$$dI = r^2 dm = \rho 2\pi r^3 w dr$$ and then integrate from 0 to R:

$$I = 2\pi\rho w \int_0^R r^3 dr = 2\pi\rho w \frac{R^4}{4} = 2\pi \frac{M_p}{\pi R^2 w} \cdot w \frac{R^4}{4} = M_p \frac{R^2}{2}$$



Now, write the energy of the machine as a function of the position of one of the masses:

$$E = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \frac{\dot{x}^2}{R^2} - m_2 g x - m_1 g(\ell - x)$$

$$= \left(\frac{1}{2} m_1 + \frac{1}{2} m_2 + \frac{1}{4} m_p\right)\dot{x}^2 + g(m_1 - m_2)x - m_1 g \ell$$

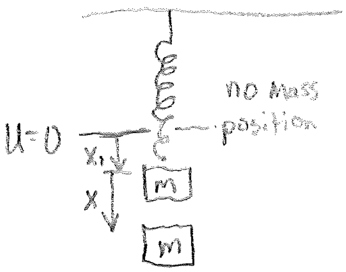
Since energy is conserved, $\frac{dE}{dt} = 0 \Rightarrow$

$$\left(m_1 + m_2 + \frac{1}{2} m_p\right)\cancel{\dot{x}}\ddot{x} + g(m_1 - m_2)\cancel{\dot{x}} = 0 \Rightarrow$$

$$\ddot{x} = \frac{g(m_1 - m_2)}{(m_1 + m_2 + \frac{1}{2} m_p)} \Rightarrow$$

$$\omega = \frac{\dot{x}}{R} = \frac{g}{R}\frac{(m_1 - m_2)}{(m_1 + m_2 + \frac{1}{2} m_p)} t$$

6) A mass m, is hanging from a spring with spring constant k. Use energy considerations to determine its equation of motion.



$$E_{eq} = \frac{1}{2}kx_1^2 - mgx_1$$

$$E = \frac{1}{2}k(x_1 + x)^2 - mg(x_1 + x) + \frac{1}{2}m\dot{x}^2$$

then $$\dot{E} = k(x_1 + x)\dot{x} - mg\dot{x} + m\dot{x}\ddot{x} = 0$$

but $mg = kx_1$ (from equillibrium position) =>

$$kx = m\ddot{x} \Rightarrow$$

$$x = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$$