Physics 303 Fall 2022 Exam 3 NAME: Solutions

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1) Consider a rocket (initial mass m_0) accelerating from rest in free space (no other forces). For what value of m is the momentum of the rocket, p, maximum?

$$V = V_{ex} \ln\left(\frac{m_0}{m}\right) \therefore p = mV = m V_{ex} \ln\left(\frac{m_0}{m}\right)$$

to find max p:

$$\frac{dp}{dt} = V_{ex}\left(\dot{m} \ln\left(\frac{m_0}{m}\right) - \cancel{m}\frac{\dot{m}}{\cancel{m}}\right)$$

$$= \dot{m} V_{ex}\left(\ln\left(\frac{m_0}{m}\right) - 1\right) = 0 \text{ for max}$$

$$\therefore \ln\left(\frac{m_0}{m}\right) = 1 \Rightarrow \frac{m_0}{m} = e \Rightarrow ($$

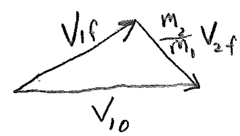
$$\boxed{m = \frac{m_0}{e}}$$

2) A mass, m_1, with initial velocity v_0 collides elastically with a stationary mass, m_2. What is the angle between the two recoiling masses after collision?

mom. cons: $\vec{P}_{1i} + \cancel{\vec{P}_{2i}}^{\,0} = \vec{P}_{1f} + \vec{P}_{2f} \Rightarrow \vec{P}_{2f} = \vec{P}_{10} - \vec{P}_{1f} \Rightarrow$

$$m_2 \vec{V}_{2f} = m_1 \vec{V}_{10} - m_1 \vec{V}_{1f} \Rightarrow$$

$$\vec{V}_{10} = \frac{m_2}{m_1}\vec{V}_{2f} - \vec{V}_{1f} \quad \therefore$$



From law of cosines:

$$V_{10}^2 = V_{1f}^2 + \left(\frac{m_2}{m_1}\right)^2 V_{2f}^2 - \frac{2m_2}{m_1} V_{1f}V_{2f}\cos\theta \qquad (1)$$

Energy cons: $\frac{1}{2}m_1V_{10}^2 = \frac{1}{2}m_1V_{1f}^2 + \frac{1}{2}m_2V_{2f}^2 \Rightarrow$

$$V_{10}^2 = V_{1f}^2 + \frac{m_2}{m_1}V_{2f}^2 \qquad (2)$$

Subtract (2) from (1):

$$0 = \left(\left(\frac{m_2}{m_1}\right)^{\cancel{2}} - \frac{m_{\cancel{2}}}{\cancel{m_1}}\right)V_{2f}^{\cancel{2}} - 2\frac{m_{\cancel{2}}}{\cancel{m_1}}V_{1f}\cancel{V_{2f}}\cos\theta$$

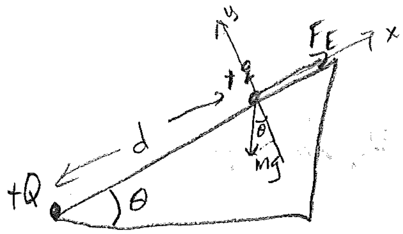
$$\therefore \boxed{\cos\theta = \frac{V_{2f}}{2V_{1f}}\left(\frac{m_2}{m_1} - 1\right)}$$

Note that if $m_2 = m_1$, the angle is $\frac{\pi}{2}$

if $m_2 \gg m_1$, then $V_{2f} \ll V_{1f}$ $(\cos\theta < 1)$

if $m_1 \gg m_2$, then $V_{1f} \approx V_{10}$ and so $V_{2f} \approx 0$ (from eq. (2))

4) A point mass with charge +q is on a frictionless inclined plane (angle θ with the horizontal). At the bottom of the incline, there is another point charge (fixed in place) with charge +Q. Find the equilibrium point for the mass and the frequency of small oscillations about this point.



At equilibrium,

$$mg\sin\theta = \frac{kqQ}{d_0^2} \Rightarrow$$

$$d_0 = \left[\frac{kqQ}{mg\sin\theta}\right]^{1/2}$$

$$U = mgh + \frac{kqQ}{d}$$

For small deviations from equil: $d = d_0 + x$, x small

$$h = d\sin\theta = (d_0 + x)\sin\theta$$

$$\therefore U = mg(d_0 + x)\sin\theta + \frac{kqQ}{(d_0 + x)}$$

Now $$(d_0 + x)^{-1} = \frac{1}{d_0}\left(1 + \frac{x}{d_0}\right)^{-1}$$

$$\approx \frac{1}{d_0}\left(1 - \frac{x}{d_0} + \frac{x^2}{d_0^2} - \dots\right)$$

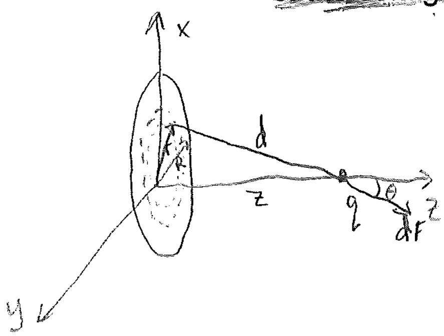
$$\therefore U = \underbrace{mgd_0}_{\text{const.}} + mgx\sin\theta + \underbrace{\frac{kqQ}{d_0}}_{\text{const}} - \frac{kqQx}{d_0^2} + \frac{kqQ}{d_0^3}x^2$$

= from equil. equat.

$$\therefore U = \text{const} + \frac{kqQ}{d_0^3}x^2 = \text{const.} + \frac{1}{2}\underbrace{\left[\frac{2(mg\sin\theta)^{3/2}}{(kqQ)^{1/2}}\right]}_{=k'}x^2$$

and $$\omega = \sqrt{\frac{k}{m}}$$

3) A thin, circular nonconducting disk of radius R has charge Q uniformly distributed on it, and is situated in the x-y plane. Find the force on a charge q located a distance z from, and on the axis of the disk, ~~and determine the potential energy of the configuration~~.



Look at a ring of charge at radius r:

$$d\vec{F} = \frac{kq\sigma(2\pi r)dr}{d^2}\cos\theta\,\hat{z} \text{ from symmetry}$$

$$= 2\pi kq\sigma \frac{z\,r\,dr}{(r^2+z^2)^{3/2}}$$

so that $$\vec{F} = 2\pi kq\sigma z\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}\hat{z} = -2\pi kq\sigma z \left.\frac{1}{(r^2+z^2)^{1/2}}\right|_0^R \hat{z}$$

$$= -2\pi kq\sigma z\left[\frac{1}{(R^2+z^2)^{1/2}} - \frac{1}{z}\right]\hat{z}$$

$$= 2\pi kq\sigma z\left[\frac{1}{z} - \frac{1}{(R^2+z^2)^{1/2}}\right]\hat{z}$$

$$= 2\pi kq\sigma\left[1 - \frac{1}{\left(\left(\frac{R}{z}\right)^2+1\right)^{1/2}}\right]\hat{z}$$

and $\sigma = \frac{Q}{\pi R^2} \Rightarrow \vec{F} = \frac{2kqQ}{R^2}\left[1 - \frac{1}{\left(\left(\frac{R}{z}\right)^2+1\right)^{1/2}}\right]\hat{z}$