Separation of Variables

We now have an equation that provides us with a means to get the wave functions, which, in turn, provide us with the means to extract the dynamic quantities of interest. Remember, that Schrödinger's equation is in quantum mechanics what F = ma is in classical mechanics. If you solve Newton's first law, knowing the potential acting on a particle, you can get a description of the behavior of that particle, x(t). So, how do we extract Ψ from Schrödinger's equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t)?$$

We notice first that this equation is a partial differential equation, consisting of terms with derivatives in time, t and position, x. The standard method of solving such an equation is the method of separation of variables in which we search for a solution for Ψ(x,t) that is a product of two functions, each of which is a function of only one variable:

$$\Psi(x,t) = \psi(x)\varphi(t)$$

putting this into Schrödinger's equation yields:

$$i\hbar\frac{d\varphi(t)}{dt}\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}\varphi(t) + V(x,t)\psi(x)\varphi(t)$$

and dividing by Ψ:

$$i\hbar\frac{d\varphi(t)}{dt}\frac{1}{\varphi(t)} = -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}\frac{1}{\psi(x)} + V(x,t)$$

Now, on the left hand side, we have only functions of time, and on the right hand side, we have only functions of position – except for the potential V! To overcome this obstacle, let's do what physicists do best and only consider potentials that do not explicitly depend on time, i.e., V(x,t) = V(x).

$$i\hbar\frac{d\varphi(t)}{dt}\frac{1}{\varphi(t)} = -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}\frac{1}{\psi(x)} + V(x)$$

OK, now that each side depends only on one variable, and these are independent, then it must be true that each side equals a constant:

$$i\hbar \frac{d\varphi(t)}{dt} \frac{1}{\varphi(t)} = E$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} \frac{1}{\psi(x)} + V(x) = E$$

I label the constant E with some feeling for the individual operators that we used for the development of the Schrödinger's equation. Now we have two ordinary differential equations that we can more easily solve:

$$i\hbar \frac{d\varphi(t)}{dt} = E\varphi(t)$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

The second of these two equations is known as the time-independent Schrödinger's equation. Before we attack its solutions, let's look at the solutions to the time-dependent part:

$$i\hbar \frac{d\varphi(t)}{dt} = E\varphi(t) \Rightarrow$$

$$\frac{d\varphi(t)}{\varphi(t)} = \frac{-iE}{\hbar} dt \Rightarrow$$

$$\int \frac{d\varphi(t)}{\varphi(t)} = \int \frac{-iE}{\hbar} dt \Rightarrow$$

$$\log \varphi(t) = \frac{-iE}{\hbar} t + C \Rightarrow$$

$$\varphi(t) = e^{\frac{-iE}{\hbar}t+C} = e^{\frac{-iE}{\hbar}t} e^C = Ce^{\frac{-iE}{\hbar}t}$$

Since we are eventually looking for solutions to $\Psi = \varphi\psi$, we will incorporate the constant C into ψ and give the solution φ as:

$$\varphi(t) = e^{\frac{-iE}{\hbar}t}$$

so that the general solution to the time-dependent wave equation is:

$$\Psi(x, t) = \psi(x)\varphi(t) = \psi(x)e^{\frac{-iE}{\hbar}t}$$

Homework: Show that E must exceed the minimum value of V in order that there are normalizable solutions Ψ. *Hint:* Consider the relative sign of Ψ and its second derivative, and the impact this has on its ability to be normalized.