Physics 405

Calculational Details for Fourier Series

I. We presume that we are given some function, $f(w)$, defined over some finite portion of the real numbers, which we decide to normalize so that the region of interest is between $-\pi$ and $+\pi$. (We will discuss how to effect, or change, that normalization below; it is basically just a re-scaling of the independent variable. If, however, the function is defined over an infinite region, such as the entire real line, then it will be necessary for the function to be periodic, with some constant period, in order for this approach to work. Otherwise, one would have to use Fourier integrals, which are not being discussed at this time.)

The Fourier series expansion for $f(w)$ utilizes the fact that the two standard trigonometric functions, sine and cosine, form a complete, orthonormal set over this region:

orthonormal:

$$\int_{-\pi}^{+\pi} dw\ \sin(nw)\sin(mw) = \pi\delta_{nm} = \int_{-\pi}^{+\pi} dw\ \cos(nw)\cos(mw)\ ,$$

$$\int_{-\pi}^{+\pi} dw\ \sin(nw)\cos(mw) = 0\ .$$

The description **complete** means that there exist two infinite sets of constants, $\{a_n \mid n = 0, 1, 2, \ldots\}$ and $\{b_n \mid n = 0, 1, 2, \ldots\}$, such that the function $f(w)$ may be written out as the following sum

$$f(w) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\ \cos(nw) + b_n \sin(nw)]\ .$$

For sufficiently "nice" functions, this is actually an equality, except possibly at the endpoints of the interval. Certainly continuous functions with period 2π qualify as the nicest sort of functions here. More generally, it may happen that there will be some finite number of points where the equality is not true, but only a finite number of specific points; this occurs when the function is only piecewise continuous, for example.

The orthogonality relationships above allow us to determine the coefficients desired:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} dt\, f(t)\cos(nt)\ , \quad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} dt\, f(t)\sin(nt)\ .$$

II. If the function in question is an even function over this range, then it is easy to see that the integral defining b_n will vanish (since $\sin(nt)$ is an odd function). Therefore, only the cosine functions are needed in the expansion. On the other hand, if the function is odd, then the integral defining a_n will vanish, so that only the sine functions are needed.

These two special cases are straightforward, and will take care of themselves. However, if the function is really only relevant (to our physics problem) between, say, 0 and $+\pi$, then we may extend its definition into the entire region $-\pi$ to $+\pi$ so that it is either even or odd, which would mean that we really only need the cosines or the sines, respectively. The decision as to which one wants, i.e., a cosine series or a sine series, will depend upon other boundary conditions in the given problem. In such a circumstance, when the function is clearly either even or odd, the integrals defining the constants may be reduced to an integration only between 0 and $+\pi$:

for functions only defined between 0 and $+\pi$:

$$\text{an odd extension:} \quad \begin{cases} a_n = 0 \ , \\ b_n = \frac{2}{\pi}\int_0^{+\pi} dt\, f(t)\sin(nt) \ . \end{cases}$$

$$\text{an even extension:} \quad \begin{cases} a_n = \frac{2}{\pi}\int_0^{+\pi} dt\, f(t)\cos(nt) \ , \\ b_n = 0 \ . \end{cases}$$

We repeat that which extension one needs, or wants, will depend upon the other boundary conditions set on the problem at the endpoints. Do note that if one chooses sine functions only then that means that the function is odd over the entire region. Consider, then, a function constant over 0 to π. An odd extension would mean that it is the negative of that constant from $-\pi$ to 0, so that it is clearly **not** continuous at the origin. In general when a function has a discontinuity the Fourier series attempts to come to the average value of the two limits. (In the particular case, above, of a constant function with an odd extension, that average value would be zero.)

III. The above discussion is given for functions, of w, defined for $w \in [-\pi, +\pi]$. If, instead, we have functions defined either between $-a$ and $+a$ or between 0 and $+a$, then we re-normalize the variables by setting new variables (y instead of w) as follows:

$$w \equiv \frac{\pi}{a} y \ .$$

This gives equivalent formulae to the ones above, where π has basically been replaced everywhere by a, and the arguments of the trigonometric functions go from nw to $n\pi(y/a)$. The resulting equations are then appropriate for some function, say $g(y)$, that is periodic for $y \in [-a, a]$:

orthonormal:

$$\int_{-a}^{+a} dw\ \sin(n\pi y/a)\sin(m\pi y/a) = a\delta_{nm} = \int_{-a}^{+a} dw\ \cos(n\pi y/a)\cos(m\pi y/a)\ ,$$

$$\int_{-a}^{+a} dw\ \sin(n\pi y/a)\cos(m\pi y/a) = 0\ ,$$

along with the equations for the coefficients:

$$g(y) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\ \cos(n\pi y/a) + b_n \sin(n\pi y/a)]$$

$$a_n = \frac{1}{a}\int_{-a}^{+a} dt\, g(t)\cos(n\pi t/a)\ , \quad b_n = \frac{1}{a}\int_{-a}^{+a} dt\, g(t)\sin(n\pi t/a)\ .$$

We may then also reconsider the case considered above, where the functions are actually only relevant to the physics problem between 0 and $+a$. We renormalize the variable as described just above, and also perform an extension of the definition of the function into the full region of $[-a, +a]$ by making it either odd or even across the boundary, depending on whether we want to utilize sine series or cosine series. Then the equations above take on the re-normalized format as follows:

for functions only defined between 0 and +a:

$$\text{an odd extension:} \quad \begin{cases} a_n = 0\ , \\ b_n = \frac{2}{a}\int_0^a dy\, g(y)\sin(n\pi y/a)\ . \end{cases}$$

$$\text{an even extension:} \quad \begin{cases} a_n = \frac{2}{a}\int_0^a dt\, g(y)\cos(n\pi y/a)\ , \\ b_n = 0\ . \end{cases}$$

Notice that the equation given for b_n in the case where only sine-series are appropriate is now exactly the same as the one given in the text.

IV. Lastly, let us consider the general case, for $w \in [-\pi, +\pi]$, but re-write the trigonometric functions in terms of exponential functions, using the standard formulae:

$$2\cos(nw) = e^{inw} + e^{-inw}\ , \quad 2i\sin(nw) = e^{inw} - e^{-inw}\ .$$

This allows us to re-consider the arbitrary, "nice" function, periodic in $[-\pi, +\pi]$, in terms of an equivalent expansion of a different set of orthogonal, complete functions, the exponentials. We express quickly both the orthogonality and the completeness as follows—which may be easily derived from the ones above using the de Moivre relationships given just above:

$$\int_{-\pi}^{+\pi} dw\, e^{inw} e^{-imw} = 2\pi\, \delta_{nm} \; ,$$

$$f(w) = \sum_{n=-\infty}^{+\infty} c_n\, e^{inw} \; ,$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dt\, f(t) e^{-int} \; .$$

The relationship between these coefficients and the a_n and b_n used for the trigonometric versions may be shown to be

$$c_0 = a_0/2 \; , \qquad c_n = \frac{1}{2}\,(a_n - i b_n) \; .$$

V. It is also useful to re-write the completeness relationship for this exponential version, which takes a very simple form:

$$\int_{-\pi}^{+\pi} dt\, f(t)\, \delta(w-t) = f(w) = \sum_{n=-\infty}^{+\infty} c_n\, e^{inw} = \int_{-\pi}^{+\pi} dt\, f(t)\, \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in(w-t)} \; ,$$

$$\Longrightarrow \qquad \sum_{n=-\infty}^{+\infty} e^{in(w-t)} = 2\pi\, \delta(w-t) \; , \qquad w, t \in [-\pi, +\pi] \; .$$

To relate this useful relationship to the ones expected from the earlier, trigonometric series, we invert de Moivre's relationship, $e^{inw} = \cos(nw) + i\sin(nw)$. Then, because the sine function is odd, the imaginary part of both sides of this equation is just $0 = 0$. But, since the Dirac delta is real-valued, and the real part of the exponential is the cosine, the real part of both sides gives the following, which one could have obtained earlier, in §I:

$$\delta(w-t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in(w-t)} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \cos[n(w-t)] = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{+\infty} \cos[n(w-t)] \right]$$

$$= \frac{1}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{+\infty} [\cos(nw)\cos(nt) + \sin(nw)\sin(nt)] \right\}$$