Physics 405

Notes: Forces and Torques between two dipoles

We consider two point dipoles with moments $\vec{p}_1$ and $\vec{p}_2$, a distance $\vec{r}$ apart. As a reasonable special case—to see the general forms involved—we take $\vec{p}_1$ perpendicular to $\vec{r}$ and $\vec{p}_2$ parallel to $\vec{r}$. We want to find for each of the two dipoles the force, and the torque, exerted on it by the other.

To begin, we first choose the $\hat{z}$-axis parallel to $\vec{p}_1$ and choose $\hat{x}$ parallel to $\vec{r}$, as well of course parallel to $\vec{p}_2$. This implies that the third Cartesian axis vector, $\hat{y}$ points into the page. We recall that the general formula for the electric field of an electric dipole is given by the last formula in Griffiths in Chapter 3:

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{k}{r^3}\Big[3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}\Big] \\ &\xrightarrow[\hat{p}=\hat{z}]{} \frac{kp}{r^3}\left[2\cos\theta\;\hat{r}+\sin\theta\;\hat{\theta}\right] .\end{aligned}$$

It is useful to also write out the general form for $\nabla\vec{E}(\vec{r})$, at least when $\hat{p}=\hat{z}$, as simplified above, although it is not quite essential. I will first write out the calculation for the general case and then specialize it for the case when $\vec{p}\propto\hat{z}$ as we have decided:

$$\begin{aligned}\nabla\vec{E}(\vec{r}) &= \frac{3k}{r^4}\Big[\hat{r}\vec{p}+\vec{p}\hat{r}+(\vec{p}\cdot\hat{r})(\mathbf{I}-5\hat{r}\hat{r})\Big] \\ &\xrightarrow[\hat{p}=\hat{z}]{} -3\frac{kp}{r^4}\Big[\cos\theta\,(3\hat{r}\hat{r}-\mathbf{I})+\sin\theta\,(\hat{\theta}\hat{r}+\hat{r}\hat{\theta})\Big] .\end{aligned}$$

However, as part of the purpose of this set of notes is to help you understand how to calculate such tensor-valued gradients of vectors, let us put in the complete derivation, assuming only that you know the form for the gradient of the location vector:

$$\nabla\vec{r}=\mathbf{I}\ , \quad \text{i.e., in components} \quad \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\begin{pmatrix}x & y & z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} ,$$

where the content of the second equation is that we have taken the actual vectors in the first equation and expressed them, via matrix notation, in terms of a matrix which contains as entries the coefficients of those vectors with respect to the standard Cartesian basis set, i.e., $\{\hat{x},\hat{y},\hat{z}\}$.

We **also** need the simpler expression, just from the definition of the gradient in spherical coordinates:

$$\nabla r=\hat{r}\ .$$

With these two notions as understood, we may now calculate a useful expression:

$$\nabla\hat{r}=\nabla\left(\frac{\vec{r}}{r}\right)=\frac{\mathbf{I}}{r}-\vec{r}\frac{1}{r^2}\hat{r}=\frac{\mathbf{I}-\hat{r}\hat{r}}{r}\ .$$

At this point we are able to determine the gradient of our given electric field, remembering, as we do the calculation to keep very good care of the order of vectors in introduced. This means, for instance, that since we are calculating $\nabla\vec{E}$, those vectors generated by ∇ acting on a scalar must be positioned to the left of those (other) vectors in $\vec{E}$, since this is the order in which they come in the expression desired:

$$\begin{aligned}
\nabla\vec{E} = k\nabla\left\{\frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{r^3}\right\} &= -3k\frac{\hat{r}}{r^4}\left[3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}\right] + \frac{k}{r^3}\left[3\vec{p}\cdot\hat{r}\left(\frac{\mathbf{I}-\hat{r}\hat{r}}{r}\right) + 3\left(\vec{p}\cdot\frac{\mathbf{I}-\hat{r}\hat{r}}{r}\right)\hat{r}\right] \\
&= -3\frac{k}{r^4}\left[3(\vec{p}\cdot\hat{r})\hat{r}\hat{r}-\hat{r}\vec{p}\right] + \frac{k}{r^4}\left\{3(\vec{p}\cdot\hat{r})(\mathbf{I}-\hat{r}\hat{r}) + 3[\vec{p}-(\vec{p}\cdot\hat{r})\hat{r}]\hat{r}\right\} \\
&= \frac{3k}{r^4}\left\{(\vec{p}\cdot\hat{r})\left[-3\hat{r}\hat{r}+\mathbf{I}-\hat{r}\hat{r}-\hat{r}\hat{r}\right]+\hat{r}\vec{p}+\vec{p}\hat{r}\right\} = \frac{3k}{r^4}\left[\hat{r}\vec{p}+\vec{p}\hat{r}+(\vec{p}\cdot\hat{r})(\mathbf{I}-5\hat{r}\hat{r})\right] .
\end{aligned}$$

(You can notice, interestingly, that $\nabla\vec{E}$, for a dipole, is a symmetric tensor; i.e., if you interchange the order of all the pairs of tensor products of vectors in the expression there is no change. When the components of the tensor are expressed in matrix form, this is the same as saying that the tensor is not changed when one takes its transpose, i.e., interchanges rows and columns.)

For this specific problem we have begun with specific choices of the orientations of the two dipoles; i.e., they are perpendicular to one another, and one is perpendicular to the vector between them while the other is parallel to that vector. Therefore, the various angles and directions in this problem are very straightforward, and there are now simplifications that we can make. I will describe those below also.

We begin by asking for the **force on 2 caused by 1**. We also choose the origin to be at the location of dipole number 1. It is good, then, to first calculate the electric field of dipole 1:

$$\vec{E}_1(\vec{r}) = \frac{kp_1}{r^3}[2\cos\theta\hat{r}+\sin\theta\hat{\theta}] .$$

For the force we need to use $\vec{p}_1 = p_1\hat{z}$, $\vec{p}_2 = p_2\hat{x}$ and the formula above for the gradient of the electric field, evaluated at our desired location, namely where $\theta = \pi/2$ and $\hat{r} = \hat{x}$. This gives us

$$\vec{F}\Big|_{\text{on 2 by 1}} = p_2\hat{x}\cdot\left\{3\frac{kp_1}{r^4}[\hat{x}\hat{z}+\hat{z}\hat{x}]\right\} = 3\frac{kp_1p_2}{r^4}\hat{z} = -3\frac{kp_1p_2}{r^4}\hat{\theta} ,$$

where we have re-introduced some spherical-coordinate basis vectors, since it is true that $\hat{\theta}$, on the equator, points directly "down." On the other hand, if one really didn't want to calculate the gradient of the vector field, we can surely notice that all that is needed is the derivative in the direction of $\vec{p}_2$, i.e., the x-derivative; therefore, we may also write:

$$\begin{aligned}
\vec{F}\Big|_{\text{on 2 by 1}} = (\vec{p}_2\cdot\nabla)\vec{E}_1(\vec{r})\Big|_{\vec{r}=x\hat{x}} &= kp_1p_2\left\{(\hat{x}\cdot\nabla)\frac{1}{r^3}[2\cos\theta\hat{r}+\sin\theta\hat{\theta}]\right\}\Big|_{r=x,\theta=\pi/2} \\
&= kp_1p_2\left\{\frac{\partial}{\partial x}\frac{1}{r^3}[2\cos\theta\hat{r}+\sin\theta\hat{\theta}]\right\}\Big|_{r=x,\theta=\pi/2} .
\end{aligned}$$

In principle we must perform the derivation before performing the evaluation; however, let's try looking at the problem in somewhat more detail instead. Since we are on the $\hat{x}$-axis, $\hat{\theta}$ points straight downward, i.e., in the direction $-\hat{z}$, while $\hat{r}$ points straight outward, in the direction $\hat{x}$. As the value of x changes slightly—relevant when asking for the derivative in the x-direction, neither of those quantities changes. In addition the value of $\theta = \pi/2$ does not change. The only thing which changes is the value of r which is equal to x. Therefore we may go ahead and evaluate first, giving us the following result, which is the same as the calculation above:

$$\vec{F}\Big|_{\text{on 2 by 1}} = kp_1p_2\Big\{\frac{\partial}{\partial x}\frac{1}{x^3}[-\hat{z}] = 3kp_1p_2\frac{\hat{z}}{x^4} = -3kp_1p_2\frac{\hat{\theta}}{r^4}\ , \ \text{“upward.”}$$

Next, we may compute the other force, i.e., **the force on 1 caused by 2**. The simplest approach is to try to re-use the calculation made above; therefore, we move the origin to the location of dipole number 2, and rotate the axes so that, now, $\hat{z}'$, i.e., the rotated $\hat{z}$-axis, points along $\vec{p}_2$. In that case the vector $\vec{r}$, now from dipole 2 to dipole 1, points in the $-\hat{z}'$ direction, while $\vec{p}_1 \propto -\hat{y}'$, while its location corresponds to $\theta = \pi$. This gives us

$$\vec{E}_2(\text{at } \vec{p}_1) = \vec{E}_2(-r\hat{z}') = 2\frac{kp_2}{r^4}\hat{z}'\ .$$

Likewise, at this point, I simply use the gradient of the field, already calculated in general above, noting that what we want is

$$\vec{F}\Big|_{\text{on 1 by 2}}(\text{at } \vec{p}_1) = p_1(-\hat{y}')\cdot\left(\frac{3kp_2}{r^4}\right)[3\hat{z}'\hat{z}' - \mathbf{I}] = 3\frac{kp_1p_2}{r^4}\hat{y}'\ .$$

Our last assignment is then to “un-do” the rotation that rotated the original axes to the “primed” ones. However, since all we want is to “un-do” the particular expression just derived, above, we can simply note that $\hat{p}_1 = \hat{z} = -\hat{y}'$. Therefore the answer above may be re-written in the original choice of axes in the form

$$\vec{F}\Big|_{\text{on 1 by 2}}(\text{at } \vec{p}_1) = -3\frac{kp_1p_2}{r^4}\hat{p}_1 = -3\frac{kp_1p_2}{r^4}\hat{z}\ .$$

We of course notice immediately that these two forces are of equal magnitude, but opposite direction, **which is what is required by Newton's Third Law.**

The next task is to calculate the torques on each of the dipoles generated by the forces just determined:

$$\vec{N}\Big|_{\text{on 2 by 1}} = \vec{p}_2 \times \vec{E}_1(\text{at } \vec{p}_2) = p_2\hat{r} \times \left(\frac{kp_1}{r^3}\right)\hat{\theta} = \frac{kp_1p_2}{r^3}\hat{\varphi} = \frac{kp_1p_2}{r^3}\hat{y}\ , \quad \text{into the page}\ ,$$

where, with our current choice of origin, at dipole 1, dipole 2 can be construed as at $\varphi = 0$, so that $\hat{\varphi}$ is into the page there, justifying our last equality, namely that at this point $\hat{\varphi} = \hat{y}$.

This calculation is very much easier because we did not need to worry about taking derivatives. Therefore, let us continue and calculate the other torque. As before we move the origin to dipole 2, but the calculation is simple enough that it doesn't seem necessary to rotate the axes. We simply note that the vector displacement from dipole 2 to dipole 1, which is $-\vec{r}$, is also just in the opposite direction to $\vec{p}_2$, so we can take the general form for the electric field, above, and replace $\hat{r}$ by $-\hat{p}$, which gives us $\vec{E}_2(\text{at } p_1) = -2kp_2(\hat{r}/r^3)$. Next the vector $\vec{p}_1$ is in the $-\hat{\theta}$ direction, so that we have

$$\vec{N}\Big|_{\text{on 1 by 2}} = \vec{p}_1 \times \vec{E}_2(\text{at } \vec{p}_1) = -p_1\hat{\theta} \times \left(\frac{2kp_2}{r^3}\right)(-\hat{r}) = -\frac{2kp_1p_2}{r^3}\hat{\varphi} = +\frac{2kp_1p_2}{r^3}\hat{y}\ , \text{into the page.}$$

In this calculation, $-\hat{\varphi}$, relative to the origin at dipole 2 is into the page, since dipole 1 is at $\varphi = \pi$; therefore $-\hat{\varphi}$ is into the page, as also is $\hat{y}$. [A different way of seeing this is to note that $-\hat{r}$, from our current origin at dipole 2, and over toward the location of dipole 1, is the same as $+\hat{x}$, while $-\hat{\theta}$ is the same as $+\hat{z}$, so that the cross product above is just $\hat{z} \times \hat{x} = \hat{y}$.]

We notice that neither the magnitudes, nor the directions, satisfy the notions of Newton's Third Law; i.e., that they should be of equal magnitude and opposite direction.

This is, however, not necessarily troublesome, since they have been computed relative to different origins. In order to compare them, we have to move at least one of them so that they are measured relative to the same origin. We decide to do this for $\vec{N}\Big|_{\text{on 2 by 1}}$, above, where the vector between the origins is then $r\hat{x}$. We use the force calculated above, to obtain the torque on dipole 2, caused by dipole 1, but measured from an origin at dipole 2 (the second choice above):

$$\vec{N}\Big|_{\text{on 2 by 1}} = \vec{p}_2 \times \vec{E}_1(\text{at } \vec{p}_2) + r\hat{x} \times \vec{F}\Big|_{\text{on 2 by 1}} = \frac{kp_1p_2}{r^3}\hat{y} + r\hat{x} \times \frac{3kp_1p_2}{r^4}\hat{z} = -\frac{2kp_1p_2}{r^3}\hat{y}\ .$$

We see that we are successful; now the two torques, when measured from the same origin, are of equal magnitude and opposite direction.
