

Physics 405

Some Thoughts on Multipole Expansions

I. Expansion of the Potential Outside the Source:

We begin with the notion of a charge density $\rho(\vec{r}')$, confined to a finite region of space. Being finite we may always conceive of it as being contained within a sphere, say of radius R . We are considering an expansion of the exact potential in powers of R/r , where $r \gg R$. We know that at any field point \vec{r} , we may write the electric potential in terms of an integral over that density:

$$V(\vec{r}) = k \int d\vec{r}' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \equiv k \left\{ \frac{M^{(0)}}{r} + \frac{M^{(1)}}{r^2} + \frac{M^{(2)}}{r^3} + \frac{M^{(3)}}{r^4} + \dots \right\} .$$

So we need to expand the integrand, the Green's function, in powers of r , which can be done with Legendre polynomials depending on the angle, γ , between the two vectors:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + (r')^2}} = \frac{1}{r} \frac{1}{\sqrt{1 - 2(r'/r) \cos \gamma + (r'/r)^2}} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos \gamma) ,$$

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi') .$$

On the other hand, the complicated form of $\cos \gamma$ makes the following a better approach:

$$\begin{aligned} \frac{1}{|\vec{r} - \vec{r}'|} &= \frac{1}{\sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + (r')^2}} = \frac{1}{r} \frac{1}{\sqrt{1 - 2(\vec{r} \cdot \vec{r}')/r^2 + (r'/r)^2}} \\ &= \frac{1}{r} - \frac{1}{2} [-2(\vec{r} \cdot \vec{r}')/r^2 + (r'/r)^2] + \frac{3}{8} [-2(\vec{r} \cdot \vec{r}')/r^2 + (r'/r)^2]^2 \\ &\quad - \frac{5}{16} [-2(\vec{r} \cdot \vec{r}')/r^2 + (r'/r)^2]^3 + \dots \\ &= \frac{1}{r} + \frac{1}{r^2} (\hat{r} \cdot \vec{r}') + \frac{1}{r^2} \frac{3(\hat{r} \cdot \vec{r}')^2 - r'^2}{2} + \frac{1}{r^3} \frac{5(\hat{r} \cdot \vec{r}')^3 - 3(r')^2 \hat{r} \cdot \vec{r}'}{2} + \dots \end{aligned}$$

We may insert this into our integral, and expand it out in some detail:

$$\begin{aligned} 4\pi\epsilon_0 V(\vec{r}) &= \frac{1}{r} \int d\vec{r}' \rho(\vec{r}') + \frac{1}{r^2} \hat{r} \cdot \int d\vec{r}' \vec{r}' \rho(\vec{r}') \\ &\quad + \frac{1}{r^3} \hat{r} \cdot \left\{ \int d\vec{r}' \frac{3\vec{r}'\vec{r}' - (r')^2 \mathbf{I}}{2} \rho(\vec{r}') \right\} \cdot \hat{r} \\ &\quad + \frac{1}{r^4} \sum_{i,j,k=1}^3 \hat{r}^i \hat{r}^j \hat{r}^k \int d\vec{r}' \frac{5(\vec{r}')^i (\vec{r}')^j (\vec{r}')^k - 3(r')^2 \delta^{ij} (\vec{r}')^k}{2} + \dots \\ &\equiv \frac{q}{r} + \frac{\hat{r} \cdot \vec{p}}{r^2} + \frac{\hat{r} \cdot \mathbf{Q} \cdot \hat{r}}{r^3} + \sum_{i,j,k} \frac{\hat{r}^i \hat{r}^j \hat{r}^k W^{ijk}}{r^4} + \dots \end{aligned}$$

Here the quantities q , \vec{p} , \mathbf{Q} , W^{ijk} , etc., are all integrals over the source of the charge, referred to as its "moments." Their function is to describe the symmetry properties of the charge distribution itself. It is customary to refer to the moments as 2^{ℓ} -moments, when they are multiplied by $r^{-(\ell+1)}$ in the expansion of the potential.

II. Explicit Properties of the lowest-order Moments

Beginning with the dipole moment, we may write it out in some detail

$$\vec{p} \equiv \int d\vec{r}' \vec{r}' \rho(\vec{r}') ,$$

$$((\vec{p}_i)) = \int d\vec{r}' \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rho(\vec{r}') = \int d\vec{r}' r' \begin{pmatrix} \sin \theta' \cos \varphi' \\ \sin \theta' \sin \varphi' \\ \cos \theta' \end{pmatrix} \rho(\vec{r}') .$$

Notice that if, for instance, ρ is independent of the azimuthal angle, then $p_x = 0 = p_y$. In that case the only one remaining is

$$p_z = \int d\vec{r}' z' \rho(\vec{r}') = 2\pi \int_0^R r'^3 r' \int_0^\pi \sin \theta' d\theta' \rho(r', \theta') P_1(\cos \theta') ,$$

which is then sometimes referred to simply as “the dipole moment.”

It is important to notice that when we have **only one** dipole under consideration, we can **always** arrange for independence of the azimuthal angle by orienting the \hat{z} -axis along the direction it points, which justifies the use of the name.

The corresponding potentials and fields are then easily written down:

$$V \Big|_{\text{pure p}} = \frac{\vec{p} \cdot \hat{r}}{r^2} ,$$

$$\vec{E} \Big|_{\text{pure p}} = -\nabla V \Big|_{\text{pure p}} = \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} .$$

Now let's continue onward and consider the quadrupole moment:

$$\mathbf{Q} \equiv \int d\vec{r}' \frac{3\vec{r}'\vec{r}' - (r')^2 \mathbf{I}}{2} \rho$$

$$((Q_{ij})) = \frac{1}{2} \int d\vec{r}' \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3(y')^2 - (r')^2 & 3y'z' \\ 3z'x' & 3z'y' & 3(z')^2 - (r')^2 \end{pmatrix} \rho(\vec{r}')$$

$$= \frac{1}{2} \int d\vec{r}' (r')^2 \begin{pmatrix} 3 \sin^2 \theta' \cos^2 \varphi' - 1 & 3 \sin^2 \theta' \cos \varphi' \sin \varphi' & 3 \sin \theta' \cos \theta' \cos \varphi' \\ 3 \sin^2 \theta' \cos \varphi' \sin \varphi' & 3 \sin^2 \theta' \sin^2 \varphi' - 1 & 3 \sin \theta' \cos \theta' \sin \varphi' \\ 3 \sin \theta' \cos \theta' \cos \varphi' & 3 \sin \theta' \cos \theta' \sin \varphi' & 3 \cos^2 \theta' - 1 \end{pmatrix} \rho(\vec{r}') .$$

Notice that if, for instance, ρ is independent of the azimuthal angle, the non-diagonal moments will vanish: $Q_{xy} = Q_{xz} = 0 = Q_{yz}$. As well, we can see that $Q_{xx} = Q_{yy}$. Since the matrix must be traceless, i.e., $Q_{xx} + Q_{yy} + Q_{zz} = 0$, it follows that we must have $Q_{zz} = -2Q_{xx} = -2Q_{yy}$, so that there is only one degree of freedom in this case. As before, it is then customary to refer to Q_{zz} as **the** quadrupole moment, Q_0 :

$$Q_0 \equiv Q_{zz} = 2\pi \int_0^R r'^4 dr' \int_0^{2\pi} \sin \theta' d\theta' \rho(r', \theta') P_2(\cos \theta') = -2Q_{xx} = -2Q_{yy} .$$

Unlike the dipole moment, however, it is **not always possible** to orient the \hat{z} -axis so that the quadrupole moment would be azimuthally symmetric. However, even though it may not have the non-diagonal terms zero, there might still be a choice of axis so that it would be much nicer. The matrix in question is symmetric, and obviously real-valued. Therefore it will always have real eigenvectors and real eigenvalues, which must sum to zero. In addition, one may always choose those eigenvectors to be orthogonal to one another, and of unit length if desired.

We now consider the special case where two of those eigenvalues are equal, then the third one must be the negative of their sum. We refer to the three of them, then, as $-\frac{1}{2}Q_0$, $-\frac{1}{2}Q_0$, and Q_0 . Let the eigenvector corresponding to Q_0 be normalized to have unit length, and called \hat{k} . It then follows that we may write \mathbf{Q} simply as

$$\text{for two equal eigenvectors, } \mathbf{Q} = \frac{1}{2}Q_0 [3\hat{k}\hat{k} - \mathbf{I}] ,$$

which may very well have non-diagonal elements. Nonetheless, were we to choose to orient our coordinate system so that $\hat{z} = \hat{k}$, then it would not have those non-diagonal elements; it would then be azimuthally symmetric with respect. The other two eigenvectors, since they correspond to identical eigenvalues, may be chosen to be **any** pair of unit vectors lying in the plane perpendicular to \hat{k} . To see all this, we may refer to either one of them as \hat{m} and write the following:

$$\begin{aligned} \mathbf{Q}\hat{k} &= \frac{1}{2}Q_0 [3\hat{k}\hat{k} - \mathbf{I}]\hat{k} = \frac{1}{2}Q_0 [3\hat{k} - \hat{k}] = Q_0 \hat{k} , \\ \mathbf{Q}\hat{m} &= \frac{1}{2}Q_0 [3\hat{k}\hat{k} - \mathbf{I}]\hat{m} = \frac{1}{2}Q_0 [3(0)\hat{k} - \hat{m}] = -\frac{1}{2}Q_0 \hat{m} . \end{aligned}$$

For this azimuthally-symmetric situation, both the electric potential and the electric field also take on a much simpler form, where we must remember that these are forms appropriate when we are **outside** the region where the charge is concentrated, i.e., for $r > R$:

$$\begin{aligned} V \Big|_{\text{pure Q, azi}} &= \frac{1}{r^3} [\hat{r} \cdot \mathbf{Q} \cdot \hat{r}] = \frac{Q_0}{2r^3} [3(\hat{k} \cdot \hat{r})^2 - 1] = \frac{Q_0}{r^3} P_2(\hat{k} \cdot \hat{r}) , \\ \implies \vec{E} \Big|_{\text{pure Q, azi}} &= -\nabla V \Big|_{\text{pure Q, azi}} = +3\frac{Q_0}{2r^4} [-2(\hat{r} \cdot \hat{k})\hat{k} + [5(\hat{k} \cdot \hat{r})^2 - 1]\hat{r}] . \end{aligned}$$

III. Origin Dependence of the Moments

The values of the moments depend, in the general case, on the choice of origin which has been made, although this is **not** true for the lowest-order one that is non-zero. Suppose, for instance, that we choose a new (primed) origin, which is effected by a move of the origin by a vector amount \vec{a} , so that the vector from the origin to an arbitrary point within the charged object is given by

$\vec{r}' = \vec{r}'' + \vec{a}$. We will refer to the original moments as above, but the (new) moments, relative to this new choice of origin, will be labelled by a subscript \vec{a} . Then we may consider the following, beginning with the (total) charge itself:

$$q \equiv \int d\vec{r}' \rho(\vec{r}') = \int d\vec{r}'' \rho(\vec{r}'') = q_{\vec{a}} .$$

The central equality occurs because the integration is over the entire region where charge is located, independent of what origin we have used to locate it. However, since the moments are statements about the symmetry of that charge distribution relative to our choice of origin they will usually change when that origin is changed:

$$\vec{p} \equiv \int d\vec{r}' \vec{r}' \rho(\vec{r}') = \int d\vec{r}'' (\vec{r}'' + \vec{a}) \rho(\vec{r}'') = \int d\vec{r}'' \vec{r}'' \rho(\vec{r}'') + \vec{a} \int d\vec{r}'' \rho(\vec{r}'') = \vec{p}_{\vec{a}} + q \vec{a} .$$

It should be obvious from the above construction that the movement of the origin will cause contributions to any given moment, say the 2^ℓ -th one, will engender additional terms in it proportional to all lower-order moments, i.e., those associated with smaller values of ℓ .

Continuing onward, we may consider the rather-more-complicated behavior of the quadrupole moment of a system, when we change the origin from which we view the system:

$$\begin{aligned} \mathbf{Q} &\equiv \int d\vec{r}' \frac{3\vec{r}'\vec{r}' - (r')^2\mathbf{I}}{2} \rho = \int d\vec{r}'' \frac{3\vec{r}''\vec{r}'' - 3\vec{r}''\vec{a} - 3\vec{a}\vec{r}'' + 3\vec{a}\vec{a} - [(r'')^2 - 2\vec{a} \cdot \vec{r}'' + a^2]\mathbf{I}}{2} \rho \\ &= \mathbf{Q}_{\vec{a}} - \left[3 \left(\frac{\vec{a}\vec{p}_{\vec{a}} + \vec{p}_{\vec{a}}\vec{a}}{2} \right) - (\vec{a} \cdot \vec{p}_{\vec{a}})\mathbf{I} \right] + \frac{1}{2} \left[3\vec{a}\vec{a} - (\vec{a}^2)\mathbf{I} \right] q . \end{aligned}$$

Clearly we could continue this sort of computations, but will forbear from doing so.

IV. Electric Moments Located in Exterior Electric Fields

An exterior electric field will exert a force, and, usually, a torque, on any electrical object that finds itself in that field. Because of this, the object will have a potential energy when placed in that field; i.e., the act of placing it there will require work.

We already know the form—Coulomb's Law—for the force exerted on a single, point charge, at a point \vec{r} , by an exterior electric field: $\vec{F} = q\vec{E}(\vec{r})$, while there is no torque on that single, point charge. As well, since the electric field in question has zero curl, there is a potential associated with it, such that $\vec{E}(\vec{r}) = -\nabla V(\vec{r})$ and which then contributes the desired potential energy, $U(\vec{r})$, which for our single point charge would simply be $U(\vec{r}) = qV(\vec{r})$.

We may use the superposition principle to consider the effects of such an electric field on an arbitrary charge distribution, and, in particular, on distributions with particular non-zero electric moments. The formulae from the superposition principle are straightforward, in principle, but will require a little effort to reduce them appropriately for the cases of interest:

$$\vec{F}(\vec{r}) = \int d\vec{r}' \rho(\vec{r}') \vec{E}(\vec{r}'), \quad \vec{N}(\vec{r}) = \int d\vec{r}' \rho(\vec{r}') \vec{r}' \times \vec{E}(\vec{r}'), \quad U(\vec{r}) = \int d\vec{r}' \rho(\vec{r}') V(\vec{r}').$$

Let's first apply these statements to a dipole, which we will treat as the system of two charges, $+q$ and $-q$, separated by a vector displacement $2\vec{a}$, in the limit as $a \rightarrow 0$ while $\vec{p} \equiv 2q\vec{a}$ stays constant. We set the center of the dipole, i.e., the point halfway between the two charges, at the location \vec{r} , relative to whatever choice of origin has been made. Therefore we may write

$$\begin{aligned} U(\vec{r}) &= qV(\vec{r} + \vec{a}) - qV(\vec{r} - \vec{a}), \\ \vec{F}(\vec{r}) &= q\vec{E}(\vec{r} + \vec{a}) - q\vec{E}(\vec{r} - \vec{a}), \\ \vec{N}(\vec{r}) &= q(\vec{r} + \vec{a}) \times \vec{E}(\vec{r} + \vec{a}) - q(\vec{r} - \vec{a}) \times \vec{E}(\vec{r} - \vec{a}). \end{aligned}$$

We now begin with the equation for the force, and expand the electric field in Taylor's theorem, about the central location \vec{r} , which gives us

$$\vec{F}(\vec{r}) = q \left[\vec{E}(\vec{r}) + (\vec{a} \cdot \nabla) \vec{E}(\vec{r}) + \dots \right] - q \left[\vec{E}(\vec{r}) - (\vec{a} \cdot \nabla) \vec{E}(\vec{r}) + \dots \right] = 2q(\vec{a} \cdot \nabla) \vec{E}(\vec{r}) + 2qO(a^3).$$

In the limit as $a \rightarrow 0$, the additional terms will vanish and we have exactly the following simple expression, for the force on a dipole centered at \vec{r} , in an external field \vec{E} :

$$\vec{F}(\vec{r}) = (\vec{p} \cdot \nabla) \vec{E}(\vec{r}).$$

Looking now at the equation for the torque on our dipole, we perform the same operations, which gives us

$$\begin{aligned} \vec{N}(\vec{r}) &= q(\vec{r} + \vec{a}) \times \left[\vec{E}(\vec{r}) + (\vec{a} \cdot \nabla) \vec{E}(\vec{r}) + \dots \right] - q(\vec{r} - \vec{a}) \times \left[\vec{E}(\vec{r}) - (\vec{a} \cdot \nabla) \vec{E}(\vec{r}) + \dots \right] \\ &= 2q\vec{a} \times \vec{E}(\vec{r}) + 2q\vec{r} \times (\vec{a} \cdot \nabla) \vec{E}(\vec{r}) + 2qO(a^2). \end{aligned}$$

Then, again in the limit as $a \rightarrow 0$ while \vec{p} remains constant, we have the formula:

$$\vec{N}(\vec{r}) = \vec{p} \times \vec{E}(\vec{r}) + \vec{r} \times \vec{F}(\vec{r}).$$

It is now worth retreating back to the formula for the potential energy of our dipole, in this external field, which has a potential $V(\vec{r})$ associated with it. We begin easily with

$$U(\vec{r}) = q[V(\vec{r}) + (\vec{a} \cdot \nabla)V(\vec{r}) + \dots] - q[V(\vec{r}) - (\vec{a} \cdot \nabla)V(\vec{r}) + \dots] = (\vec{p} \cdot \nabla)V(\vec{r}) + qO(a^2).$$

In the limit as $a \rightarrow 0$ the ignored terms all vanish; therefore, we may write explicitly the already-well-known formula for this potential energy, of an ideal, point dipole, at location \vec{r} :

$$U(\vec{r}) = (\vec{p} \cdot \nabla)V(\vec{r}) = -\vec{p} \cdot \vec{E}(\vec{r}) .$$

However, an interesting phenomenon occurs at this point. We know that there is a straightforward relationship between the potential energy and the associated force: $\vec{F} = -\nabla V$. Therefore, we should be able to calculate the force on the dipole in this way. This gives us

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) = +\nabla(\vec{p} \cdot \vec{E}) .$$

Unfortunately, this does **not appear** to be the same as the formula derived above, namely that

$$\vec{F}(\vec{r}) = (\vec{p} \cdot \nabla)\vec{E}(\vec{r}) .$$

Of course the last formula above—the one we should believe—was derived directly from (the superposition of) Coulomb’s Law, while the one above it came from the introduction of the potential. We therefore recall that the existence of the potential requires that the vector field should have zero curl, which is therefore necessary for the derivation of that (other) formula. To see that this is what makes them the same, we consider the formula your textbook gives you for the gradient of the dot product of two vector fields:

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} .$$

Applying this to our problem, we choose \vec{A} to be \vec{p} , which we recall is taken as **constant**; likewise we choose \vec{E} to be \vec{E} , and then obtain

$$\nabla(\vec{p} \cdot \vec{E}) = \vec{p} \times (\nabla \times \vec{E}) + (\vec{p} \cdot \nabla)\vec{E} .$$

This expression then tells us **the exact thing** that we needed: for arbitrary values of the (constant) vector \vec{p} , we find that $\nabla(\vec{p} \cdot \vec{E})$ is the same as $(\vec{p} \cdot \nabla)\vec{E}$ if and only if $\nabla \times \vec{E} = 0$. We will therefore have to consider such questions much more carefully when, next semester, we consider time-dependent problems where this is no longer true.

To consider these same notions for a pure quadrupole is rather more complicated, algebraically; therefore, we only begin the discussion with the calculation of the energy. We first require that the location of the charge distribution that has the quadrupole moment should be “well outside” the location of the charge distribution that is causing the external field; i.e., it

should truly be “external.” Under these circumstances a detailed derivation can show the following formulae for the behavior of this quadrupole:

$$U \Big|_{\text{pure Q}}(\vec{r}) = -\frac{1}{3} \sum_{i,j=1}^3 Q^{ij} \partial_i E_j(\vec{r})$$

$$\xrightarrow{\text{azimuthal symmetry}} -\frac{1}{2} Q_0 (\hat{k} \cdot \nabla) [\hat{k} \cdot \vec{E}(\vec{r})],$$

where, in the last line, we must remember that \hat{k} is constant. Therefore, the field must be non-uniform in order for there to be any potential energy relative to location of a quadrupole. One may use this to calculate the associated force and torque, which will of course depend on the second derivatives of the electric field.