

The Transformation of Rigid Objects

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Abstract

There are some seemingly simple problems in special relativity that are not so easy to solve once you get past the surface. One group of these problems has to do with rigid objects. There is a standard “rigid” rod problem that people use for this example, which this paper will deal with. There is also the idea of a rigid sphere or disk. One might ask, what happens to these fictitious round objects when they are rotated about an axis with very large angular velocities (ω). What is meant by very large ω ? Very large ω means that there are parts of the object that are moving at velocities comparable to the speed of light (c) with respect to the object’s rest frame. This would be somewhat annoying, considering that any real object that we know of would fly apart long before such an angular velocity was obtained, but remember, this is a hypothetical object, not a real one.

One item that this paper will address is the contradictions that arise when an object is assumed to be truly rigid. These are, of course, contradictions that arise from special relativity. It will also address how the rigid rod and rigid sphere/disk problems are related.

There is one last topic to cover. What does all of this have to do with anything real? There is an article titled “A Crucial Test for Einstein’s Special

Theory of Relativity Against the Lorentz-Poincare Ether Theory of Relativity.” It actually proposes a test of special relativity that involves rotating objects. This is, of course, related to a good portion of this paper.

The Rigid Rod

First, let us start by defining what a rigid rod is. It is a rod that does not deform when a force is applied to it, and if a force is applied to one end, the other end will start to accelerate the instant the force was applied. The rod could also be nailed down in the center, so that it had a pivot point. If a torque were applied at one end to start the rod spinning about this pivot point, the other end would start moving at the same instant the torque was applied. The rod would not flex; it would be truly rigid. It is an idealized object, just like the frictionless, massless pulleys that are used in problems given in freshman physics classes.

Since the rigid rod is an idealized object, it makes sense to see if there are any problems with such an object actually existing. There are some problems, and they come directly out of special relativity. First of all, a force being applied to one end will cause the other end to accelerate the very instant the force is applied. This implies that a signal telling the other end to accelerate has propagated down the rod at a velocity faster than the speed of light, i.e., instantaneously. This is in direct violation of the special theory of relativity.

To make things even more interesting, we will boost into other reference frames. We will start with a rigid rod of length L , which is initially at rest, and oriented such that it lies along the x -axis. We will then apply a force to one end,

pushing it in the x-direction. Since the rod is rigid, all points on the rod start to accelerate simultaneously. This is important, because now everything lies on a line of simultaneity, which will not be the same in another reference frame, and Lorentz transformations can easily be performed on any point on this line. It also makes it easy to construct space-time diagrams. As matter of course, it is fine to do the Lorentz transformations with $t=0$, since it is possible to zero a watch for any time. For all reference frames of interest, $t=0$ will be the time when the end of the rod which the force is applied starts to accelerate. All that really needs to be done is to see what happens to the time as you move away from the origin. All of this put together causes

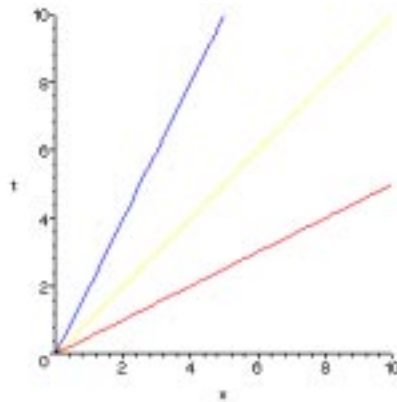
$$t' = \gamma(t - vx) \text{ to become } -\gamma x$$

Now, all of a sudden, if we have boosted into another frame of reference, the time that a part of the rod starts accelerating depends on the position along the rod. This means that the rod is no longer rigid in all reference frames, because not all parts start accelerating simultaneously. This also means that there will be a “wave” traveling down the rod.

Now, let us look at the case where v is negative. Here, t' gets bigger with x , so the “wave” is traveling down the rod, starting at $x=0$, and moving out, getting to the end of the rod at $t' = -\gamma L$. This “wave” is still moving faster than the speed of light, though. This can be seen, using a space-time diagram where a line of simultaneity of a moving reference frame has a slope of v . To see why this implies that this “wave” moves faster than c , we need a frame S , and a frame S' , where the

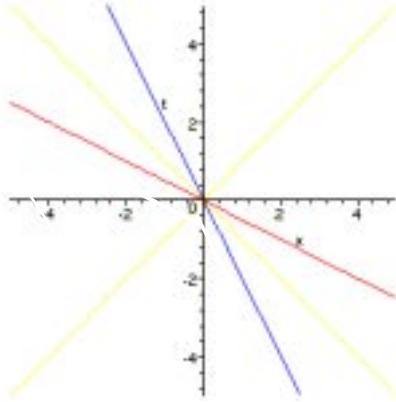
velocity of S' with respect to S is v . Now, if we place the rod initially at rest in S , an observer in S' sees it moving with velocity $-v$. Also, when the force is applied, all points start to accelerate simultaneously in S . In S' , however, all points on the rod do not start accelerating simultaneously, as shown by $t' = \gamma vx$. Now, if we look at a space-time diagram where S' is not moving, the line of simultaneity of S has a slope of $-v$ on a space-time diagram. This means that the “wave” traveling down the rod has a world line in S' represented by a line which has a slope of $-v$. Since the velocity is equal to $\frac{1}{\text{slope}}$ on a space diagram, the velocity of the wave is $-\frac{1}{v}$. Since $|v|$ is always less than 1, $\left|\frac{1}{v}\right|$ is always greater than 1, thus, the “wave” travels faster than the speed of light in all frames. This is result is because events space-like separated in one reference frame are space-like separated in all reference frames.

It is beneficial to actually look at some space-time diagrams of the rigid rod having a force applied to it. Below is a space-time diagram (S' stationary) for the case where the value of v is negative. The blue line is the world line of the front of the rod, and the red is the line of simultaneity for the frame S . This is the line which represents the “wave” traveling down the rod.



This shows how one would expect the “wave” to travel, from $x=0$ to $x=L$. It may seem strange that this is the case where v is negative, but this arises from the way v was defined. “ v ” was defined to be negative in the S frame, and the diagram is in the S' frame, where the negative value of v is taken.

The next logical step is to consider the case where v is positive. This makes things even more inconsistent with reality. For this case, the “wave” is traveling in the opposite direction, starting at the end where the force was not applied, and ending at the end where it was applied. This means that the end where the force is applied will accelerate after the other end starts to accelerate. The wave travels from $x=L$ to $x=0$. This obviously cannot be. Below is the space-time diagram for this case. As in the last one, blue is the world line for the front of the rod, and red is the line of simultaneity for S . It is easy to see the direction that the “wave” is traveling.



One last subject of interest with rigid rods is the acceleration. More specifically the acceleration of a rod being pushed verses a rod being pulled. It is useful to see if one case produces a higher acceleration than another. Much of the discussion, including equations, from this part will closely follow the paper titled “Does a Rod, Pushed by a Force, Accelerate Less than the Same Rod Pulled by the Same Force. It is first necessary to start defining reference frames and variables. First, x and t will be the variables for space and time in the lab frame, S . In the S' frame, which is instantaneously co-moving with the rod, x' will be the space coordinate; it is not necessary for a t' coordinate yet. $x'=0$ is the point at which the force is being applied. It will also be useful to introduce the infinitesimals dx' and dt so that may see how things change with time or distance. v will be the velocity of S' with respect to S . Now it is necessary to see how two points on the rod, x' , and $x'+dx'$ change in a given time interval, dt .. This is given by ⁽¹⁾

$$[v(x'+dx',t) - v(x',t)]dt$$

This equation looks very close to the definition of a derivative, which is

$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$. In fact the equation above is $\frac{\partial v(x', t)}{\partial x'} dx' dt$. One more thing to consider

is

how dx' changes, in the S frame. This is just a simple length contraction problem. This is given by ⁽²⁾

$$\left[\frac{dx'}{\gamma} \right]$$

where γ is the standard $\frac{1}{\sqrt{1 - v^2}}$. Next, we equate the two equations. To do this we need to

differentiate the second one with respect to time, as well as multiplying by it dt . This gives us the equation ⁽³⁾

$$\frac{\partial v(x', t)}{\partial x'} dx' dt = dx' \frac{\partial}{\partial t} \left(\frac{1}{\gamma} \right) dt = dx' \frac{\partial}{\partial t} \sqrt{1 - v^2} dt$$

Now, all that is left to be done is the differentiation of the $\sqrt{1 - v^2}$ term, realizing that v is a function of time, and is also the first derivative of $x'(t)$. The result that this yields is ⁽⁴⁾

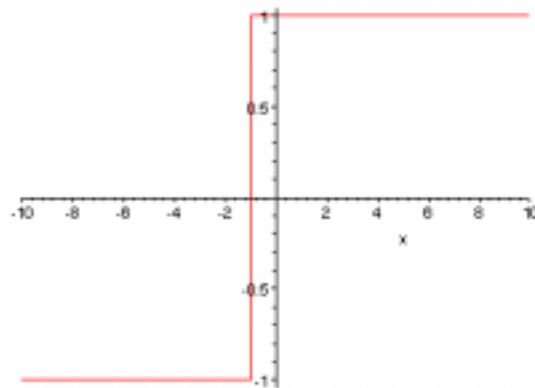
$$\frac{\partial v}{\partial x'} = - \frac{v}{\sqrt{1 - v^2}} \frac{\partial v}{\partial t}$$

which is a first order, non-linear partial differential equation, that does have a solution. Now the solution for this equation looks somewhat ugly. ⁽⁵⁾

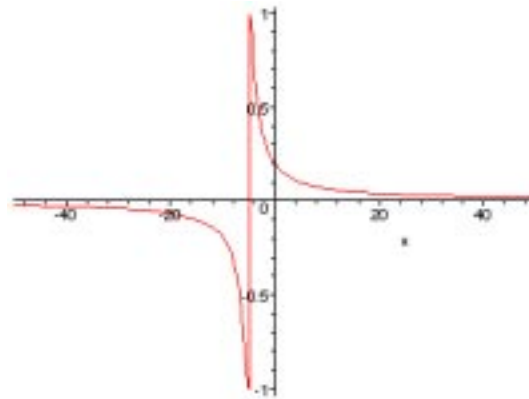
$$v(x',t) = \frac{\frac{at}{1+ax'}}{\left[1 + \left(\frac{at}{1+ax'}\right)^2\right]^{\frac{1}{2}}}$$

Here, a is supposed to be the acceleration of the point where the force is applied to the rod. To gain a grasp of what is going on, it will be very helpful to plot this function. Holding t constant and plotting v versus x' will give the best understanding of how the rod behaves if it is being pulled or pushed. This plot will show the velocity of the points on the rod verses the one point on the rod that has the force applied to it. There is one discrepancy, though. The authors of “Does a Rod, Pushed by a Force Accelerate Less than the Same Rod Pulled by the Same Force?” have made the assumption that the rod has been feeling the force for a long time. Below is a plot where the same assumption has been used by plotting

$$\lim_{t \rightarrow \infty} \frac{\frac{at}{1+ax}}{\sqrt{1 + \left(\frac{at}{1+ax}\right)^2}}$$



There is a problem here. This is a step function that steps from -1 to 1 at $x = -\frac{1}{a}$. That is a strange result. That says that in the limit in which t goes to ∞ , one part is going in the plus x direction at the speed of light, and another part is going in the minus x direction at the speed of light. Of course nothing can go the speed of light, unless a finite force has been applied forever (which is what we have done with our limit as t goes to ∞), but even the results for finite values of t are disturbing, as shown in the plot below. They do not agree with what the paper says about the given equation, which is “The speeds having abscissa $x' > 0$... are lower than the speeds of the points with $x' < 0$. As will be proved in Sect. 2, the point with $x' = 0$ corresponds to the application point of the external force.”



There is still a discontinuity at $x = -\frac{1}{a}$. The derivative at that point is a delta function. In section two of “Does a Rod, Pushed by a Force, Accelerate Less than

the Same Rod Pulled by the Same Force?” the authors use the equation for v to solve for p , the stress. Here is what they do:⁽⁶⁾

$$(\mu_0 + p) \frac{d(\gamma v)}{dt} = - \frac{dp}{dx'}$$

where μ_0 is the proper mass density. The differentiation of γv assumes that v is correct. If it is not, then the whole thing is flawed. When the differentiation of γv is carried out, and the equation solved for p , the result is ⁽⁷⁾

$$p \propto \frac{1}{1 + ax'} + C$$

What happens to the stress when $x = -\frac{1}{a}$? It blows up to ∞ . These are some

discrepancies that should not be overlooked. One way to check the discrepancies is to set $x'= 0$ to be the center of the rod, as opposed to the application point of the force on the rod. Why would this be a good check? The paper claims that this effect is to set up the Lorentz contraction of the rod. It claims that the velocity of each point on the rod is such that the rod contracts towards the application point of the force. Setting the center of the rod $x'= 0$, instead of the application point of the force will determine if the effect that G. Cavalleri and G Spinelli calculated is real, or just an artifact of their choice for a coordinate system. Wolfgang Rindler’s new textbook does say that all points move with constant proper acceleration in a case like this. This means that it should be possible to set $x'= 0$ as any point on the rod, and still get a valid result. If the coordinate system is changed so $x' = 0$ is the center of the rod, and the rod undergoes a Lorentz contraction, the center is going to be at a

constant value of x' as measured in the S frame. Now everything should appear to contract towards that center point, and it would appear that a rod being pushed accelerates at the same rate as a rod being pulled. Of course, this is not a rigorous mathematical argument, but it is definitely something to think about.

It is worth noting that close to $x'=0$, then the behavior, while using x' and t as coordinates in the manner prescribed in the paper, is exactly as G. Cavalleri and G Spinelli claim; a rod being pulled accelerates faster than a rod being pushed. The real problem comes when, as stated before, $x = -\frac{1}{a}$. This is actually a very large distance for any a which is not much, much greater than 1. The end of the paper does arrive at a sensible conclusion, though. It says that $a = \frac{f}{m_0}$ where m_0 is the rest mass of the object, and f is the force on the object. This makes perfect sense, as long as a is the acceleration in the instantaneous rest frame of the rod.

Rigid Spheres and Disks

To discuss rigid spheres and rigid disks, everything could be done in terms of spheres, but in many cases a disk is adequate for the demonstration of an idea, and is simpler than the sphere. Again, it is necessary to state what is meant by rigid. First, either the disk or sphere may be rotated about an axis at enormous angular velocities without flying apart. Second, just like the rigid rod, there can be no deformation of the rigid sphere or disk.

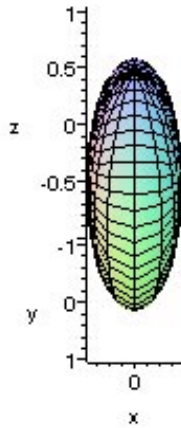
The first question to answer about a rigid sphere is how it transforms when boosted up to a large velocity. Since length contraction only occurs along the direction of motion, the best description is that it starts to look like a pancake as v increases. In the limit that $v = c$, the sphere becomes a disk. To show this, it is best to start with the equation of a sphere in Cartesian coordinates, solve for x (since the boost will be in the x -direction), and divide by γ . This yields

$$x^2 = \frac{R^2 - y^2 - z^2}{\gamma}$$

When can be transformed to spherical coordinates $[r, \theta, \phi]$.

$$r \leq \frac{R}{\sqrt{1 + \sin(\phi)^2(\gamma - 1) - \sin(\phi)^2 \sin(\theta)^2(\gamma - 1)}}$$

Where R is the radius of the sphere before it is transformed. The function in the radical on the bottom is always greater than or equal to 1, making r always less than or equal to R . When v is zero, γ is one, and the equation above goes to $r \leq R$, showing that when the sphere is measured in its own frame of rest, it is a sphere, as it should be. Below is a plot of a sphere moving at $.99c$, and $\gamma = 7$.



As stated before, it starts to pancake out. To make things more interesting with the relativistically moving sphere above, picture it with stripes on it so that its orientation could be described. Ideally, these stripes would start at the points where a line parallel to the y -axis passes through the center of the sphere, and radiate outward, becoming thickest when they have traversed one quarter of the circumference. They then end at the point on the other side of the sphere from where they originated. If the sphere were slowly spinning, it would be observable. Now, it is important to discuss what would happen to the stripes as the sphere spins slowly about the y -axis. It is spinning slowly, so there are no points on the sphere with a tangential velocity comparable to the overall velocity of the sphere. (Remember, the sphere is traveling at $0.99 c$ with respect to us.) These stripes are stuck on the surface of the sphere, so they transform accordingly under a boost. Any stripe with its centerline parallel to the x - y plane is going to keep the same thickness, but is going to have a different curvature than in the sphere's rest frame. Any stripe with its centerline parallel to the y - z plane has the same curvature, but

gets narrower. The stripes will transform to match the surface of the sphere. Since the sphere is rotating, these stripes change orientation, and thus change shape.

To describe these stripes mathematically, a new coordinate system is needed to make it simpler to understand. This coordinate system is a spherical coordinate system, (like $[r, \theta, \phi]$), that has been rotated so that the y-axis has taken the place of the z-axis in the normal formulation of spherical coordinates. To differentiate between standard spherical coordinates where ϕ is measured with respect to the z-axis, and this new set of spherical coordinates, a new set of variables will be used, $[R, \Theta, \Phi]$, the capital versions of the standard variables (Θ must still be measured from the x-axis, with the z-axis corresponding to $\Theta = \frac{\pi}{2}$). In the sphere's rest frame, these stripes have a constant R , their Φ values range from 0 to π , and their Θ values range from Θ_1 to Θ_2 . In the moving frame, they have to match the surface of the sphere.

$$R = \frac{\rho}{\sqrt{1 + \sin(\Phi)^2(\gamma - 1) - \sin(\Phi)^2 \sin(\Theta)^2(\gamma - 1)}}, \text{ where } \Theta \in [\Theta_1, \Theta_2]$$

This is the equation of the stripes on the surface of the sphere, where ρ is the radius of the sphere in its rest frame. This equation looks almost exactly the same as the equation for the shape of a sphere moving at relativistic speeds. The only differences are the change of variables into the rotated coordinate system, and the change of the \leq to an $=$. To make this equation that of a stripe on a spinning

sphere, Θ_1 and Θ_2 need to be functions of time. More specifically, $\Theta_1(t) - \Theta_2(t) = \Delta\Theta$, where $\Delta\Theta$ is constant.

$$\Theta_1(t) = \omega t + \Delta\Theta + \xi \text{ and } \Theta_2(t) = \omega t + \xi$$

These equations fit nicely with this requirement, and they are consistent with the idea that the sphere is rotating with an angular velocity ω . The ξ is simply a phase constant so that more than one stripe can be placed on the sphere at a time. What happens when t changes? The shape of the stripes will change as well. The shape of the stripes changing implies that the shape of the rigid sphere is changing. The sphere is supposed to be rigid, and not be able to change shape, so, again, like the rod, a sphere that is rigid in its own rest frame is no longer rigid in another reference frame where it is not at rest.

Now it is time to deal with spheres/disks rotating with large angular velocities. The best place to start is to solve for the tangential velocity of each point on the object in its frame of rest. This is very simple to do, but there is an apparent contradiction that must be dealt with afterwards. The tangential velocity of any point on an object rotating about an axis is the distance from that axis of rotation (r_{\perp}) times the angular velocity (ω) of the rotation. This is easily calculated with vectors, using \mathbf{r} and \mathbf{w} as the representation of the distance vector and angular velocity vector respectively (bold denotes an object as a vector). \mathbf{u} is defined to be the tangential velocity of the point on the object.

$$\mathbf{w} \times \mathbf{r} = \mathbf{u}$$

There is a limit, though, and that limit is that $\omega \leq \frac{1}{r_{\perp}}$, where r_{\perp} is the component of \mathbf{r} that is perpendicular to \mathbf{w} . This limit says that v can never be larger than c .

Here is the apparent contradiction. Length contraction only happens along the direction of motion. This means that if a rigid disk is spinning very fast about an axis that is perpendicular to its surface and running through its center, its radius should not shrink, at least not if it is being observed in its own reference frame. It might seem that its circumference should shrink, though. This is because the velocity of a point is in the $\hat{\mathbf{e}}_{\theta}$ direction, the same direction with which the circumference is oriented. If the circumference shrinks, then the radius must shrink with it. Now we are left to figure out which case is right. One way to find out is to start by claiming that a section of disk's circumference, $r d\theta$, is contracted by a factor of $\frac{1}{\gamma}$, and then finding out what this implies. γ is now a function of r , and on a disk that is spinning as described, \mathbf{w} and \mathbf{r} are always perpendicular, reducing the equation for the tangential velocity to a scalar equation.

$$\omega r = u$$

Plugging this into the explicit form of γ gives

$$\gamma = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-(\omega r)^2}}$$

The limit on ω here becomes $\omega \leq \frac{1}{r}$. Now it is possible to integrate around the

circumference of the disk.

$$cir = \int_0^{2\pi} \sqrt{1 - (\omega r)^2} r d\theta = 2\pi r \sqrt{1 - (\omega r)^2}$$

If this were correct, then as $\omega \rightarrow \frac{1}{r}$, r would go to zero. This means that ω would

have to change, because r would be changing, and $u = \omega r$.

Let us further examine the behavior of this function to see if it represents something real. A helpful place to start is by taking the derivative with respect to r .

$$\frac{\partial}{\partial r} cir = \frac{\pi(2r - 4\omega^2 r^3)}{\sqrt{r^2 - \omega^2 r^4}}$$

When the derivative is set equal to zero, and then r solved for, the result is

$r = \frac{1}{\sqrt{2\omega}}$. This point is either a maximum or a minimum as well as being less than

the maximum allowed value for r for a given ω . Using the second derivative with

respect to r , we find that this point is maxima. This means that it would be possible

for parts of the disk, which are on the outside when the disk is not spinning, to no

longer be on the outside. They have contracted to the point that parts of the disk

have changed position relative to each other, i.e. the disk has partially inverted

itself. This is a non-physical result, showing that the circumference cannot shrink,

and thus the radius stays the same, at least for the case when the sphere is being

measured in its own rest frame.

Now for a more challenging topic: The velocity of each point on a spinning object in a moving reference frame. When things start moving fast, the velocities do not add as they would in Galilean relativity. This is what makes solving for the velocity of a point on a spinning object in a moving frame a much more complicated task. We will first start with the equations out of Wolfgang Rindler's new textbook for velocity transformation. ⁽⁷⁾

$$u_x = \frac{u_x' + v}{1 + u_x' v}, \quad u_y = \frac{u_y'}{\gamma(1 + u_x' v)}, \quad u_z = \frac{u_z'}{\gamma(1 + u_x' v)}$$

These equations assume that the velocity \mathbf{v} is in the x-direction. The equation for the tangential velocity of a rotating object needs to be manipulated so that it is compatible with the equations above. The first thing that needs to be done is to change r to r' . Also, since the formulae are set up so that the velocity is in the x-direction, $\hat{v} = \hat{e}_x$, which makes things a bit easier to do. When the equation for \mathbf{u} is plugged into the three relativistic velocity addition equations above, the results are

$$u_x = \frac{\mathbf{w} \times \mathbf{r}' \cdot \hat{\mathbf{e}}_x + v}{1 + \mathbf{w} \times \mathbf{r}' \cdot \hat{\mathbf{e}}_x v}, \quad u_y = \frac{\mathbf{w} \times \mathbf{r}' \cdot \hat{\mathbf{e}}_y}{\left(1 + \mathbf{w} \times \mathbf{r}' \cdot \hat{\mathbf{e}}_x v\right) \gamma}, \quad u_z = \frac{\mathbf{w} \times \mathbf{r}' \cdot \hat{\mathbf{e}}_z}{\left(1 + \mathbf{w} \times \mathbf{r}' \cdot \hat{\mathbf{e}}_x v\right) \gamma}$$

These are not very nice to deal with, so, for some examples, \mathbf{w} will be said to be perpendicular to \mathbf{v} . This will make many of the terms disappear, and make it easier to see what happens to an object when it is spinning very fast, and moving by at relativistic speeds. Since it is convenient, we will set $\mathbf{w} = \omega \hat{\mathbf{e}}_y$. Once this simplification has been made,

$$\mathbf{w} \times \mathbf{r} = \omega r_z' \hat{\mathbf{e}}_x - \omega r_x' \hat{\mathbf{e}}_z$$

This has no y-component, and the x and z components only have one term. With this the velocity of a point on a spinning object moving at a relativistic velocity becomes

$$u_x = \frac{\omega r_z' + v}{1 + \omega r_z' v}, \quad u_y = 0, \quad u_z = \frac{\omega r_x'}{(1 + \omega r_z' v)\gamma}$$

Before anything else is dealt with, it is useful talk about ω . The ω that has been in all of the equations up to this point has been the angular velocity of the object in the object's frame of rest. The best way to discuss ω is to set up two coordinate systems, S and S', where S' is the object's frame of rest, and it is moving with velocity v with respect to frame S. Since we know the period of the revolution of the object in S', (T_0), we can solve for the period, T in S.

$$T = \gamma T_0 = \frac{\gamma 2\pi}{\omega} \Rightarrow \omega' = \frac{\omega}{\gamma}$$

The observer in S measures ω' to be slower than the observer in S' measures ω . Now let us look at the velocity of a point on a sphere. It is going to be useful to determine how it changes as \mathbf{r}' changes. This means that it is time to take some derivatives of u_x and u_z .

$$\frac{\partial u_x}{\partial r_z'} = \frac{\omega}{1 + \omega r_z' v} - \frac{(\omega r_z' + v)\omega v}{(1 + \omega r_z' v)^2}, \quad \frac{\partial u_z}{\partial r_x'} = \frac{\omega}{(1 + \omega r_z' v)\gamma}, \quad \frac{\partial u_z}{\partial r_z'} = \frac{\omega r_x' v}{(1 + \omega r_z' v)^2 \gamma}$$

$\frac{\partial u_x}{\partial r_z'}$ has a behavior that is very well behaved, until r_2' gets close to $-\frac{1}{\omega}$, where

it is very large. This means that the velocity climbs very fast with just a small

increase in r_2' . $\frac{\partial u_z}{\partial r_x'}$ is constant at constant r_2' . As r_2' increases, $\frac{\partial u_z}{\partial r_x'}$ decreases like $\frac{1}{r}$,

which is in accordance with the way $\frac{\partial u_x}{\partial r_z'}$ behaves. $\frac{\partial u_z}{\partial r_z'}$ is always negative, so the

higher its value for r_z' , the smaller u_z is. To really see what it is like, we can take

the magnitude of the velocity at any point on the sphere.

$$|v| = \sqrt{\left(\frac{\omega r_z' + v}{1 + \omega r_z' v}\right)^2 + \left(\frac{\omega r_x'}{(1 + \omega r_z' v)\gamma}\right)^2}$$

This can easily be converted to spherical coordinates with a computer program. Once converted, it is obvious that the magnitude of the velocity does not match what one would expect if Galilean relativity were true. The result is the combination of how relativistic velocities add, and how the sphere transforms under a boost. At the point on the sphere where $\mathbf{w} \times \mathbf{r}$ is parallel to \mathbf{v} , there is not that much of a difference between the velocity of the sphere, and the velocity of the point on the sphere (assuming ω and v are large). Where $\mathbf{w} \times \mathbf{r}$ is anti-parallel to \mathbf{v} , the difference can be enormous. This is a result of how velocities add relativistically.

Let us choose to call the part of the sphere where $\mathbf{w} \times \mathbf{r}$ is parallel to \mathbf{v} the top of the sphere, and the part of the sphere where $\mathbf{w} \times \mathbf{r}$ is anti-parallel to \mathbf{v} the bottom. If the tangential velocity of the surface of the sphere at the top is $.99c$ in its

frame, and the velocity of the sphere is $.99c$ as well, then the velocity of a point on the top is $.99995c$. On the other hand, if we measure the velocity of a point on the bottom of the same sphere, we get zero. Galilean relativity would demand that the bottom point have velocity zero, as it did in this case, but it also would demand that the top point have a velocity of $1.98c$. Since this is not possible, something has to happen to the shape of the sphere to allow this. At the top of the sphere, it has to have length contracted to just such an amount that velocity of $.99995c$ is the same if Galilean relativity were true, (not really, but close), and it were moving with a velocity of $1.98c$.

There is one thing that makes this easier. The sphere can only undergo length contraction along the direction of motion, despite the fact that different points on the sphere have velocities that are not parallel to the overall velocity sphere. This is consistent with the discussion earlier about the how radius of a spinning disk in its own reference frame stays constant, as opposed to the circumference shrinking. If the sphere is spinning with a large angular velocity it behaves just like the disk; it does not deform in its own frame of rest. If the spinning sphere were to undergo length contraction in a direction not parallel with the velocity that the sphere is moving, then it would be possible to have an angular velocity large enough to partially invert the sphere, just like in the case of the disk.

There is a simple reason why it will not invert if the length contraction happens only along the direction of v . The if the sphere is moving in the x -direction, and the angular velocity is in the y -direction, the velocity of any point on the sphere

only depends on the value of z . In fact, the greater the value of z , the greater the velocity of a point on the sphere. This means that the sphere will be “pinched” at the top, and be large on the bottom. To see this, an expression for the shape of the sphere will be necessary. The simplification that \mathbf{v} is in the x -direction, and \mathbf{w} is in y -direction will be made. This makes it possible to use a spinning disk, and get the same result as a sphere. Now, u_x only depends on the value of z . Under these circumstances, the cross section of a sphere in the x - z plane has the same shape as the disk would. It is necessary to know what γ is at any point on the disk.

$$\gamma = \frac{1}{\sqrt{1 - (u_x)^2}} = \frac{1}{\sqrt{1 - \left(\frac{\omega z + v}{1 + \omega z v}\right)^2}}$$

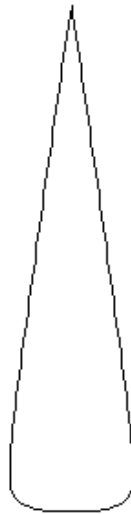
The next step is to convert the equation of a disk into Cartesian coordinates. Here, we can solve for the x -value of the edge of the disk, and perform the Lorentz transformation on it. This is quite simple, since γ only depends on z .

$$x = \sqrt{R^2 - z^2} \sqrt{1 - \left(\frac{\omega z + v}{1 + \omega z v}\right)^2}$$

This is the distance between the edge of the disk given in terms of z . R is the proper radius of the disk. To generalize this to a sphere, it is necessary to understand that the distance of the edge of the sphere to the x - y plane is the important parameter. This makes the generalization to a sphere simple.

$$x = \sqrt{R^2 - z^2 - y^2} \sqrt{1 - \left(\frac{\omega z + v}{1 + \omega z v}\right)^2}$$

It will be nice to see what a sphere looks like. Just a plot of a cross section in the x-z plane will suffice to see this. Below is the plot of the cross section of a sphere with $R = 0.99$, $\omega = 1$, and a velocity of $0.99c$.



This is an extreme case, but it illustrates the point very well. The sphere now has the same shape as a carrot.

An Application

An article by F. Winterberg called “A Crucial Test for Einstein’s Special Theory of Relativity Against the Lorentz-Poncare Ether Theory of Relativity” is

related to this paper through rotating objects. Though the Lorentz-Poincare Ether Theory of Relativity is beyond the scope of this paper, some of its predictions are stated in this article, and will be used in this section of the paper.

What do rapidly rotating objects have to do with a test of special relativity? According to F. Winterberg, the Lorentz-Poincare theory requires time for length contraction to take place, where as in Einstein's theory it happens instantaneously. This means that a moving, spinning object would have what Winterberg calls relativity violating effects if the Lorentz-Poincare theory were correct. These relativity violating effects would only be observable when the acceleration of an object was great enough that the change in velocity was comparable to the velocity of the stress waves that would be traveling through the object. This is because the length which contraction is predicted by the Lorentz-Poincare theory takes place at the same speed at which the stress waves travel through the object.

There is an experiment that would show observable effects of the delayed length contraction, if the Lorentz-Poincare theory were correct. A cross is needed with four arms of equal length, l . If the Lorentz-Poincare theory were correct, the cross, in an Earth laboratory frame of reference, "is moving with absolute velocity $v \cong 300 \frac{km}{s}$ against the substratum. ⁽⁸⁾" If the cross were not rotating, both theories would give the same results. If it were rotating, the results would be quite different for each frame. Einstein's theory says that, in the rest frame of the cross (the lab on the Earth), the cross will not distort. If it were in a moving frame, the angles that the four arms make with each other (α) will change as a function of ωt . Einstein's

theory says that $\alpha(t)$ will look like $\sin(2\omega t)$. The Lorentz-Poincare theory, on the other hand states that in the Earth frame, which is moving relative to the Ether, the angle will distort, but not like $\sin(2\omega t)$, as Einstein's theory would predict. The angle here is described by a second order, homogeneous, forced ordinary differential equation. ⁽⁹⁾

$$\frac{d^2\alpha}{dt^2} + 2\omega_1 \frac{d\alpha}{dt} + \omega_0^2 \alpha = \frac{1}{4} v^2 \omega_0^2 \sin(2\omega t)$$

Here, ω_0 is what is called the circular frequency of the bending waves, and ω_1 is given by ⁽¹⁰⁾

$$\omega_1 = \frac{2 \chi T \varepsilon^2 E}{9 r^2 \rho c_p}$$

where χ is the heat conduction coefficient, T is the temperature in Kelvin, ε is the thermal expansion coefficient, c_p is the constant pressure specific heat, r is the radius of the rod, and E is the Young Module.

The equation above shows that there is an angular velocity such that a resonance would be set up. This would produce observable effects in real materials, in an experiment that could actually be carried out. Winterberg gives an example of the types of numbers that would be involved. He claims that to achieve resonance with a steel cross with arms having a length of 10cm, and a radius of .5cm the tangential speed of the arms would be comparable to 250 m/s, or about the same speed of an ultracentrifuge.

Conclusion

Both the rigid sphere/disk, and rod problems are related by the topics discussed in both parts of this paper. The “wave” traveling down the rigid rod in a frame moving with respect to the rod, shows that the rigid rod is not rigid in all reference frames. This is the same as the stripes on the moving, slowly spinning sphere deforming. These stripes would not deform in the rigid sphere’s own frame of rest, just like the rigid rod would not deform in its own frame of rest. This is a convenient way of showing that truly rigid objects cannot exist according to special relativity.

The fact that the radius of a spinning disk should not shrink due to length contraction of its circumference is not a hard concept to grasp. There is a more simplistic approach to see this that was not taken in this paper. One could imagine that it does shrink. Then either the velocity at a given point would have change, or ω would have to change. If the velocity changed, then there would be a different gamma factor, and the circumference would not have had to contract as much as it did. If ω were to change instead, then once the circumference had length contracted, a given point would still have the same velocity as before, and thus would have to undergo another contraction. This would repeat until the disk was a

point with an infinite ω . These arguments are simpler than the argument used in the body of the paper, but the argument used there is more complete.

The spinning disk/sphere moving at relativistic velocities is a bit more challenging topic. The results are interesting too. The way velocities add relativistically is at the root of this problem. The idea that only the components of the tangential velocity of a point on a spinning sphere which are parallel to the motion of the sphere affect the length contraction of that sphere is strange. It is easy to imagine what this does to the concept of a rigid sphere, considering that a sphere spinning slowly gives rise to contradictions.

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The other, new, relativity book by Rindler that I need the info for

Wolfgang Rindler

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Does a Rod, Puushed by a Force, Accelerate Less than the Same Rod Pulled by the Same Force?

G. Cavalleri, G. Spinelli

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