Applying Lie Group Symmetries to Solving Differential Equations

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Symmetry analysis based on Lie group theory may be used to simplify a system of equations, thereby making it a valuable asset for solving non-linear problems. Since the method of determining symmetries is applicable to almost any system of differential equations, a general method for solving any differential equation may be formulated from this analysis, which is an improvement over the common practice of employing special “tricks” to solve an equation of known type. Although most of us are familiar with several techniques for determining exact solutions to differential equations, what is often not recognized is that these methods are usually special cases of symmetry analysis. Further impetus for the development of a general approach is its applicability to equations of unfamiliar type, for which there is no known special “trick”.

A general integration procedure for differential equations based on their invariance under a continuous group of symmetries was developed by Sophus Lie near the mid nineteenth century when he discovered a relationship between the assortment of special techniques being used to solve seemingly unrelated equations.

This paper will provide an introduction to symmetry analysis, where the word introduction is used in its truest sense since only the surface of this monumental theory will be presented. The paper begins with a discussion of terminology and selected theorems in order to provide the necessary framework for the eventual discussion of the symmetries of the heat equation.

The author would like to credit the text by Olver, which is listed as a reference, for the overall structure of this paper, as much has been borrowed from his introduction to the topic.

Terminology and Definitions

Two important concepts, which must be clearly distinguished in the reader's mind before continuing this study, are the group and the manifold. The distinction is important to make because a Lie group is connected to both ideas.

Loosely speaking, manifolds are generalizations of the familiar concepts of curves and surfaces in 3-D space. In general, they are spaces which locally look Euclidean, but may be quite different globally. The objects that will be discussed in this paper (differential equations, symmetry groups, etc.) are defined on open subsets of Euclidean space, and despite any particular co-ordinate system used to describe these subsets, their underlying geometrical features are defined to be co-ordinate independent. So let us define a manifold[(Olver)] as a set M that contains a countable number of subsets U_i (called co-ordinate charts) and one-to-one functions X_i, which map the U_i onto connected open subsets of Euclidean space.

A simple example of a manifold is Euclidean space: R^m. There is a single co-ordinate chart equal to R^m and so any open subset U is an m-dimensional manifold whose local co-ordinate map is the identity.

Throughout this paper, the assumption that all manifolds are connected will be made. A topological space is connected if it cannot be written as the disjoint union of two open sets.

A group[(Olver)] is a set G for which there is a group operation (usually called multiplication) defined such that for any two elements of the group, g and h, their product, g*h, is also a group element. Furthermore, the group operation must satisfy the following axioms:

i) Associativity. $g*(h*k) = (g*h)*k$

ii) Identity Element. The group must contain an identity element, e, with the following behaviour under group multiplication: $e*g = g = g*e$

iii) Inverses. For each group element g there is an inverse g^{-1} which returns the identity under group multiplication: $g*g^{-1} = e = g^{-1}*g$

As an example, consider the set of integers: the group operation may be defined as addition, whereby the identity element is the integer zero and the inverse of a given element is its negative. Since addition is an associative operation, the set is a group.

A Lie group is further defined to carry the structure of a smooth manifold so that its group elements can be continuously varied. The assumption of connectedness made for manifolds also applies to Lie groups. The set of real numbers is a group which is analogous to the one described above, with addition serving as the group operation, zero being the identity and the negative of an element being its inverse. Since this group is also a smooth manifold, it is a Lie group.

In general we will be discussing Lie sub-groups, so it is worthwhile to mention them. Since subgroups satisfy all of the group properties, they are also groups. The proper definition of a Lie sub-group is modeled on that of a sub-manifold. Following from this definition is the theorem that if H is a closed sub-group of a Lie group G then H is also a regular sub-manifold of G and hence a Lie group in its own right.

If one is only interested in group elements close to the identity element, as is often the case, a local Lie group can be defined by the local co-ordinate expressions for the group operations. A local Lie group consists of connected open sub-sets that contain the origin. It also satisfies the ordinary Lie group requirements, however they do not need to be satisfied everywhere (i.e. they may only be defined near zero).

Lie groups often arise as transformations on some manifold, as opposed to as abstract, self-contained entities. For example, the group SO(2) arises as the group of rotations in the plane $M = R^2$. In general, Lie groups will be represented here as transformation groups of some manifold. The transformation group does not need to be defined for all elements of the group or for all points on the manifold (i.e. it can act locally).

Symmetry Groups

The present section will present a comprehensive method for solving differential equations via the use of symmetry groups. A symmetry group of a system of differential equations transforms solutions of the system to other solutions and is the largest local group of transformations acting on the independent and dependent variables of the system. Before attempting to determine the symmetry groups of systems of differential equations, it is helpful to introduce the concept by studying simple algebraic equations.

Given a system of equations defined for one or several variables in a manifold, we define a symmetry group of the system to be a local group of transformations that acts on the manifold by transforming solutions of the system to other solutions.

Consider the system of lines: $x = cy + d$ and the one parameter group of translations, G_c: $(x,y) \rightarrow (x + cd, y + d)$; d ? R, c = fixed constant. The lines are clearly G_c invariant, and so G_c is a symmetry group of the system of lines.

The above example looks at symmetries of the solution set of a system of equations. We can also examine the invariance of a function under a group of transformations. Given a group of transformations acting on a manifold, and a function which maps from that manifold to another manifold, if for all group elements and all points on the manifold the function yields the same mapping for a point on the manifold as it does for that same point acted on by the group operation, the function is said to be an invariant function for that group. To clarify this, look again at the group G_c. The function ?(x,y) = x – cy is an invariant function since ?(x + cd, y + d) = ?(x,y) for all d. In other words, the transformation of the variables by the group did not affect the functional mapping.

In Lie group theory, one can replace the above criteria for invariant functions and subsets by an *equivalent* linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action. Given G, a connected group of transformations acting on the manifold M, a smooth real-valued function ? that maps points on M to R is an invariant function if and only if ☝(?) = 0 for all x ? M and for all ☝, where ☝ are the infinitesimal generators of G.

For the translation group G_c, the infinitesimal generator is ☝ = $c?_x + ?_y$. We found above that the function ?(x,y) = x-cy is an invariant function of the group. The same conclusion can easily be derived from the condition of infinitesimal invariance since:

☝(?) = $c?_x$(x – cy) + $?_y$(x – cy) = c – c = 0

In counting the number of invariants that a given group of transformations has, we count only the functionally independent invariants. Once this criteria has been established, we may begin to construct the set of invariants of a given group action. For a one parameter group of transformations G acting on M, with infinitesimal generator ☝ = $?_1$ $?/?x^1$ + … + $?_m$ $?/?x^m$, a local invariant is a solution of the linear, homogeneous partial differential equation ☝(?) = 0. If the generator is not zero when evaluated on the manifold, then there are (m-1) functionally independent solutions to the differential equation[(Olver)]. One may obtain the general solution by integrating the characteristic system of ordinary differential equations corresponding to the above equation:

$$dx^1/d?_1(x) = \ldots = dx^m/d?_m$$

To illustrate this, we can determine the invariants of the rotation group SO(2). The infinitesimal generator is ☝ = -y?$_x$ + x?$_y$. Thus m = 2 here and so there is one invariant function of the group. The corresponding characteristic system: -dx/y = dy/x has the general solution $x^2 + y^2 = c$, where c is an arbitrary constant. So, any function of ?(x,y) = $x^2 + y^2$ is an invariant of the rotation group.

As demonstrated earlier with regards to invariant functions and subsets, an important element of Lie group theory and transformation groups is the infinitesimal transformation. The concept of a vector field on a manifold is used to provide insight regarding this topic. Consider a smooth curve C on a manifold M, parametrized by f : I → M, where I is a subinterval of R. We may thus describe C by a collection of smooth functions f (e). If we define the local co-ordinates to be represented by the variable x, then at each point x = f (e) on C there is a tangent vector given by df /de. If we define the local co-ordinates x = (x^1, …, x^m) then we can express the tangent vectors as:

$$☝|_x = df/de = df^1/de\ ?/?x^1 + \ldots + df^m/de\ ?/?x^m$$

Symmetries of Differential Equations

Throughout the discussion of systems of differential equations, we will define the system of equations as 🌢, with p independent variables x^i and q dependent variables u^i. Also, let $X = R^p$ and $U = R^q$ be the spaces representing the respective groups of variables. Then we may define a symmetry group of the system of equations as a local group of transformations G which acts on some open subset M, contained in XxU, that transforms solutions of 🌢 to other solutions of 🌢. For example, consider the ordinary differential equation $u_{xx} = 0$ for which we know the solutions to be straight lines. The rotation group SO(2) is a symmetry group then, since it transforms linear functions to other linear functions.

The final topic to be addressed before studying the symmetries of differential equations is the process of prolongation. The somewhat ill defined notion of a system of differential equations needs to be replaced by a concrete geometric object that is characterized by the vanishing of certain functions. The aforementioned space XxU needs to be extended, or prolonged, to include not only the variables under consideration, but also the other partial derivatives that exist in the system. Let U_k represent the Euclidean space of all different kth order derivatives of the function and let $U^{(n)} = U_1 \times U_2 \times \ldots \times U_n$ be the Cartesian product space of all the different derivatives of the function from order 0 to n.

Consider a system where p = 2 and q = 1. Then we have that $X = R^2$ and $U = R^1$. The space U_1 has co-ordinates (u_x, u_y). The space U_2 has co-ordinates (u_{xx}, u_{xy}, u_{yy}). The space that contains all partial derivatives of u, up to second order, is U^2 with co-ordinates $(u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$.

The nth prolongation of a function is directly related to the co-ordinates obtained above. The prolongation is a vector function from the space of the independent variables to the space

U^n, whose entries represent the values of f and all its derivatives up to order n. For system described above, the second prolongation is given by:

$$(u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = (f; ?f/?x, ?f/?y; ?^2f/?x^2, ?^2f/?x?y, ?^2f/?y^2)$$

The total space $XxU^{(n)}$ whose co-ordinates represent the independent and dependent variables, as well as the derivatives of the dependent variables is called the nth order jet space of XxU.

For cases where we are only interested in differential equations defined in some open subset M contained in XxU, we define the n-jet space of M as $M^{(n)} = M \times U_1 \times \dots \times U_n$.

We are now in a position to reformulate the basic problem of finding symmetries of systems of differential equations. A system of r nth order differential equations can be viewed as a map from the jet space $XxU^{(n)}$ to an r dimensional Euclidean space. Since the differential equations determine where the map vanishes on the jet space, they determine a sub-variety of the total space. The abstract set of differential equations can thus be taken to be a concrete geometrical subset of the jet space. Also, the graph of the nth prolongation of a solution of the system is required to be contained in the sub-variety.

As an example, look at Laplace's equation in the plane: $u_{xx} + u_{yy} = O$.

The equation is a linear sub-variety in the jet space $XxU^{(2)}$ with co-ordinates $(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$. Using the requirement that the graph of the second prolongation ($pr^{(2)}$) of any solution must lie in the sub-variety, which is defined by the differential equation, we can check if the function $f(x,y) = x^3 - 3xy^2$ is a solution:

$$pr^{(2)} f(x,y) = (x^3 - 3xy^2; 3x^2 - 3y^2, -6xy; 6x, -6y, -6x)$$

This lies in the sub-variety since the fourth and sixth entries sum to zero as required by Laplace's equation.

The group actions must also be prolonged. Given G, a local group of transformations acting on an open subset M contained in the space of variables, the action of G on the nth jet space $M^{(n)}$ is its nth prolongation. This prolongation transforms the derivatives of the initial functions into those of the prolonged functions.

The problem of finding a symmetry group of a differential equation may now be reduced to the equivalent task of demonstrating that the corresponding sub-variety is invariant under the prolonged group action.

The method of solving equations, however, hinges upon one final theorem: consider a system of nth order differential equations: 💧$_i$ $(x,u^{(n)}) = 0$ defined over a manifold M contained in XxU, and a local group of transformations G acting on M. If the nth prolongation of every infinitesimal generator of G acting on the system vanishes, then G is a symmetry group of the system.

The Heat Equation

A simple equation describing the conduction of heat in a one-dimensional rod is:

$$u_t = u_{xx}$$

To illustrate the unification of the theory that has been presented, as well as to demonstrate how it is applied, the solution for the heat equation will be presented.

The independent variables in the equation are x and t, and the sole dependent variable is u. The equation has a corresponding linear sub-variety contained in the jet space $XxU^{(2)}$ defined by: 💧$(x,t; u; u_x, u_t; u_{xx}, u_{xt}, u_{tt}) = u_t - u_{xx} = 0$.

A general vector field on XxU has the following form:

☝ = ?(x,t,u) ?/?x + t(x,t,u) ?/?t + f (x,t,u) ?/?u

We need to find all possible values of the variables ?, t and f such that the one-parameter group exp(e☝) is a symmetry group of the differential equation. Thus we need to find every set of variables for which the second prolongation of ☝ acting on the system vanishes.

To find the second prolongation of ☝, we employ the general prolongation formula(Olver), which is derived from the following formula:

$$pr^{(n)}\ ☝|_{x,u(n)} = d/de|_{e=0}\ pr^{(n)}\ [\exp(e☝)]\ (x,u^{(n)})$$

→ $pr^{(1)}\ ☝ = ☝ + f^{x}\ ?/?u_x + f^{t}\ ?/?u_t$ → $f^{x} = f_{x} + (f_{u} - ?_x)\ u_x - t_{x}\ u_t - ?_u\ u_x^2 - t_u\ u_x\ u_t$

→ $f^{t} = f_{t} - ?_t\ u_x + (f_{u} - t_t)\ u_t - ?_u\ u_x\ u_t - t_u\ u_t^2$

(Sub-scripts denote partial derivatives).

→ $pr^{(2)}\ ☝ = pr^{(1)}\ ☝ + f^{xx}\ ?/?u_{xx} + f^{xt}\ ?/?u_{xt} + f^{tt}\ ?/?u_{tt}$

Where the first co-efficient f^{xx} is given by:

$$f^{xx} = f_{xx} + (2f_{xu} - ?_{xx})\ u_x - t_{xx}\ u_t + (f_{uu} - 2?_{xu})\ u_x^2 - 2t_{xu}\ u_x\ u_t - ?_{uu}\ u_x^3 - t_{uu}u_x^2\ u_t$$

$$+ (f_{u} - 2?_x)\ u_{xx} - 2t_{x}\ u_{xt} - 3?_u\ u_x\ u_{xx} - t_u\ u_t\ u_{xx} - 2t_u\ u_x\ u_{xt}$$

Now, applying the expression for $pr^{(2)}$ ☝ to the heat equation gives $f^{t} = f^{xx}$. Thus we can set the above expressions for f^{t} and f^{xx} equal to each other in order to solve for the co-efficients of the terms involving partial derivatives of u, also imposing the constraint that $u_t = u_{xx}$.

This yields the following results:

Term	**Co-efficient**
i) $u_x\ u_{xt}$	$0 = -2t_u$
ii) u_{xt}	$0 = -2t_x$
iii) u_{xx}^2	$- t_u = t_u$
iv) $u_x^2\ u_{xx}$	$0 = - t_{uu}$
v) $u_x\ u_{xx}$	$-?_u = -2t_{xu} - 3?_u$

vi) u_{xx}	$f_u - t_t = -t_{xx} + f_u - 2?_x$
vii) u_x^3	$0 = -?_{uu}$
viii) u_x^2	$0 = f_{uu} - 2?_{xu}$
ix) u_x	$-?_t = 2f_{xu} - ?_{xx}$
x) 1	$F_t = f_{xx}$

Now the above information may be used to give general expressions for the variables ?, t and f . Since the subscripts denote partial derivatives, it follows from the first two expressions that t is a function of t alone. Thus the fourth expression reduces to one of ? alone, showing that ? must be independent of u. Continuing in this fashion, one finds that:

$? = c_1 + c_4 x + 2c_5 t + 4c_6 xt$

$t = c_2 + 2c_4 t + 4c_6 t^2$

$f = (c_3 - c_5 x - 2c_6 t - c_6 x^2) u + a(x,t)$

Where the c's are arbitrary constants and a is an arbitrary solution of the heat equation.

By applying these solutions to the original expression for a general vector field on XxU, it follows that six vector fields span the Lie algebra of infinitesimal symmetries of the heat equation. These fields are given directly by using the six arbitrary constants defining the above equations (i.e. by alternately setting all but one equal to zero):

From c_1: ☝$_1 = ?_x$

From c_2: ☝$_1 = ?_t$

From c_3: ☝$_1 = u\, ?_u$

From c_4: ☝$_1 = x\, ?_x + 2t\, ?_t$

From c_5: ☝$_1 = 2t\, ?_x - xu\, ?_u$

From c_6: ☝$_1$ = 4tx ?$_x$ + 4t^2 ?$_t$ – (x^2 + 2t) u ?$_u$

From the term a(x,t) there is also the infinite dimensional sub-algebra ☝a = a(x,t) ?$_u$.

Each infinitesimal generator generates a one-parameter transformation group under whose operation the heat equation is invariant. The transformations that they generate on an initial point (x, t, u) are given by exp(e☝) (x, u, t) = (x', u', t'):

☝$_1$ = (x+e, t, u) ☝$_2$ = (x, t+e, u) ☝$_3$ = (x, t, exp(e)u)

☝$_4$ = (exp(e)x, exp(2e)t, u) ☝$_5$ = (x +2et, t, u exp(-ex – e^2 t))

☝$_6$ = (x / (1-4et), t / (1-4et), uv(1-4et) exp(-ex^2/1-4et))

☝$_a$ = (x, t, e a(x,t))

The functional invariance under each symmetry group follows as usual. For example, if u = f(x,t) is a solution, then by the first symmetry group, u = f(x – e, t) is also a solution.

It is now possible to construct a general solution of the heat equation by studying the symmetries under which its solutions are invariant.

Clearly the method of symmetry analysis of differential equations allows one to rigorously constrain the solution set of a particular problem, thereby simplifying it and facilitating the search for a solution. Using the method developed by Lie, the equations are seemingly forced to reveal their symmetries. Obviously much more can be done using symmetry analysis than was demonstrated in this paper, however hopefully this glimpse will whet the reader's appetite for more.

References:

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