Electromagnetic Fields of (single) Moving Charges, (Monopoles)

for Physics 495, *An Introduction to Special Relativity*

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I. Electric Fields of a Moving Charge

Our project is to understand the electric and magnetic fields caused by some charge, a monopole, with charge q. We will base our efforts on an understanding of the forces on some test charge, q_0, as measured in different reference frames. We will distinguish the forces felt by this test charge by claiming that when it is at rest, all the forces it feels are caused by the electric field, $\vec{E} = \vec{E}(\vec{r}, t)$, in its vicinity, while when it is moving with velocity $\vec{u}$, it feels, also, a force due to a magnetic field, $\vec{B} = \vec{B}(\vec{r}, t)$; therefore, the total force will satisfy the Lorentz force law:

$$\vec{F} = q_0\{\vec{E} + \vec{u} \times \vec{B}\} \ , \tag{1.1}$$

We will also take as given that the electric caused by a single monopole *at rest* is given by Coulomb's Law. Since we are interested in something more "exciting" than just Coulomb's Law with everything at rest, this means we will need at least 2 reference frames to study things. Therefore, we will normally take the reference frame S' as one in which the causative agent q is at rest. We will of course then have a different frame, S, which sees reference frame S' as moving with velocity $\vec{v}$, which means that S will also see q as moving with velocity $\vec{v}$. Therefore, we recall the important transformational properties of the force, namely that the 3-dimensional force is given by $\vec{F} = d\vec{p}/dt$, where of course $\vec{p} \equiv \gamma_u m\vec{u}$ is the relativistically-correct momentum for a particle of mass m moving with 3-velocity $\vec{u}$. That allowed us to determine the transformation laws between reference frames as

$$\begin{aligned} \vec{F}_{||} &= \frac{\vec{F}'_{||} + (\vec{u}\,' \cdot \vec{F}\,')\,\vec{v}}{1 + \vec{v} \cdot \vec{u}\,'} \ , \qquad \vec{F}_{\perp} = \frac{\gamma^{-1}\vec{F}'_{\perp}}{1 + \vec{v} \cdot \vec{u}\,'} \ , \\ \text{or} \quad \vec{F} &= \frac{\gamma^{-1}\vec{F}\,' + (1-\gamma^{-1})(\hat{v} \cdot \vec{F}\,')\,\hat{v} + (\vec{u}\,' \cdot \vec{F}\,')\,\vec{v}}{1 + \vec{v} \cdot \vec{u}\,'} \quad . \end{aligned} \tag{1.2}$$

To find the electric field caused by a moving charge, q, we now choose S' so that q is at rest, and choose S so that q_0 is at rest, so that it will only feel the force due to the electric fields caused by q; therefore, q_0 is moving with velocity $-\vec{v}$ in frame S'. In S', q is at rest, so that the only field it creates is the Coulomb field. If at time t' we ask for the electric field it creates at a vector distance $\vec{r}'$, it will be given by

$$\vec{E}'(\vec{r}', t') = \frac{kq}{r'^3}\vec{r}' \ , \tag{1.3}$$

where k is some constant that depends upon one's choice of units. As the test charge, q_0 is moving, the point at which it feels the force will be different at different values of t', so we may write

$$\vec{F}' = \frac{kqq_0}{[r'(t')]^3}\vec{r}'(t') \ . \tag{1.4}$$

It is important to recall that the vector $\vec{r}'(t')$ is the vector distance between the two charges at the same time, as simultaneity is measured in S', i.e.,

$$\vec{r}'(t') \equiv \vec{r}_0{}'(t') - \vec{r}_q{}'(t') \ . \tag{1.5}$$

We now Lorentz transform this equation for the force to frame S, where q_0 is at rest, which means that $\vec{u} = 0$. It is therefore simplest to use the inverse formulation of Eqs. (1.2), solving them for $\vec{F}'$ in terms of $\vec{F}$ and $\vec{u}$, which is of course done by simply interchanging primed quantities and not-primed quantities everywhere and also changing the sign of the relative velocity $\vec{v}$ of the two frames. Then, inserting the fact that $\vec{u} = 0$, we obtain the simple relationships:

$$\begin{aligned} \vec{F}'_{||} &= \left.\frac{\vec{F}_{||} - (\vec{u}\cdot\vec{F})\,\vec{v}}{1 - \vec{v}\cdot\vec{u}}\right|_{\vec{u}=0} = \vec{F}_{||} \ , \qquad \vec{F}'_{\perp} = \left.\frac{\gamma^{-1}\vec{F}_{\perp}}{1 - \vec{v}\cdot\vec{u}}\right|_{\vec{u}=0} = \gamma^{-1}\vec{F}_{\perp} \\ \Longrightarrow \quad \vec{F}_{||} &= \vec{F}'_{||} = \frac{kqq_0}{r'^3}\vec{r}'_{||} \ , \quad \vec{F}_{\perp} = \gamma\,\vec{F}'_{\perp} = \gamma\,\frac{kqq_0}{r'^3}\vec{r}'_{\perp} \ , \end{aligned} \tag{1.5}$$

where the last inversion of the equations was desired since we knew the values for the forces in S', i.e., we know $\vec{F}'$ and desired to find $\vec{F}$.

We now have the forces in S, but they are still written in terms of quantities measured in S'. We therefore also need to transform those quantities between frames as well, which means $\vec{r}\,'(t') \equiv \vec{r}_1{}'(t') - \vec{r}_2{}'(t')$. An important difficulty, however, is that the two events $(\vec{r}\,'_1, t')$ and $(\vec{r}\,', t')$, which were simultaneous in S' will **not be simultaneous** in S. In order to have a reasonable quantity for a force in S, we need a measurement of the difference in positions in S at the same time, t, as measured in S. **Happily**, for this case that is not a serious problem since the charge causing the field, q, is at rest in S'. Therefore, we choose, instead, a different event on that worldline, say $(\vec{r}_q{}', t'_<)$, with the property that when transformed to S it and the event on q_0's worldline are simultaneous there. Since q is at rest, the given value of $\vec{r}_q{}'$ will be the same at both events, so that the difference, $\vec{r}\,'(t')$, still has the same value. Then, to completely define, and also simplify, the associated algebra at the moment, let us choose that time of simultaneity in the S frame as $t = 0$, and also locate the charge q at rest at the origin in S'. This identifies the event with coordinates $(\vec{r}_q{}', t'_<)$ as the origin in S' and therefore also in S; it also identifies the event with coordinates $(\vec{r}_0{}'(t'), t')$ as the event $(\vec{r}, 0)$ in S, where we may calculate $\vec{r}$ by using the Lorentz transformation equations for that event:

$$\vec{r}_{||} = \gamma \vec{r}\,'_{||} \;, \quad \vec{r}_\perp = \vec{r}\,'_\perp \;. \tag{1.6}$$

Inserting these statements into Eqs. (1.5) we obtain the following remarkably simple result:

$$\vec{F}_{||} = \frac{kqq_0}{r'^3}\vec{r}\,'_{||} = \frac{kqq_0}{r'^3}\gamma\vec{r}_{||} \;, \quad \vec{F}_\perp = \gamma\,\frac{kqq_0}{r'^3}\vec{r}\,'_\perp = \gamma\,\frac{kqq_0}{r'^3}\vec{r}_\perp \quad \Longrightarrow \quad \vec{F} = \gamma\,\frac{kqq_0}{r'^3}\vec{r} \;, \tag{1.7}$$

although we must still replace the magnitude r'. This, however, is quite straightforward since

$$(\vec{r}\,')^2 = (\vec{r}\,'_{||} + \vec{r}\,'_\perp)^2 = (\gamma\vec{r}_{||} + \vec{r}_\perp)^2 = \gamma^2(\vec{r}_{||})^2 + (\vec{r}_\perp)^2 = r^2[1 + (\gamma^2 - 1)\cos^2\theta] \;, \tag{1.8}$$

where θ is the angle between $\vec{r}$ and $\vec{v}$. This then becomes the equation for the force felt by q_0—which then also determines the electric field created by q, which is moving with speed $\vec{v}$—at the time $t = 0$ when it is at the origin in S:

$$\vec{F} = \frac{kqq_0}{r^3}\left\{\frac{\gamma}{[1 + (\gamma^2 - 1)\cos^2\theta]^{3/2}}\right\}\vec{r} \Longrightarrow \quad \vec{E} = \frac{kq}{r^3}\left\{\frac{\gamma}{[1 + (\gamma^2 - 1)\cos^2\theta]^{3/2}}\right\}\vec{r}\;. \tag{1.9}$$

Noting that $\gamma^2 - 1 = v^2\gamma^2$ and exchanging $\cos^2\theta$ for $1 - \sin^2\theta$, we may rewrite this form for the electric field in a different, and sometimes simpler form:

$$\vec{E} \;=\; \frac{kq}{r^3}\left\{\frac{1-v^2}{[1-v^2\sin^2\theta]^{3/2}}\right\}\vec{r}\;. \tag{1.9a}$$

Also, for this simple case, we may write the transformation equation between frames for the electric field:

$$\vec{E}_{||} = \vec{E}\,'_{||}\,,\; \vec{E}_\perp = \gamma\vec{E}\,'_\perp\,,\quad \text{or } \vec{E} = \gamma\vec{E}\,' - (\gamma-1)(\hat{v}\cdot\vec{E}\,')\hat{v}\;. \tag{1.10}$$

2. Magnetic Fields of a Moving Charge

To determine the magnetic fields generated by our monopole q because of its motion, we will have to allow our test charge to be moving in the frame S where we want our fields described. Therefore, as before, we will have a frame S' where q is at rest, so that we may use the straightforward Coulomb's Law to determine its field, and we will also take S' as being measured by S as moving with velocity $\vec{v}$. However, in S we want our test charge to be moving with non-zero velocity $\vec{u}$, and therefore, a non-zero velocity $\vec{u}\,'$ in S', which will complicate the transformation of the force between frames since the test charge is no longer at rest in either frame!

To recapitulate, as the source charge is at rest in S' we may still use the (usual) Coulomb's Law force as given by Eq. (1.4) to determine the force felt by the test charge in S'; however, we must use the full formulation of the force transformation equation to determine the desired force, $\vec{F}$ as measured in S, where q is moving with velocity $\vec{v}$ and q_0 is moving with velocity $\vec{u}$. It is therefore desirable to first recall the transformation equation for $\vec{u}\,'$ in terms of $\vec{v}$ and $\vec{u}$, where it will turn out simpler to use the (lengthy) full vector form:

$$\vec{u}\,' = \frac{\vec{u}_{||} - \vec{v} + \gamma^{-1}\vec{u}_\perp}{1 - \vec{v}\cdot\vec{u}} = \frac{(\hat{v}\cdot\vec{u})\,\hat{v} - \vec{v} + \gamma^{-1}\,\hat{v}\times(\vec{u}\times\hat{v})}{1-\vec{v}\cdot\vec{u}}\;. \tag{2.1}$$

Then, as already noted, we may not use the simple algebraic device used in Eqs. (1.5) to obtain the desired force, but must rather insert things into the full form in Eqs. (1.2). We proceed by first noting that the Coulomb force is just in the direction $\vec{r}\,'$, so that everything else may be factored out front, giving

$$\vec{F} = \frac{kqq_0}{r\,'^3} \left\{ \frac{\gamma^{-1}\hat{v} \times (\vec{r}\,' \times \hat{v}) + (\hat{v} \cdot \vec{r}\,')\,\hat{v} + (\vec{u}\,' \cdot \vec{r}\,')\,\vec{v}}{1 + \vec{v} \cdot \vec{u}\,'} \right\} \tag{2.2}$$

We first notice that the first term in the numerator, which multiplies γ^{-1}, is just the perpendicular component of $\vec{r}\,'$, which of course transforms to the perpendicular component of $\vec{r}$, without change, so that we may simply drop the prime on the $\vec{r}\,'$ in that term. Similarly, the second term is just the parallel component of $\vec{r}\,'$, which we know is equal to γ times the parallel component of $\vec{r}$. In addition, using Eq. (2.1) for $\vec{u}\,'$, we may transform the denominator into its form in terms of quantities measured in S:

$$1 + \vec{v} \cdot \vec{u}\,' = 1 + \frac{\vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v}}{1 - \vec{v} \cdot \vec{u}} = \frac{1 - v^2}{1 - \vec{v} \cdot \vec{u}} = \frac{1/\gamma^2}{1 - \vec{v} \cdot \vec{u}} \ . \tag{2.3}$$

This allows us to rewrite the first two terms of the equation for $\vec{F}$ as

$$\frac{kqq_0}{r\,'^3} \left\{ \frac{\gamma^{-1}\hat{v} \times (\vec{r}\,' \times \hat{v}) + (\hat{v} \cdot \vec{r}\,')\,\hat{v}}{1 + \vec{v} \cdot \vec{u}\,'} \right\} = \gamma^2 \frac{kqq_0}{r\,'^3} [1 - \vec{v} \cdot \vec{u}] \left\{ \gamma^{-1}\hat{v} \times (\vec{r} \times \hat{v}) + \gamma(\hat{v} \cdot \vec{r})\,\hat{v} \right\}$$

Now we consider the third term of the equation for $\vec{F}$, where we must use Eq. (2.1) to replace $\vec{u}\,'$:

$$\frac{kqq_0}{r\,'^3} \frac{(\vec{u}\,' \cdot \vec{r}\,')\,\vec{v}}{1 + \vec{v} \cdot \vec{u}\,'} = \gamma^2 \frac{kqq_0}{r\,'^3} \left\{ (\hat{v} \cdot \vec{r}\,')(\hat{v} \cdot \vec{u} - v) + \gamma^{-1}\vec{r}\,' \cdot [\hat{v} \times (\vec{u} \times \hat{v})] \right\} \vec{v} \ .$$

We may now recombine the two portions of the equation for the force and cancel various terms:

$$\begin{aligned}
\vec{F} &= \gamma^2 \frac{kqq_0}{r'^3} \Big\{ \gamma\hat{v}(\hat{v} \cdot \vec{r})(1 - v^2) \\
&\quad + \gamma^{-1}[\vec{r} - (\hat{v} \cdot \vec{r})\hat{v} - (\vec{v} \cdot \vec{u})\vec{r} + (\vec{v} \cdot \vec{u})(\hat{v} \cdot \vec{r})\hat{v} + (\vec{r} \cdot \vec{u})\vec{v} - (\vec{r} \cdot \hat{v})(\hat{v} \cdot \vec{u})\vec{v}] \Big\} \\
&= \gamma \frac{kqq_0}{r'^3} \left[(\hat{v} \cdot \vec{r})\hat{v} + \vec{r} - \hat{v}(\hat{v} \cdot \vec{r}) - (\vec{v} \cdot \vec{u})\vec{r} + (\vec{r} \cdot \vec{u})\vec{v} \right] \\
&= \gamma \frac{kqq_0}{r'^3} \left[\vec{r} + \vec{u} \times (\vec{v} \times \vec{r}) \right] \ .
\end{aligned} \tag{2.4}$$

From Eq. (1.7) the first term in this last equality is just the electric force viewed in this frame, S. It is of course also the only remaining part of this force if we were to slow the test charge to rest, i.e., to set $\vec{u}$ to 0; therefore, we can conclude that the remainder is the magnetic force:

$$\vec{F}_M \equiv \vec{F} - \vec{F}_E = \gamma \frac{kqq_0}{r'^3} \vec{u} \times (\vec{v} \times \vec{r}) \equiv q_0 \vec{u} \times \vec{B} \ . \tag{2.5}$$

Therefore we may write out explicitly a form for the magnetic field generated by a single monopole of charge q and moving with velocity $\vec{v}$, all of which has been generated simply from Coulomb's Law and the validity of Lorentz transformations:

$$\vec{B} = \gamma \frac{kq}{r'^3} \, \vec{v} \times \vec{r} = \frac{kq}{r^3} \left\{ \frac{1 - v^2}{[1 - v^2 \sin^2 \theta]^{3/2}} \right\} \vec{v} \times \vec{r} = \vec{v} \times \vec{E} \, , \tag{2.6}$$

where the $\vec{E}$ in question is of course given in either of Eq. (1.9) or Eq. (1.9a).

It is well known that the electric and magnetic fields may be written in terms of a magnetic vector potential, $\vec{A}$, and a scalar potential, V. We will proceed, below, to determine those for our single, moving, charged particle, and to show that they do indeed generate the desired $\vec{E}$ and $\vec{B}$ fields.

On the other hand, from the point of view of using these calculations to generate the correct Lorentz transformations for the electric and magnetic fields, we may now re-write the above in the following way:

$$\vec{E}_{||} = \vec{E}'_{||} \, , \quad \vec{E}_{\perp} = \gamma \, \vec{E}'_{\perp} \, , \qquad \vec{B} = \gamma \, \vec{v} \times \vec{E}' \, . \tag{2.7}$$

However, we must **still be wary** of this result. Although it is definitely now more general than Eqs. (1.10), it is still transforming from a particular frame where there was no magnetic field; i.e., in the frame in which q is at rest, namely S', there is no magnetic field, so that this is probably not still the general transformation law which we are hunting, although certainly it must be a special case of it. It is of course true that it does nicely describe the electromagnetic fields for a single, moving charged particle—where there would indeed not be any magnetic field in the rest frame. Although we will not go ahead and look for the more general transformation law, we will eventually come back to look at this question of the single charged particle's fields some more.

Another quite interesting thing to do might be to show that these fields we have found do actually still satisfy Maxwell's equations. I say that it might be interesting because, after all, we know that this is the Lorentz transform of the electric field generated by Coulomb's Law for a charged particle at rest; therefore, one could easily argue that it "obviously" satisfies Maxwell's equations. On the other hand, that argument of course assumes that Maxwell's equations are preserved by Lorentz transformations; this is certainly true, but we have not really shown it.

The vector $\vec{r}$ in the equation is actually the vector from the current location of the charged particle to the field point in question; since the charged particle is moving, $\vec{r}$ depends on time. To describe this in more detail, we again follow the notation of Eq. (1.5), where the field point is fixed:

$$\vec{r}(t) = \vec{r}_0 - \vec{r}_q(t) \ . \tag{2.8a}$$

Therefore if we think of the electric field as $\vec{E}[\vec{r}(t)]$, then the action of a time derivative is as follows:

$$\frac{\partial \vec{E}[\vec{r}(t)]}{\partial t} = -\vec{v}\cdot\nabla\vec{E}[\vec{r}(t)] \quad \text{where } \vec{v} = \frac{d\vec{r}_q(t)}{dt} \ . \tag{2.8b}$$

a. To show Gauss' Law, we need to calculate the integral of $\vec{E}$ over some Gaussian surface, which contains the charge. I choose a sphere of radius R, centered on the charge at some particular instant, using $\eta \equiv \cos\theta$ as a variable for an integration:

$$\oint d\vec{A}\cdot\vec{E} = \frac{kq(1-v^2)}{R^2}(2\pi)R^2\int_0^\pi d\theta\,\frac{\sin\theta}{(1-v^2\sin^2\theta)^{3/2}} = 2\pi kq(1-v^2)\int_{-1}^{+1}\frac{d\eta}{(1-v^2+v^2\eta^2)^{3/2}} \tag{2.9}$$

The value of this last integral is straightforward, and is given by $2/(1-v^2)$ so that the result is $4\pi kq$, as desired for Gauss' Law.

b. To show Faraday's Law, we define $\vec{E} \equiv \mathcal{E}\vec{r}$ and begin with

$$\frac{\partial \vec{B}[\vec{r}(t)]}{\partial t} = -\vec{v}\cdot\nabla\vec{B} = -\vec{v}\cdot\nabla(\vec{v}\times\vec{E}) = -\vec{v}\times(\vec{v}\cdot\nabla)\vec{E} = -\vec{v}\times[\vec{r}(\vec{v}\cdot\nabla)\mathcal{E}+\mathcal{E}(\vec{v}\cdot\nabla)\vec{r}]. \tag{2.10}$$

Now here it is straightforward to determine that $(\vec{v}\cdot\nabla)\vec{r} = \vec{v}$ so that that term disappears from the sum, leaving us with

$$\frac{\partial \vec{B}[\vec{r}(t)]}{\partial t} = (\vec{r}\times\vec{v})(\vec{v}\cdot\nabla)\mathcal{E} \ . \tag{2.11}$$

Now we proceed to calculate the other side of this Maxwell equation:

$$\nabla \times \vec{E} = \mathcal{E}\nabla \times \vec{r} + (\nabla\mathcal{E}) \times \vec{r} = -\vec{r} \times \nabla\mathcal{E} \ , \tag{2.12}$$

where the last equality is true because the curl of location is zero, i.e., $\nabla \times \vec{r} = 0$.

We now want to show that these two quantities will add to zero, i.e., that Faraday's Law is satisfied. Therefore, we will actually have to determine the value of $\nabla\mathcal{E}$, in some detail. We do this via a chain of simple calculations, using its various parts in such a way as to completely avoid having to use spherical coordinates, which would be "a pain":

$$\begin{aligned}
\nabla\frac{1}{r^3} = -3\frac{\hat{r}}{r^4} \ , \quad \text{while} \quad \nabla\hat{r} = \nabla\frac{\vec{r}}{r} = \frac{I_3 - \hat{r}\hat{r}^{\,T}}{r} \ , \\
\text{We next need} \ \ \nabla\cos\theta = \nabla(\hat{v}\cdot\hat{r}) \ = \hat{v}\cdot(\nabla\hat{r}) = \frac{\hat{v} - (\hat{v}\cdot\hat{r})\hat{r}}{r} = \frac{\hat{v} - \cos\theta\,\hat{r}}{r} \\
\text{and then } \nabla\{(1 - v^2\sin^2\theta)^{-3/2}\} = -(3/2)(1 - v^2\sin^2\theta)^{-5/2}(2v^2)\cos\theta\,\nabla(\cos\theta) \\
= -\frac{3v^2\cos\theta}{r(1 - v^2\sin^2\theta)^{5/2}}(\hat{v} - \cos\theta\,\hat{r}) \\
.
\end{aligned} \tag{2.13}$$

Now we may insert this quantity back into the two equations, finding that both of them result in a vector that is in the direction perpendicular to both $\hat{r}$ and $\hat{v}$, i.e., in the direction $\hat{r} \times \hat{v}$. We begin the curl of the electric field:

$$\nabla \times \vec{E} = -\vec{r} \times \nabla\mathcal{E} = +3\frac{kq_0(1 - v^2)}{r^3}\frac{v^2\cos\theta}{(1 - v^2\sin^2\theta)^{5/2}}[\hat{r} \times \hat{v}] \ . \tag{2.14}$$

The similar calculation for the time derivative of the magnetic field gives us

$$\begin{aligned}
\frac{\partial\vec{B}[\vec{r}(t)]}{\partial t} &= (\vec{r} \times \vec{v})(\vec{v}\cdot\nabla)\mathcal{E} = rv^2[\hat{r} \times \hat{v}]\,(\hat{v}\cdot\nabla\mathcal{E}) \\
&= -3\frac{kq_0(1 - v^2)v^2}{r^3}[\hat{r} \times \hat{v}]\left(\frac{(1 - v^2\sin^2\theta)\cos\theta + v^2\cos\theta(1 - \cos^2\theta)}{(1 - v^2\sin^2\theta)^{5/2}}\right) \\
&= -3\frac{kq_0(1 - v^2)}{r^3}[\hat{r} \times \hat{v}]\frac{v^2\cos\theta}{(1 - v^2\sin^2\theta)^{5/2}} \ .
\end{aligned} \tag{2.15}$$

We can easily see that these two cancel out when added together, thus verifying Faraday's Law for these charges moving, at constant velocity.

3. General Transformation Law for the Electromagnetic Fields

In principle we could try to invert the equation and hope to use this to discover the form of the missing terms. Another approach might be to just go to yet some third inertial frame; however, this would appear to be rather messy. Instead, let us try to use our transformation equations for the force to just pretend we did, and see what the result should look like.

We suppose that we begin in a reference frame in which both the electric and magnetic fields are non-zero, and we transform from there to another frame. For now we will, again, refer to that initial frame as S'—although it is obviously a different one than the earlier discussions. In that frame we have the full Lorentz force equation:

$$\vec{F}\,' = q_0(\vec{E}\,' + \vec{u}\,' \times \vec{B}\,') \,. \tag{3.1}$$

Then we transform this equation back to the frame S, which measures S' to be moving with velocity $\vec{v}$, using Eq. (1.2) and also Eq. (2.3):

$$\begin{aligned}
\vec{F}_\perp &= \gamma^{-1} q_0 \frac{\vec{E}\,'_\perp + (\vec{u}\,' \times \vec{B}\,')_\perp}{(1-v^2)/(1-\vec{v}\cdot\vec{u})} = \gamma\, q_0(1-\vec{u}\cdot\vec{v})[\vec{E}\,'_\perp + (\vec{u}\,' \times \vec{B}\,')_\perp] \,, \\
\vec{F}_{||} &= q_0 \frac{\vec{E}\,'_{||} + (\vec{u}\,' \cdot \vec{E}\,')\vec{v} + (\vec{u}\,' \times \vec{B}\,')_{||}}{1+\vec{v}\cdot\vec{u}\,'} \\
&= \gamma^2 q_0 (1-\vec{v}\cdot\vec{u})[\vec{E}\,'_{||} + (\vec{u}\,' \cdot \vec{E}\,')\,\vec{v} + \hat{v}\cdot(\vec{u}\,' \times \vec{B}\,')\,\hat{v}] \,.
\end{aligned} \tag{3.2a}$$

We also insist that the Lorentz force equation hold directly in the frame S, i.e.,

$$\vec{F}_\perp = q_0[\vec{E}_\perp + (\vec{u}\times\vec{B})_\perp] \quad \text{and} \quad \vec{F}_{||} = q_0[\vec{E}_{||} + (\vec{u}\times\vec{B})_{||}] \,. \tag{3.2b}$$

We may immediately pick out of these relations the transformation law for the electric field, since we know that we would have **only** an electric contribution to the force in frame S if the velocity of the test charge, $\vec{u}$, would vanish there. We also know that if $\vec{u}$ were zero, then $\vec{u}\,'$ would just be $-\vec{v}$, as it was in our first considerations in these notes. Therefore, making those insertions into the comparison of the two forms for $\vec{F}$ above, we have the following two conclusions, where we use the fact that the cross product is completely perpendicular to $\vec{v}$, i.e., has no parallel component:

$$\begin{aligned} \vec{E}_\perp &= \gamma[\vec{E}\,'_\perp + (-\vec{v}\times\vec{B}\,')_\perp] = \gamma(\vec{E}\,'_\perp - \vec{v}\times\vec{B})\ , \\ \vec{E}_{||} &= \vec{E}\,'_{||}\frac{1-v^2}{1-v^2} = \vec{E}\,'_{||}\ . \end{aligned} \tag{3.3}$$

We may now proceed forward to find that information within the equation which is valid when $\vec{u}\neq 0$, beginning with the equation for the parallel component of $\vec{F}$. Inserting the information just obtained for the transformation of $\vec{E}_{||}$, we have

$$\begin{aligned} \vec{E}\,'_{||} + \hat{v}\cdot(\vec{u}\times\vec{B})\hat{v} &= \frac{\vec{E}\,'_{||} + (\vec{u}\,'\cdot\vec{E}\,')\vec{v} + \hat{v}\cdot(\vec{u}\,'\times\vec{B}\,')\hat{v}}{1+\vec{v}\cdot\vec{u}\,'} \\ &= \gamma^2(1-\vec{v}\cdot\vec{u})[\vec{E}\,'_{||} + (\vec{u}\,'\cdot\vec{E}\,')\vec{v} + \hat{v}\cdot(\vec{u}\,'\times\vec{B}\,')\,\hat{v}] \\ &= \gamma^2(1-\vec{v}\cdot\vec{u})(\vec{E}\,'_{||} + (\vec{u}\,'\cdot\vec{E}\,')\vec{v}\ +\ \gamma\,[\hat{v}\times\vec{u}\cdot\vec{B}\,']\hat{v}\ . \end{aligned}$$

Since the parallel component of $\vec{B}$ makes no contribution to $\hat{v}\times\vec{u}\cdot\vec{B}$ we may replace $\vec{B}$ there by $\vec{B}_\perp$. As well, since it is all in the direction $\hat{v}$, we may simply write out the (scalar) coefficient of $\hat{v}$, in the form

$$\begin{aligned} \hat{v}\times\vec{u}\cdot(\vec{B}-\gamma\vec{B}\,')_\perp &= \gamma^2\left\{(v^2-\vec{v}\cdot\vec{u})(\vec{E}\,'\cdot\hat{v}) + v(\vec{u}_{||} - \vec{v} + \gamma^{-1}\vec{u}_\perp)\cdot\vec{E}\,'\right\} = \ldots = \gamma\,v\,\vec{u}_\perp\cdot\vec{E}\,' \\ &= \gamma[\vec{v}\times(\vec{u}\times\hat{v})]\cdot\vec{E}\,' = \gamma(\hat{v}\times\vec{u})\cdot(\vec{v}\times\vec{E}\,') \end{aligned}$$

Since $\vec{v}$ and $\vec{u}$ are arbitrary we may infer from this that the transformation law for $\vec{B}_\perp$ is given by

$$\vec{B}_\perp = \gamma\,\vec{B}\,'_\perp + \gamma\,\vec{v}\times\vec{E}\,'\ . \tag{3.5}$$

Having now the majority of the transformation equations desired, we may return to the equation for $\vec{F}_\perp$ and consider the case when $\vec{u}\neq 0$, to see what information still remains there. When we do this it is convenient to remember that the "operator" that picks out the perpendicular part of some vector may be written in terms of a cross product that may be expanded:

$$(\vec{u}\times\vec{B})_\perp = \hat{v}\times[(\vec{u}\times\vec{B})\times\hat{v}] = \hat{v}\times[(\vec{u}\cdot\hat{v})\vec{B} - (\hat{v}\cdot\vec{B})\vec{u}] = (\vec{u}\cdot\hat{v})\,\hat{v}\times\vec{B}_\perp - (\hat{v}\times\vec{u}_\perp)\vec{B}_{||}\ , \tag{3.6}$$

where we have replaced $\vec{v}\times\vec{B}$ by $\vec{v}\times\vec{B}_\perp$, since only that part of $\vec{B}$ which is perpendicular to $\vec{v}$ contributes to the cross product, with the same being said as well for $\vec{u}$. We also note that exactly the same expansion may be written out beginning with the primed version, i.e., with $\vec{u}\,'\times\vec{B}\,'$. Returning now to the comparison of Eq. (3.2a) and Eq. (3.2b) for non-zero $\vec{u}$, we first write the comparison and insert our knowledge about the transformation of $\vec{E}_\perp$:

$$\gamma\,(1-\vec{u}\cdot\vec{v})[\vec{E}\,'_\perp+(\vec{u}\,'\times\vec{B}\,')_\perp]=\vec{E}_\perp+(\vec{u}\times\vec{B})_\perp=\gamma(\vec{E}\,'_\perp-\vec{v}\times\vec{B}\,')+(\vec{u}\times\vec{B})_\perp\ .$$

The terms with $\gamma\vec{E}\,'_\perp$ cancel and we use the expansion above in Eq. (3.4) for both the unprimed and the primed versions, to obtain

$$(\vec{u}\cdot\hat{v})\,\hat{v}\times\vec{B}_\perp-(\hat{v}\cdot\vec{B})\,\hat{v}\times\vec{u}=-\gamma\,(\vec{u}\cdot\vec{v})\vec{E}\,'_\perp+\gamma(\vec{u}\cdot\hat{v})\,\hat{v}\times\vec{B}\,'_\perp-(\hat{v}\times\vec{u})(\hat{v}\cdot\vec{B}\,')\ .$$

We now, finally, can insert our equation for the transform of $\vec{B}_\perp$, Eq. (3.5), into this equation, with the result that

$$\begin{aligned}(\hat{v}\times\vec{u})(\vec{B}-\vec{B}\,')_{||}&=0\ ,\\ \Longrightarrow\qquad \vec{B}_{||}&=\vec{B}\,'_{||}\ .\end{aligned}\tag{3.7}$$

4. Summary of the Transformation Laws in the 2-form called Faraday

We first summarize the transformation equations obtained above:

$$\begin{gathered}\vec{E}_{||}=\vec{E}\,'_{||}\ ,\qquad \vec{B}_{||}=\vec{B}\,'_{||}\ ,\\ \vec{E}_\perp=\gamma(\vec{E}\,'_\perp-\vec{v}\times\vec{B})\ ,\quad \vec{B}_\perp=\gamma(\vec{B}\,'_\perp+\vec{v}\times\vec{E})\\ \text{or}\\ \begin{aligned}\vec{E}&=\gamma\,\vec{E}\,'-(\gamma-1)(\hat{v}\cdot\vec{E}\,')\,\hat{v}-\gamma\,\vec{v}\times\vec{B}\,'\ ,\\ \vec{B}&=\gamma\,\vec{B}\,'-(\gamma-1)(\hat{v}\cdot\vec{B}\,')\,\hat{v}+\gamma\,\vec{v}\times\vec{E}\,'\ ,\end{aligned}\end{gathered}\tag{4.1}$$

Now we want to define a 2-form over spacetime. However, since we have been dealing with vectors over 3-dimensional space, we must first acknowledge that one may simply use the metric to transform tangent vectors into 1-forms and vice versa and also that, in 3 dimensions, we may create directly 2-forms from the components of tangent vectors:

$$\begin{aligned}\vec{E} = E^a\,\widetilde{e}_a \iff \underset{\sim}{E} \equiv E_a\,\underset{\sim}{\boldsymbol{\omega}}^a \equiv g_{ab}E^b\,\underset{\sim}{\boldsymbol{\omega}}^a\ , \quad \underset{\underset{2}{\sim}}{E} = \tfrac{1}{2}\eta_{abc}E^c\,\underset{\sim}{\boldsymbol{\omega}}^a \wedge \underset{\sim}{\boldsymbol{\omega}}^b\ ,\\ \vec{B} = B^a\,\widetilde{e}_a \iff \underset{\sim}{B} \equiv B_a\,\underset{\sim}{\boldsymbol{\omega}}^a \equiv g_{ab}B^b\,\underset{\sim}{\boldsymbol{\omega}}^a\ , \quad \underset{\underset{2}{\sim}}{B} = \tfrac{1}{2}\eta_{abc}B^c\,\underset{\sim}{\boldsymbol{\omega}}^a \wedge \underset{\sim}{\boldsymbol{\omega}}^b\ .\end{aligned} \tag{4.2}$$

Typically we will be able to decide from whence a given set of components comes, either from a tangent vector or from a 1-form, by the position of the indices. (Of course in 3-dimensions in Cartesian coordinates it doesn't really matter, although, for instance, in spherical polar coordinates, it does matter. In that case we must perform a transformation from the usual Cartesian, 3-dimensional metric matrix, $\eta \Longrightarrow I_3$ to g; in the spherical polar case, we would have $g_{ab} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$, with the notation "diag" indicating the elements of a diagonal matrix. This gives, for instance $E_\theta = r[\cos(\theta)(\cos\varphi\,E_x + \sin\varphi\,E_y) - \sin\theta\,E_z] = r^2\,E^\theta$.)

Returning to the general case, we have now defined sufficient quantities to create a presentation for a 2-form in terms of the components from our two 3-dimensional tangent vectors; however, since the "physical behavior" of the magnetic field is truly more like that of a (3-dimensional) 2-form, it enters into the Faraday in terms of the dual of its vectorial form, either as the 3-dimensional 2-form denoted by $\underset{3}{*}\underset{\sim}{B}$, promoted to a 4-dimensional 2-form, or directly as a 4-dimensional 2-form $*(\underset{\sim}{B} \wedge dt)$, while the electric field is more straightforward; in the simplest presentation just below this definition, one sees this difference by the differing positions of their indices.

$$\begin{aligned}\underset{\sim}{F} \equiv \tfrac{1}{2}\,F_{\mu\nu}\,\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \ \equiv\ -\underset{3}{*}\underset{\sim}{B} + \underset{\sim}{E} \wedge dt = -i * (\underset{\sim}{B} \wedge dt) + \underset{\sim}{E} \wedge dt\ ,\\ \text{or}\quad F_{ab} \equiv \eta_{abc}\,B^c\ , \quad F_{a4} \equiv E_a\ ,\end{aligned} \tag{4.3}$$

If we then present these components in $\{x, y, z, t\}$ coordinates, we have

$$F_{\mu\nu} \Longrightarrow \begin{pmatrix} 0 & +B^z & -B^y & +E_x \\ -B^z & 0 & +B^x & +E_y \\ +B^y & -B^x & 0 & +E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \equiv \begin{pmatrix} -\vec{B}\times & \vec{E} \\ -\vec{E}^{\,T} & 0 \end{pmatrix}\ , \tag{4.3a}$$

where the notation $-\vec{B}\times$ is yet another convenient notation for a matrix that contains the components of the 3-dimensional 2-form $-\underset{3}{*}\underset{\sim}{B}$, which has components as shown, since that 3×3 matrix operating on an arbitrary 3-vector, say $\vec{w}$, will have as result $-\vec{B}\times\vec{w}$.

As a useful aside we note that we have now presented three different ways to describe the "ubiquitous" fact that the magnetic field "should," originally, have been written as a 2-form, by giving three different presentations of how to write down that 2-form:

$$\underset{3}{*}\underset{\sim}{B} = i * (\underset{\sim}{B}\wedge dt) = \vec{B}\times \ . \tag{4.3b}$$

It is also worthwhile to determine how this transforms into basis sets based on alternate coordinate systems. A homework problem will ask you to do this in spherical polar coordinates, where you should find the following presentation:

$$F_{\alpha\beta} \Longrightarrow \begin{pmatrix} 0 & \sqrt{g}\,B^\varphi & -\sqrt{g}\,B^\theta & E_r \\ -\sqrt{g}\,B^\varphi & 0 & \sqrt{g}\,B^r & E_\theta \\ \sqrt{g}\,B^\theta & -\sqrt{g}\,B^r & 0 & E_\varphi \\ -E_r & -E_\theta & -E_\varphi & 0 \end{pmatrix} , \qquad \sqrt{g} \equiv \sqrt{\det(g_{ab})} = r^2\sin\theta\ . \tag{4.4}$$

To proceed to the question of the behavior of this 2-form under Lorentz boosts, we recall that a standard boost, $B(\vec{v})$, has the property that it converts contravariant coordinates (of a 4-vector) from the primed frame, x'^α, into those for the unprimed frame, x^μ, i.e., $x^\mu = B^\mu{}_\alpha\, x'^\alpha$. Therefore one would expect this 2-form to have its components transform as

$$F_{\mu\nu} = A^\alpha{}_\mu\, A^\beta{}_\nu\, F'_{\alpha\beta} \quad\Longleftrightarrow\quad F = A^T F' A \ \text{ and } \ AB = I\ , \ \text{i.e.,}$$

$$\begin{pmatrix} -\vec{B}\times & \vec{E} \\ -\vec{E}^{\,T} & 0 \end{pmatrix} = \begin{pmatrix} I_3 + (\gamma-1)\hat{v}\hat{v}^{\,T} & -\gamma\vec{v} \\ -\gamma\vec{v}^{\,T} & \gamma \end{pmatrix} \begin{pmatrix} -\vec{B}'\times & \vec{E}' \\ -\vec{E}'^{T} & 0 \end{pmatrix} \begin{pmatrix} I_3 + (\gamma-1)\hat{v}\hat{v}^{\,T} & -\gamma\vec{v} \\ -\gamma\vec{v}^{\,T} & \gamma \end{pmatrix} . \tag{4.5}$$

The general algebra will not be done here, but the reader should consider performing at least one special case in order to provide a basis to believe that this transformation law is identical with the one given in Eq. (4.1). A different approach toward verification might also consist in showing that the 0 in the lower-right corner is indeed preserved by the transformation.

5. Maxwell is the dual of Faraday, and the Maxwell Equations as 1-forms

Given the form of the Faraday 2-form in Eqs. (4.3), we may immediately construct its dual 2-form:

$$
\begin{aligned}
\underset{\sim}{M} \;\equiv\; *\underset{\sim}{F} \;&= *\Big\{ - i * (\underset{\sim}{B} \wedge dt) + \underset{\sim}{E} \wedge dt \Big\} = -i\Big\{ i * (\underset{\sim}{E} \wedge dt) + \underset{\sim}{B} \wedge dt \Big\} \, , \\
\Longrightarrow \quad \underset{\sim}{\Omega} \equiv \underset{\sim}{F} + *\underset{\sim}{F} &= *(\underset{\sim}{E} - i\underset{\sim}{B}) \wedge dt + (\underset{\sim}{E} - i\underset{\sim}{B}) \wedge dt \, ,
\end{aligned}
\tag{5.1}
$$

where we notice that this formulation for the "self-dual part" of the Faraday 2-form is "obviously" self-dual, i.e., equal to its own dual. As well we notice that one may easily create $i\underset{\sim}{M} \equiv i * \underset{\sim}{F}$ by beginning with $\underset{\sim}{F}$ and changing $\underset{\sim}{B}$ to $-\underset{\sim}{E}$ while simultaneously changing $\underset{\sim}{E}$ to $+\underset{\sim}{B}$, which is the "standard" definition of the electromagnetic duality transformation first given by Maxwell.

We may want to quickly calculate the action of the exterior derivative on $\underset{\sim}{F}$ and $\underset{\sim}{M}$. As this will generate 3-forms, which truly have only 4 independent components, it is more expedient to go ahead and take the dual of these 3-forms, ending up with appropriate 1-forms. We begin the Faraday:

$$
*d\underset{\sim}{F} = *d\Big\{ - \underset{3}{*}\, \underset{\sim}{B} + \underset{\sim}{E} \wedge dt \Big\} = - * \underset{3}{d}\, \underset{3}{*}\, \underset{\sim}{B} - *[dt \wedge \underset{3}{*}\, \dot{\underset{\sim}{B}} - \underset{3}{d}\, \underset{\sim}{E} \wedge dt] \, ,
\tag{5.2}
$$

where we have expanded the 4-dimensional exterior derivative into its spatial and temporal parts, and have noticed that the action of the 4-dimensional derivative on a 2-form including dt will only have a 3-dimensional effect. Next we need to recall two lemmas about the behavior of the Hodge dual, the exterior derivative, and the 3-dimensional divergence and curl operations:

$$
\begin{aligned}
- \underset{3}{*}\, \underset{3}{d}\, \underset{3}{*}\, \underset{\sim}{B} * \; = +\nabla \cdot \vec{B} \;\Longrightarrow\; - \underset{3}{d}\, \underset{3}{*}\, \underset{\sim}{B} = +(\nabla \cdot \vec{B})\, \underset{3}{\underset{\sim}{V}} \;\Longrightarrow\; - * \underset{3}{d}\, \underset{3}{*}\, \underset{\sim}{B} = +(\nabla \cdot \vec{B}) * \underset{3}{\underset{\sim}{V}} = -(\nabla \cdot \vec{B})\, dt \, , \\
\underset{3}{*}\, \underset{3}{d}\, \underset{\sim}{E} = \; - (\nabla \underset{\sim}{\times} \vec{E}) \;\Longrightarrow\; \underset{3}{d}\, \underset{\sim}{E} = - * \left\{ (\nabla \underset{\sim}{\times} \vec{E}) \right\} \, ,
\end{aligned}
\tag{5.3}
$$

where we use the under-tilde under $\nabla \times \vec{E}$ to indicate that we mean that 1-form whose components are the components of $\nabla \times \vec{E}$ lowered by the use of the metric tensor (acting as a map from tangent vectors to 1-forms). Using these lemma we may now rewrite our desired expression as follows:

$$
*d\underset{\sim}{F} = -(\nabla \cdot \vec{B})\, dt - [\dot{\underset{\sim}{B}} + \nabla \underset{\sim}{\times} \vec{E}] \, ,
\tag{5.4}
$$

where we have also used the fact that $\underset{3}{d}\,\underset{\sim}{E}$ is a 2-form so that it commutes with dt and the fact that

$$*[dt \wedge \underset{3}{*}\, dx^i] = - * [dt \wedge dx^j \wedge dx^k] = +dx^i \ , \ \text{with } i, j, k \text{ in cyclic order.}$$

As $\dot{\underset{\sim}{B}} + \nabla \underset{\sim}{\times} \vec{E}$ is a purely spatial 1-form, we have indeed separated out the spatial and temporal portions of the (4-dimensional) 1-form $*d\underset{\sim}{F}$, as was desired. In order to obtain the (exterior) derivative of the Maxwell, we recall that $i * \underset{\sim}{F}$ is obtained by just sending $\underset{\sim}{B}$ to $-\underset{\sim}{E}$ and, simultaneously, $\underset{\sim}{E}$ to $+\underset{\sim}{B}$. Therefore we may do that in the expression above giving us those derivatives:

$$i * d * \underset{\sim}{F} = (\nabla \cdot \vec{E})\, dt - [-\dot{\underset{\sim}{E}} + \nabla \underset{\sim}{\times} \vec{B}] \ , \tag{5.5}$$

At this point we recall the experimental results contained in Maxwell's Equations; they tell us that the 4 expressions contained in $*d\underset{\sim}{F}$ should vanish, while the 4 expressions contained in $*d\underset{\sim}{M}$ should be $-4\pi k(-\rho dt + \underset{\sim}{J}_3)$, where ρ is the total charge per unit volume and $\vec{J}$ is the current density, i.e., the total current passing through a unit area, with its direction in the direction of the normal to that area. [It should also be noted that in standard MKS units $k = 1/(4\pi\epsilon_0)$ while $1/(\mu_0\epsilon_0) = c^2 = 1$ implies that, in our standard choice of units $\mu_0 = 1/\epsilon_0$. We therefore intend now to create a tangent vector, and associated 1-form,

$$\widetilde{J} \Longrightarrow \quad (\vec{J}, \rho)^T \quad \Longrightarrow \quad \underset{\sim}{J} = J_i\, dx^i - \rho\, dt \ , \tag{5.6}$$

which allows us to now write Maxwell's equations in the form

$$* d * \underset{\sim}{F} = 4\pi i k \underset{\sim}{J} \ , \quad * d\underset{\sim}{F} = 0 \ . \tag{5.7}$$

We may now invoke, locally, Poincare's Lemma, which tells us of the (non-unique) existence of some 1-form, $\underset{\sim}{A}$, locally, such that

$$\underset{\sim}{F} = d\underset{\sim}{A} \ , \qquad \widetilde{A} \Longrightarrow \quad (\vec{A}, V)^T \ , \quad \underset{\sim}{A} \Longrightarrow \quad (\underset{\sim}{A}_3, -V) \ , \tag{5.8}$$

where the presentation in terms of $\vec{A}$ and V simply defines the spatial and temporal parts in the forms we are used from more elementary approaches to the electric and magnetic fields. We may even make that representation more explicit as follows.

$$-\underset{3}{*}\underset{\sim}{B} + \underset{\sim}{E}\wedge dt = \underset{\sim}{F} = d(\underset{\sim}{A} - V\,dt) = \underset{3}{d}\underset{\sim}{A} + dt\wedge\dot{\underset{\sim}{A}} - \underset{3}{d}V\wedge dt = \underset{3}{d}\underset{\sim}{A} - \{\dot{\underset{\sim}{A}} + \underset{3}{d}V\}\wedge dt\,. \tag{5.9}$$

Comparing the two "ends" of the equation chain we may read off the (standard, well-known) relationships:

$$\begin{aligned} \underset{\sim}{E} &= -\dot{\underset{\sim}{A}} - \underset{3}{d}V \qquad \text{or} \qquad \vec{E} = -\dot{\vec{A}} - \nabla V\,, \\ \text{and} \quad \underset{3}{*}\underset{\sim}{B} &= -\underset{3}{d}\vec{\underset{\sim}{A}} \qquad \text{or} \qquad \vec{B} = \nabla\times\vec{A}\,. \end{aligned} \tag{5.10}$$

At this point we should note that the existence of this potential of course satisfies half of the Maxwell equations, but the other half remain, now in the form

$$*\,d*\,d\underset{\sim}{A} = 4\pi i k \underset{\sim}{J}\,, \tag{5.11}$$

and that the so-created vector potential is not unique, but, rather, may be modified more or less at will by adding the gradient of a scalar, i.e.,

- $\underset{\sim}{A}$ and $\underset{\sim}{A} + d\chi$, for any function $\chi \in \mathfrak{F}$, create exactly the same Faraday.

These two facts may be interrelated since it is usual to make certain choices of χ so that the equation involving $\underset{\sim}{J}$ is simplified. We will discuss these shortly, but, first, let's discuss the role of $\underset{\sim}{A}$ in the single monopole case.

6. The 4-Vector Potential for the Single Moving Charge

We first want to spend a little more time "understanding" the forms we now have for the electric and magnetic fields of a moving monopole, as given in Eqs. (1.9a) and Eqs. (2.6):

$$\vec{E} \;=\; \frac{kq}{r^3}\left\{\frac{1-v^2}{[1-v^2\sin^2\theta]^{3/2}}\right\}\vec{r}\,, \qquad \vec{B} \;=\; \vec{v}\times\vec{E}\,. \tag{6.1}$$

This form may be understood somewhat differently by introducing the *retarded position* and *retarded time* associated with the field point, and field time, and also the motion of the charge causing the field. We first recall, from the discussion near Eq. (1.5), that Coulomb's Law involves a difference of the two vectors describing the location of the field point and the location of the charged particle, both at a given, single time, t, as measured in the observer's rest frame. We now generalize those notions to 4-dimensional notation, and introduce

- **the event of the measurement of the field**, with coordinates $\widetilde{r}_0 \quad \Longrightarrow \quad (\vec{r}_0, t)^T$, and
- **the worldline of our charged particle,** parametrized by the time of our observer, t, which has coordinates given by a time-dependent 4-vector, $\widetilde{r}_q(t) = (\vec{r}_q(t), t)^T$, so that
- we may create the event which is **the intersection of this worldline with the past lightcone of the measurement event**. That intersection occurs at an earlier time, which we denote by $[t]$, and refer to as *the retarded time for the field event*, and satisfies the following equation:

$$\Big|[\vec{r}_0 - \vec{r}_q([t])\Big| = t - [t] \geq 0 \ . \tag{6.2}$$

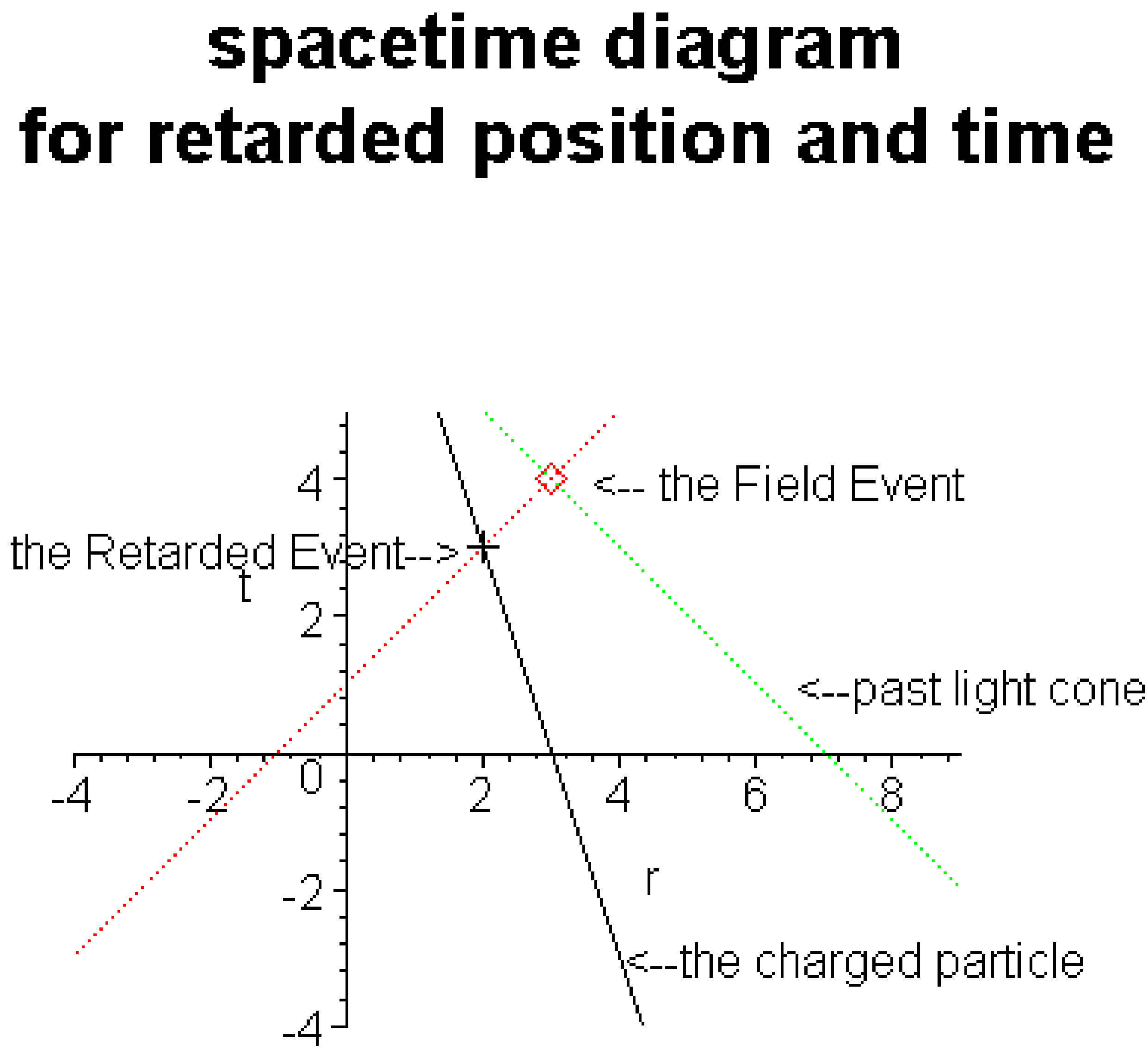
- Lastly, we introduce a "perhaps fictitious" event. We suppose the charged particle has been travelling at constant velocity, $\vec{v}$, at least since the retarded time, $[t]$, and look at the position the charged particle would then have at the time of the measurement, so that we create a spatial vector

$$\vec{r}_f(t) \equiv \vec{r}_q([t]) + (t - [t])\vec{v} \ , \tag{6.3}$$

  so that the coordinates for event N are $\widetilde{r}_f(t) = (\vec{r}_f(t), t)^T$. Of course if the particle has indeed been travelling at constant velocity this is the actual, current location of that particle, $\vec{r}_q(t)$, even though our field event could have no knowledge of this at that time.

We now recall, following Eq. (1.5), that the vector direction, $\vec{r}$, for the electric field is given by the (spatial) difference of the location of the field point and the (presumed) location of the charged particle

$$\vec{r} \equiv \vec{r}_0 - \vec{r}_f(t) = \vec{r}_0 - \vec{r}_q([t]) - (t - [t])\vec{v} \tag{6.4}$$



It is then quite convenient to introduce the spatial difference between the field point at the measurement time and the charged particle at the retarded time, $[\vec{r}]$, which we refer to as *the retarded position*, which of course also depends on time:

$$\begin{aligned}[\vec{r}](t) &\equiv \vec{r}_0 - \vec{r}_q([t]) = \vec{r} + (t-[t])\vec{v}\ ,\\ &\Longrightarrow\quad [r](t) \equiv \big|[\vec{r}]\big|(t) = \big|\vec{r}_0 - \vec{r}_q([t])\big| = t - [t]\ ,\\ &\Longrightarrow\quad \vec{r} = [\vec{r}] - [r]\vec{v} = [\vec{r} - r\vec{v}]\ .\end{aligned} \tag{6.5}$$

where the last equality in the first line follows from Eq. (6.4) while the last one in the second line follows from Eq. (6.2). We may then proceed further by considering the angle between the two vectors $\vec{r}$ and $[\vec{r}]$, which we call α:

$$r\cos\alpha = \vec{r}\cdot[\hat{r}\,] = [\vec{r}-r\vec{v}]\cdot[\hat{r}\,] = [r - \vec{r}\cdot\vec{v}] = [r(1-v\cos\theta)]\ . \tag{6.6}$$

However, we also look at the first line of Eq. (6.5), which we may conceive of as telling us that the three vectors, $[\vec{r}]$, $\vec{r}$, and $(t-[t])\vec{v}$ form a triangle, with $[\vec{r}]$ as the hypotenuse. We now apply the *Law of Sines*, from plane geometry, to this triangle. It requires that we have the various angles "across from" various sides of the triangle. Therefore we begin with the side created by $[\vec{r}]$; the angle across from it is $\pi-\theta$, where θ is the angle between $\vec{v}$ and $\vec{r}$. Similarly, the angle across from the side with length $(t-[t])\vec{v} = [r]\vec{v}$, is the angle between $[\vec{r}]$ and $\vec{r}$, which we have already labelled as α. Therefore the Law of Sines tells us that the first line below is true:

$$\begin{aligned}\frac{[r]}{\sin\theta} = \frac{[rv]}{\sin\alpha}\quad &\Longrightarrow v\sin\theta = \sin\alpha\\ \Longrightarrow\ 1 - v^2\sin^2\theta\ &= 1-\sin^2\alpha = \cos^2\alpha\\ \Longrightarrow r' = r\sqrt{1-v^2\sin^2\theta} = r\cos\alpha &= \vec{r}\cdot[\hat{r}\,] = [r-\vec{r}\cdot\vec{v}\,] = [r(1-v\cos\theta)] \equiv \mathcal{S}\ ,\end{aligned} \tag{6.7}$$

where we do not bother to put "brackets" around $\mathcal{S}$—to indicate that it should be evaluated at the retarded time—since it will **always** be just the retarded quantity indicated above. However, the left-hand side of the last line is just the denominator in the electric field; therefore, an alternative mode to write that equation is

$$\vec{E} = \frac{kq(1-v^2)}{\mathcal{S}^3}[\vec{r}-r\vec{v}\,]\ ,\qquad \vec{B} = \vec{v}\times\vec{E} = [\hat{r}\,]\times\vec{E}\ , \tag{6.8}$$

where the last equality is true because $\vec{v} = [\vec{v}]$ and $(\vec{v} - \hat{r}) \times (\vec{r} - r\vec{v}) = \vec{0}$.

As a last property to consider concerning this important quantity, I want to show that it is intimately related to a 4-dimensional, Lorentz-invariant quantity. We calculate the following (Lorentz-invariant) scalar product where $\widetilde{u}$ is the 4-velocity of the charged particle:

$$\widetilde{u} \cdot (\widetilde{r}_0 - \widetilde{r}_q) = \gamma\Big\{\vec{v} \cdot \{\vec{r}_0 - \vec{r}_q\} - 1(t - [t])\Big\} = \gamma\{\vec{v} \cdot [\vec{r}] - [r]\} = \widetilde{u} \cdot [\widetilde{r}] = \gamma[\vec{v} \cdot \vec{r} - r] = -\gamma\mathcal{S} \;, \quad (6.9)$$

which tells us that our important (retarded) denominator, $\mathcal{S}$, varies under a Lorentz transformation inversely to γ_v, so that their product is an invariant. We also note that the 4-vector created there, namely $[\widetilde{r}] \equiv ([\vec{r}], [r])^T$ is of course the 4-vector tangent to the path of the light ray that propagates from the retarded event to the field event. We could also use this, now, as a way to rewrite, yet one more time, the form of the electric field; however, it will be much more useful in a few minutes when we begin to consider the form for the electromagnetic 4-vector potential function, $\widetilde{A}$.

We begin in the frame S', where the charged particle is at rest; there we know that $\vec{A}$ is zero while $V' = kq/r'$. In that frame we also know that $\vec{v}$ is zero, which implies that $\widetilde{u}^4 = +1$, so that one could write $\widetilde{A} = V'\widetilde{u}$ and it would, at least, be true in this rest frame. On the other hand, this equality has the appearance of a "tensor equation," since there are 4-tensors on both sides, in which case it would be true in all frames. However, in order for this to actually be a tensor equation, the quantity V' must be a scalar. Let us consider that thought: Obviously k is a scalar, i.e., it is just a number. That the charge q is a scalar is an (experimentally-verified) assumption of our theory. On the other hand, r' is the same as the quantity we were considering in Eq. (6.7), where we showed that it was the same as $\mathcal{S}$, while we also showed that $\gamma\mathcal{S}$ is in fact an invariant. However, of course, $\gamma = 1$ in the rest- frame S' so that we may now postulate a correct tensor equation defining the 4-potential in any frame: For a particle with charge q moving with velocity $\vec{v}$, we have the following, in an arbitrary frame, where, of course V is the associated scalar potential and θ is the angle between $\vec{r}$ and $\vec{v}$:

$$\widetilde{A} = \frac{kq}{\gamma r'}\widetilde{u} = -\frac{kq}{\gamma\mathcal{S}}\widetilde{u} = -\frac{kq\,\widetilde{u}}{\widetilde{u} \cdot [\widetilde{r}]} = \frac{kq}{\gamma r\sqrt{1 - v^2\sin^2\theta}}\widetilde{u} = \frac{kq}{r\sqrt{1 - v^2\sin^2\theta}}\begin{pmatrix}\vec{v}\\1\end{pmatrix} = V\begin{pmatrix}\vec{v}\\1\end{pmatrix} . \quad (6.10)$$

Having “created” this 4-potential, it is worthwhile to actually prove that it does generate the desired electromagnetic tensor, Faraday. It is intended to do this calculation **without** assuming that the velocity is constant; nonetheless, it will turn out that the final result splits nicely into a part proportional to the velocity and another part proportional to the acceleration. Therefore, hopefully, for ease of understanding, this will be done in two parts: first, with the velocity still constant, and the calculation done in 3-space with fairly ordinary mathematical techniques of 3-vector analysis; and, then, the calculation will be redone, allowing the velocity to vary, with that calculation being done with techniques of 4-vector differential calculus and differential forms.

Considering now the case of constant velocity, we note that $\widetilde{A}$ is basically determined by the potential function, V, which contains r and $\sin^2\theta$; therefore, we need only seriously consider the calculation involving those quantities. Furthermore, we only to calculate ∇V since, from Eqs. (5.10), the two quantities needed may be written as

$$\nabla\times\vec{A} = \nabla\times V\vec{v} = -\vec{v}\times\nabla V\ ,\quad \dot{\vec{A}} = -\vec{v}\cdot\nabla\vec{A} = -\vec{v}(\vec{v}\cdot\nabla V) = -v^2(\hat{v}\cdot\nabla V)\hat{v}\ , \tag{6.11a}$$

where the calculation with the time derivative notes that V depends only on $[\vec{r}] = \vec{r}_0 - \vec{r}_q(t)$ which depends on t only through the motion of the charged particle. This tells us that

$$\frac{\partial V([\vec{r}])}{\partial t} = \frac{d[\vec{r}]}{dt}\cdot\frac{\partial V}{\partial[\vec{r}]} = -\vec{v}\cdot\nabla V\ . \tag{6.11b}$$

Therefore we proceed as follows:

$$\begin{aligned}
\nabla\vec{r} = \mathbf{I}_3\ ,\ \nabla r = \hat{r}\ ,\ \nabla\hat{r} = \frac{\mathbf{I}_3 - \hat{r}\hat{r}^T}{r} \quad &\Longrightarrow \quad \nabla\cos\theta = \hat{v}\cdot\nabla\hat{r} = \frac{\hat{v} - \cos\theta\,\hat{r}}{r}\ ,\\
\nabla\sin^2\theta = \nabla(1-\cos^2\theta)\ &= -\nabla\cos^2\theta = -2\cos\theta\nabla\cos\theta\ ,\\
\Longrightarrow\quad \nabla(r\sqrt{1-v^2\sin^2\theta}) &= \sqrt{1-v^2\sin^2\theta}\,\hat{r} + \tfrac{1}{2}r\frac{(-v^2)(-2\cos\theta)(\hat{v}-\cos\theta\,\hat{r})}{r\sqrt{1-v^2\sin^2\theta}}\\
\Longrightarrow\quad \nabla V &= \frac{-kq}{r^2(1-v^2\sin^2\theta)^{3/2}}\left\{(1-v^2\sin^2\theta)\hat{r} - v^2\cos^2\theta\,\hat{r} + v^2\cos\theta\,\hat{v}\right\}\\
&= -kq\frac{(1-v^2)\hat{r} + v^2\cos\theta\,\hat{v}}{r^2(1-v^2\sin^2\theta)^{3/2}}\ .
\end{aligned} \tag{6.12a}$$

Therefore,

$$\dot{\vec{A}} = -v^2(\hat{v}\cdot\nabla V)\hat{v} = v^2\frac{\{(1-v^2)\cos\theta + v^2\cos^2\theta}{r^2(1-v^2\sin^2\theta)^{3/2}} \ , \tag{6.12b}$$

which gives us an expression for $\vec{E}$ where the terms in $\hat{v}$ all cancel so that we obtain the expected results:

$$\vec{E} = kq\frac{(1-v^2)\hat{r}}{r^2(1-v^2\sin^2\theta)^{3/2}} \quad , \qquad \vec{B} = \vec{v}\times\vec{E} \ .$$

To proceed further, I propose to work with the 4-vector formulation of $\widetilde{A}$ as given in Eq. (6.10). [I should note that this discussion follows that in Anderson's text, *The Electromagnetic Field.*] The spatial and temporal derivatives of $\widetilde{A}$ are needed, as shown in Eq. (5.10), to determine the electric and magnetic fields. This derivatives are the rates of change as we vary (the spatial and temporal parts of) the field event, which we have labelled as $\widetilde{r}_0$. Therefore, although we will simply write ∇ and $\partial/\partial t$, we really mean something more like ∇_0 and $\partial/\partial t_0$; this distinction will occasionally be of some importance to see our way clearly. In addition, since $\widetilde{A}$ really depends both on $\widetilde{r}_0$ and on $\widetilde{r}_q([t])$, we have to pay careful attention to the fact that $\widetilde{r}_q([t])$ will vary as we vary $\widetilde{r}_0$; therefore, we will exercise some care below to determine the value of the derivatives of the one with respect to the other.

It is also worth noting that, if we intend to let the velocity vary, at the field event, the only velocity we can know, as propagated to us by some device which can move no faster than does light, is the velocity of the particle back at the retarded event, so that the velocity in our equations will now be denoted by $[\vec{v}\,]$.

The goal of the next several paragraphs is now to determine the derivative of $\widetilde{r}_q([t])$ with respect to $\widetilde{r}_0$. Proceeding ahead we first note that $[\vec{r}\,]$ depends on time only through $[t]$, so we may write

$$\frac{\partial[r]}{\partial t} = \frac{\partial|[\vec{r}\,]|}{\partial t} = \left(\frac{d[t]}{dt}\right)\left(\frac{\partial\sqrt{[\vec{r}\,]^2}}{\partial[t]}\right) = -\left(\frac{d[t]}{dt}\right)\left(\frac{-[\vec{v}\,]\cdot[\vec{r}\,]}{[r]}\right) . \tag{6.13a}$$

However, we also know, because of the transmission via light rays, that

$$t-[t]=[r] \quad \Longrightarrow \quad \frac{d[t]}{dt} = 1-\frac{\partial[r]}{\partial t} \ . \tag{6.13b}$$

This gives us two linear equations for the two unknowns, $d[t]/dt$ and $\partial[r]/\partial[t]$, which may be solved immediately to obtain

$$\begin{aligned} \frac{\partial[r]}{\partial t} &= \frac{L}{L-1} , \quad \frac{d[t]}{dt} = \frac{1}{L-1} , \\ \text{where } L \equiv \frac{[\vec{v}]\cdot[\vec{r}]}{[r]} &\Longrightarrow 1 - L = \frac{1}{[r]}\{[r] - [\vec{v}]\cdot[\vec{r}]\} = -\frac{[\widetilde{u}]\cdot[\widetilde{r}]}{\gamma[r]} = \frac{\mathcal{S}}{[r]} , \\ \Longrightarrow \quad \frac{d[t]}{dt} &= \frac{[r]}{\mathcal{S}} , \quad \text{and} \quad \frac{\partial[r]}{\partial t} = -\frac{[\vec{v}]\cdot[\vec{r}]}{\mathcal{S}} . \end{aligned} \tag{6.15}$$

Now, since $\vec{r}_q([t])$ depends on $[t]$, which depends on $\vec{r}_0$ and therefore on $\vec{r}$, while $[\vec{r}]$ depends on $\vec{r}$ directly and also through $\vec{r}_q([t])$, we need to calculate appropriate gradients of our quantities as well:

$$\begin{aligned} \nabla[t] &= \nabla(t - [r]) = -\nabla[r] , \\ \nabla[r] &= \frac{[\vec{r}]}{[r]} + (\nabla[t])\frac{\partial[r]}{\partial[t]} = -(\nabla[t])\frac{[\vec{v}]\cdot[\vec{r}]}{[r]} , \\ \Longrightarrow \quad \nabla[r] = \frac{[\vec{r}]}{[r]-[\vec{v}]\cdot[\vec{r}]} &= \frac{[\vec{r}]}{\mathcal{S}} = -\nabla[t] \text{and} \quad \nabla[\vec{r}] = \mathbf{I}_3 + (\nabla[t])\frac{\partial[\vec{r}]}{\partial[t]} = \mathbf{I}_3 + \frac{[\vec{r}][\vec{v}]^T}{\mathcal{S}} , \\ \frac{\partial[\vec{r}]}{\partial t} &= \frac{\partial(\vec{r}_0 - \vec{r}_q([t]))}{\partial t} = -\left(\frac{\partial[t]}{\partial t}\right)\frac{[r]}{\mathcal{S}} , \end{aligned} \tag{6.16}$$

where in the very last calculation above, of course ∇ really means the derivative as you change the location of the field point, only, i.e., ∇_0, so that $\partial/\partial t$ requires that $\vec{r}_0$ is held fixed.

We now want to summarize the above in a useful form. This is most easily done by using the notation $\partial \equiv (\nabla, \partial/\partial t) \Longrightarrow \partial_\mu \equiv \partial/\partial x^\mu$ to indicate the 4-dimensional gradient. (Note that the exterior derivative may be written in the form $d = dx^\mu \partial_{x^\mu}$ so that we are actually dealing with the components of the "1-form-like operator," d.) We have

$$\begin{aligned} \partial[\widetilde{r}]^T &= \begin{pmatrix} \mathbf{I}_3 + [\vec{r}\vec{v}^T]/\mathcal{S} & [\vec{r}]/\mathcal{S} \\ -[r\vec{v}]/\mathcal{S} & 1 - [r]/\mathcal{S} \end{pmatrix} = \mathbf{I}_4 - \frac{[\widetilde{r}][\widetilde{u}]^T}{[\ \widetilde{r}]\cdot[\widetilde{u}]} , \\ \text{or} \quad \partial_\mu[r^\nu] &= \delta^\mu_\nu + \frac{[r_\mu u^\nu]}{\gamma\mathcal{S}} = \delta^\mu_\nu - \frac{[r_\mu u^\nu]}{[r_\lambda u^\lambda]} . \end{aligned} \tag{6.17}$$

Since $[\widetilde{r}] = \widetilde{r}_0 - \widetilde{r}_q([t])$, and our derivatives are actually rates of change as $\widetilde{r}_0$ varies, we would expect the result, above, to be of the form $\delta^\mu_\nu - \partial_\mu[r^\nu]$, which allows us to identify a quantity we will soon need explicitly, namely

$$\partial_\mu[(\widetilde{r})^\nu_q] = \frac{[r_\mu u^\nu]}{[r_\eta u^\eta]} \ . \tag{6.18}$$

Now we have enough background to calculate the derivatives of the 4-potential, $\widetilde{A} = \widetilde{A}(\widetilde{r}_0, \widetilde{r}_q([t]))$, where, as well, $[t] = [t](\widetilde{r}_0)$. As already noted, the derivative of $\widetilde{A}$ contains two terms:

$$\partial_\mu \widetilde{A} = \left(\partial_\mu \widetilde{A}\right)_{\tilde{r}_q \text{constant}} + (\partial_\mu [\widetilde{r}^\nu_q]) \left(\frac{\partial}{\partial(\widetilde{r}_q)^\nu}\right) \widetilde{A} \ . \tag{6.19}$$

When Eq. (6.18) is inserted into the second term above we have a part of it that contains

$$u^\nu \frac{\partial}{\partial r^\nu_q} = \left(\frac{dr^\nu_q}{d\tau}\right)\left(\frac{\partial}{\partial r^\nu_q}\right) = \frac{d}{d\tau} \ , \tag{6.20}$$

where τ is the proper time measured along the trajectory of the charged particle. As well, we easily see that the first term of Eq. (6.19) is

$$\partial_{r^\mu_0}\left(\frac{-kq[\widetilde{u}]}{[\widetilde{u}\cdot(\widetilde{r}_0 - \widetilde{r}_q)]}\right) = \frac{kq[\widetilde{u}][u_\mu]}{[\widetilde{u}\cdot(\widetilde{r}_0 - \widetilde{r}_q)]^2} \ . \tag{6.21}$$

Therefore, we can put it all together to obtain a fairly simple formula for the (4-dimensional) gradient of $\widetilde{A}$:

$$\partial_\mu \widetilde{A} = kq \left\{ \frac{[\widetilde{u} u_\mu]}{[\widetilde{u}\cdot\widetilde{r}]^2} - \frac{[r_\mu]}{[\widetilde{u}\cdot\widetilde{r}]} \frac{d}{d[\tau]} \left(\frac{[\widetilde{u}]}{[\widetilde{u}\cdot\widetilde{r}]}\right) \right\} = \frac{-kq}{[\widetilde{u}\cdot\widetilde{r}]} \frac{d}{d[\tau]} \frac{[r_\mu \, \widetilde{u}]}{[\widetilde{u}\cdot\widetilde{r}]} \ , \tag{6.22}$$

where the last equality comes from $(d/d[\tau])[\widetilde{r}] = (d/d[\tau])(\widetilde{r}_0 - [\widetilde{r}_q]) = -[\widetilde{u}]$.

At last we may now take the skew part of this quantity, i.e., use it to calculate $d\underset{\sim}{A}$, which is the Faraday:

$$\begin{aligned} (\underset{\sim}{F})_{\mu\nu} &= (d\underset{\sim}{A})_{\mu\nu} = \frac{kq}{[\gamma\mathcal{S}]} \frac{d}{d[\tau]} \frac{[r_\mu u_\nu - r_\nu u_\mu]}{[\widetilde{u}\cdot\widetilde{r}]} \ , \\ \text{or } \ \underset{\sim}{F} &= d\underset{\sim}{A} = \frac{kq}{[\gamma\mathcal{S}]} \frac{d}{d[\tau]} \frac{[\underset{\sim}{r} \wedge \underset{\sim}{u}]}{[\widetilde{r}\cdot\widetilde{u}]} = \left[\frac{kq}{\gamma\mathcal{S}} \frac{d}{d\tau} \frac{\underset{\sim}{r}\wedge\underset{\sim}{u}}{\widetilde{r}\cdot\widetilde{u}}\right] \ . \end{aligned} \tag{6.23}$$

Although this is indeed a rather nice, simple form, now we must actually calculate the derivative along the worldline. We know that $[d\widetilde{r}/d\tau] = -[\widetilde{u}]$ and that $[d\widetilde{u}/d\tau] \equiv [\widetilde{a}]$, so we obtain

$$\begin{aligned} \underset{\sim}{F} &= \frac{kq}{[\gamma\mathcal{S}]} \left\{ \frac{-[\underset{\sim}{u}\wedge\underset{\sim}{u}] + [\underset{\sim}{r}\wedge\underset{\sim}{a}]}{[\widetilde{r}\cdot\widetilde{u}]} - \frac{[\underset{\sim}{r}\wedge\underset{\sim}{u}]}{[\widetilde{r}\cdot\widetilde{u}]^2}\{-\widetilde{u}\cdot\widetilde{u} + \widetilde{r}\cdot\widetilde{a}\} \right\} \\ &= -kq \left[\frac{\gamma\mathcal{S}\,\underset{\sim}{r}\wedge\underset{\sim}{a} + (1 + \widetilde{r}\cdot\widetilde{a})\,\underset{\sim}{r}\wedge\underset{\sim}{u}}{(\gamma\mathcal{S})^3} \right] \ . \end{aligned} \tag{6.24}$$

To study this further, we note that, when the value of $\gamma\mathcal{S}$ is inserted into the numerator, it splits easily into two parts, one which exists even when the acceleration is zero, i.e., when the velocity is constant, and the other which is linear in the 4-acceleration:

$$\begin{aligned} \underset{\sim}{F} &\equiv \underset{\sim}{F}_{\rm vel} + \underset{\sim}{F}_{\rm acc} \ , \\ \underset{\sim}{F}_{\rm vel} &= \ -kq \left[\frac{\underset{\sim}{r} \wedge \underset{\sim}{u}}{(\gamma\mathcal{S})^3}\right] \ , \qquad\qquad \gamma\mathcal{S} = -\widetilde{u}\cdot\widetilde{r} \ , \\ \underset{\sim}{F}_{\rm acc} &= \ -kq \left[\frac{\widetilde{r}\cdot(\underset{\sim}{u}\underset{\sim}{a} - \underset{\sim}{a}\underset{\sim}{u})\wedge\underset{\sim}{r}}{(\gamma\mathcal{S})^3}\right] = -kq \left[\frac{\widetilde{r}\cdot(\underset{\sim}{u}\wedge\underset{\sim}{a})\wedge\underset{\sim}{r}}{(\gamma\mathcal{S})^3}\right] \ . \end{aligned} \tag{6.25}$$

In the process of doing this we have defined the "dot product" of a vector and a 1-form, and even the dot product of a vector and a 2-form. The dot product of a vector and a 1-form is of course the same as the action of that 1-form on that vector, or any number of ways of saying the same thing: $\widetilde{r}\cdot\underset{\sim}{u} \equiv \underset{\sim}{u}(\widetilde{r}) \equiv u_\mu\, r^\mu$. This sort of action is then extended to the 2-form by remembering that it is the skew-symmetric sum of the tensor product of two 1-forms; therefore, in each one of those two parts we allow the 1-form "nearest" to the vector to "contract its indices" with that vector, i.e.,

$$\widetilde{\alpha}\rfloor(\underset{\sim}{\beta}\wedge\underset{\sim}{\gamma}) \equiv (\underset{\sim}{\beta}(\widetilde{\alpha}))\underset{\sim}{\gamma} - (\underset{\sim}{\gamma}(\widetilde{\alpha}))\underset{\sim}{\beta} = (\beta_\mu\alpha^\mu)\underset{\sim}{\gamma} - (\gamma_\mu\alpha^\mu)\underset{\sim}{\beta} \equiv \widetilde{\alpha}\cdot(\underset{\sim}{\beta}\wedge\underset{\sim}{\gamma}) \ . \tag{6.26}$$

This is more often denoted as a "step product" because of the symbol on the far left.

Having this very simple form, we can first notice that the two powers of r in the numerator cause this field to decrease only as $1/r$ as the magnitude of r becomes very large, which is what we expect from a "radiation field," i.e., a field that manages to "survive" at very large distances when integrated over an entire sphere. However, it is also probably useful to pull out the electric and magnetic contributions separately from this form. To do this we must recall the $3+1$-decomposition of several different 4-dimensional quantities:

$$u_i = \gamma\, v_i \ \ u_4 = -\gamma \ , \quad [r_i] = [\vec{r}]_i \ , [r_4] = -[r] \ , \quad a_i = \gamma^2\vec{a}_i - a_4 v_i \ , \ a_4 = -\gamma^4(\vec{a}\cdot\vec{v}) \ ,$$

$$B^k = \tfrac{1}{2}\eta^{ijk}F_{ij} \ , \quad E_i = \tfrac{1}{2}(F_{i4} - F_{4i}) \ ,$$

$$\Longrightarrow \quad [\widetilde{r}\cdot\widetilde{a}] = [\gamma^2(\vec{r}\cdot\vec{a} - \gamma^2\vec{v}\cdot\vec{a}\,\mathcal{S})] \ .$$

We may then pull out the part of the Faraday, in Eq. (6.25), that we need for the electric field:

$$
\begin{aligned}
\vec{E}_{\rm acc} &= -\frac{kq}{[\gamma\mathcal{S}]^3}\left[-\gamma^5\mathcal{S}(\vec{a}\cdot\vec{v})\vec{r}+\gamma^3\mathcal{S}r\vec{a}+\gamma^5\mathcal{S}r(\vec{a}\cdot\vec{v})\vec{v}+\gamma^3(\vec{a}\cdot\vec{r}-\mathcal{S}\gamma^2\vec{a}\cdot\vec{v})(-\vec{r}+r\vec{v})\right] \\
&= \frac{kq}{\mathcal{S}^3}[(\vec{a}\cdot\vec{r})\vec{r}-\mathcal{S}r\vec{a}-r(\vec{a}\cdot\vec{r})\vec{v}] = \frac{kq}{\mathcal{S}^3}[\vec{r}\times\{(\vec{r}-r\vec{v})\times\vec{a}\}] \\
\vec{E} = \vec{E}_{\rm vel}+\vec{E}_{\rm acc} &= \frac{kq}{\mathcal{S}^3}\Big\{[1-v^2+\vec{a}\cdot\vec{r}]\Big\}\,\vec{r}-\frac{kq}{\mathcal{S}^2}\,[r\vec{a}]\,, \qquad \mathcal{S}\equiv r' = r\sqrt{1-v^2\sin^2\theta}\,.
\end{aligned} \tag{6.27}
$$

Remembering that $[\vec{r}-r\vec{v}]=\vec{r}$, the vector pointing toward the field point from that point where the charged particle would have been if it had travelled from the retarded event to the current time with the velocity it had then, we see that there is indeed another contribution to the electric field in that same direction—although proportional to the acceleration—and also a contribution in the direction of that acceleration. As well we can even more easily see that the magnitude of the accelerated portion of the field does indeed decrease like kq/r as r becomes very large while the non-accelerated portion of course decreases like kq/r^2, all as expected.

Now we want to determine the magnetic field associated with this form of the Faraday:

$$
\begin{aligned}
(\vec{B}_{\rm acc})^k &= -\frac{kq}{[\gamma\mathcal{S}]^3}\,\eta^{kij}\left[\gamma^3\mathcal{S}r_i(a_j+\gamma^2(\vec{a}\cdot\vec{v})v_j)+\gamma^3(\vec{a}\cdot\vec{r}-\gamma^2\mathcal{S}\,\vec{a}\cdot\vec{v})\,r_iv_j\right] \\
&= -\frac{kq}{\mathcal{S}^3}\Big[\mathcal{S}\,\vec{r}\times\vec{a}+(\vec{a}\cdot\vec{r})\,\vec{r}\times\vec{v}\Big] \\
\text{or}\quad \vec{B}_{\rm acc} &= [\hat{r}]\times\vec{E}_{\rm acc} = -\frac{kq}{\mathcal{S}^3}\Big[\mathcal{S}\,\vec{r}\times\vec{a}+(\vec{a}\cdot\vec{r})\,\vec{r}\times\vec{v}\Big]\,, \\
\text{or}\quad \vec{B} &= \vec{B}_{\rm vel}+\vec{B}_{\rm acc} = [\hat{r}]\times\vec{E}\,.
\end{aligned} \tag{6.28}
$$

where we had to use a cross-product identity to justify the form above, namely $-[\hat{r}]\times[(\vec{a}\cdot\vec{r})\,\vec{r}-\mathcal{S}\,r\vec{a}-r(\vec{a}\cdot\vec{r})\,\vec{v}]=[\mathcal{S}\,\vec{r}\times\vec{a}+(\vec{a}\cdot\vec{r})\vec{r}\times\vec{v}]$. We see immediately that the magnetic field of the moving monopole charge is always perpendicular to the electric field, as expected since there is no magnetic field in the rest frame which causes the invariant $\vec{E}\cdot\vec{B}$ to vanish. Since $\vec{B}_{\rm acc}$ is perpendicular to $[\vec{r}]$, the other invariant also vanishes for the accelerated fields, i.e., $\vec{B}^2_{\rm acc}-\vec{E}^2_{\rm acc}=0$, as one would expect for the fields far from the source, trying to behave like true radiation fields.