Lie Algebra for the Poincaré, and Lorentz, Groups

I. Relations for the Groups themselves

The Poincaré group is the set of all Lorentz transformations and changes of origin in 4-dimensional spacetime, also referred to as the affine transformations of spacetime. We describe them in a straightforward way by giving their action on the location 4-vector, $\widetilde{x}$. The 4-vector $\widetilde{x}$ has components x^α as measured in some frame $\mathcal{O}$ while in an alternate frame, possibly also with a different choice of origin, $\mathcal{O}'$, it has components x'^μ, where of course the two reference frames have agreed to utilize the same choice of basis vectors, which for the moment we suppose to be simply Cartesian ones, also referred to as Minkowski basis vectors: $\{\hat{x}, \hat{y}, \hat{z}, \hat{t}\}$. These two reference frames are related via a Lorentz transformation L_1 and change of origin $\widetilde{a}_1$, most easily described by explaining the relationship between their two distinct sets of components for $\widetilde{x}$:

$$x'^\mu = (L_1)^\mu{}_\alpha x^\alpha + a_1^\mu \ . \tag{1.1}$$

If now we suppose some third frame, $\mathcal{O}''$, also possibly with a different choice of origin, who measures location with the components x''^τ, and is related to $\mathcal{O}'$ by a Lorentz transformation L_2 and change of origin $\widetilde{a}_2$, we may write the following:

$$x''^\tau = (L_2)^\tau{}_\mu x'^\mu + a_2^\tau = (L_2)^\tau{}_\mu \left[(L_1)^\mu{}_\alpha x^\alpha + a_1^\mu\right] + a_2^\tau = (L_2 L_1)^\tau{}_\alpha x^\alpha + \left[(L_1)^\tau{}_\mu a_1^\mu + a_2^\tau\right] \ . \tag{1.2}$$

We see that this multiple transformation has the same generic form as a single Poincaré transformation, which would take us directly from $\mathcal{O}$ to $\mathcal{O}''$, with their location measurements related as follows:

$$\begin{gathered} x''^\tau = (L_{21})^\tau{}_\alpha x^\alpha + a_{21}^\tau \ , \\ L_{21} = L_2 L_1 \ , \qquad \widetilde{a}_{21} = L_2 \widetilde{a}_1 + \widetilde{a}_2 \ . \end{gathered} \tag{1.3}$$

This is all sensible, i.e., this product is again a Poincaré transformation since the product $L_2 L_1$ is of course again a Lorentz transformation, and the action of L_2 on a 4-vector is a 4-vector while of course the sum of two 4-vectors is a 4-vector. It is however useful to invent some simpler notation to describe the elements of this group themselves, via their parameters L and

$\widetilde{a}$, rather than demonstrating explicitly their action on (location) 4-vectors over spacetime. Therefore we agree to denote an element of this group by the symbols: $\{L, \widetilde{a}\}$. As well we see from the example above that the "multiplication" of two of these group elements is simply what occurred when we allowed two of them to act successively on the location 4-vector; in what follows, for the elements themselves as labeled above, we will denote multiplication simply by putting them adjacent to one another. In the transcription below for that multiplication, I am switching the labels 1 and 2 only so that then they are in a "more natural" order:

$$\{L_1, \widetilde{a}_1\}\{L_2, \widetilde{a}_2\} = \{L_1 L_2, L_1\widetilde{a}_2 + \widetilde{a}_1\} \ . \tag{1.4}$$

Do please notice that this is rather a more complicated instance of "a group multiplication" than we have had before, when all we were really considering was matrix groups. A group of course should have an identity operation and an inverse for each of its elements. The identity is "obvious," namely one simply does not adjust the origin and does not perform any transformation, so that the identity is symbolized by $\{\mathbf{I}_4, \widetilde{0}\}$. We may then consider the product above for the special case when the left-hand side is the product of an element with its inverse, and the right-hand side is this identity, which gives us the following rule to determine the inverse:

$$\{L, \widetilde{a}\}^{-1} = \{L^{-1}, -L^{-1}\widetilde{a}\} \ . \tag{1.5}$$

An interesting simple way to look at these groups that include both matrix operations with a fixed origin and a translation of that origin can be presented via a 5-dimensional matrix representation, that acts on 5-dimensional vectors related in a simple way to the 4-dimensional vectors observed above. We first inject our 4-dimensional spacetime into a 5-dimensional vector space by simply appending a 1 as the 5-th component. We pick an arbitrary location 4-vector, $\widetilde{x}$, and then represent it by the following 5-dimensional column matrix/vector:

$$x^\mu \ \Longrightarrow \ \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ 1 \end{pmatrix} \ . \tag{1.6a}$$

We then represent the Poincaré group element by a 5×5 matrix:

$$\{L, \widetilde{a}\} \Longrightarrow \begin{pmatrix} L & \widetilde{a} \\ \widetilde{0}^T & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} L & \widetilde{a} \\ \widetilde{0}^T & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ 1 \end{pmatrix} = \begin{pmatrix} L\widetilde{x} + \widetilde{a} \\ 1 \end{pmatrix} . \tag{1.6b}$$

To continue it is useful to summarize what we already know about Lorentz transformations themselves. Recall that the Lorentz transformation relates to reference frames (with the same choice of origin) which are related to one another either by some rotation of the spatial coordinates or by having a non-zero relative velocity between them, or both. This generic statement is most easily summarized by saying that there is a 4-dimensional scalar product for 4-vectors which has the same value when measured by any of the reference frames related by Lorentz transformations—often called "inertial reference frames." Therefore a Lorentz transformation is to be originally perceived as an operator transforming measurements of some 4-vector between two reference frames, keeping the scalar product of that vector with itself invariant:

$$\begin{gathered} (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 \equiv \vec{x}^{\,2} - (x^4)^2 \equiv \widetilde{x} \cdot \widetilde{x} \equiv \eta_{\mu\nu}\, x^\mu\, x^\nu \equiv \widetilde{x}^T \eta \widetilde{x} \; ; \\ L \in SO(3,1) \quad \Longleftrightarrow \quad L^T \eta L = \eta \;\text{ or }\; L^\mu{}_\alpha \eta_{\mu\nu} L^\nu{}_\beta = \eta_{\alpha\beta} \; , \\ \Longrightarrow \quad L^T \eta = \eta L^{-1} \quad \Longrightarrow \quad L^{-1} = \eta^{-1} L^T \eta \quad \text{or} \quad (L^{-1})^\beta{}_\mu = \eta^{\beta\gamma} L^\nu{}_\gamma \eta_{\nu\mu} = L_\mu{}^\beta \; , \end{gathered} \tag{1.7}$$

where we have used $\widetilde{x}$ to denote **both** the actual 4-vector and the matrix which has its components as elements. As well we have used $\eta = ((\eta_{\alpha\beta}))$ to denote the components of the metric tensor, introducing a notation to show the relationship between the set of elements of a matrix and a name for the entire matrix itself, which notation allows us also to note the tensor character of the elements of that matrix under some transformation of coordinates. Here the fact that η is a type $\left[{0 \atop 2}\right]$ tensor can be seen easily when we write it in terms of its components with an indication as to whether they transform in a contravariant way—upper indices—or a covariant way—lower indices. On the last line we have also solved the constraining equation for L in a way to give an explicit expression for its matrix inverse in terms of itself and the metric, η, and have also used the additional property ascribed to η, of raising and lowering indices, i.e., changing the transform type, to create a useful, simple form for the components of that inverse, namely the following, a formula worth repeating:

$$(L^{-1})^\beta{}_\mu = L_\mu{}^\beta \equiv \eta^{\beta\gamma} L^\nu{}_\gamma \eta_{\nu\mu} \ . \tag{1.7$'$}$$

Lastly we have also introduced a standard notation for the Lorentz group, namely SO(3,1):

a. the symbol O means that it is an "orthogonal" group, i.e., one that preserves some scalar product,

b. the entries (3,1) tell us that the scalar product in question has 3 positive eigenvalues and 1 negative one, and

c. the symbol S in the front tells us the matrices in question are "special," which, from "olden times," means that they have determinant +1.

For the purposes of studying Lie groups it is usual to attempt to characterize them in terms of their behavior when they are not much different from no transformation at all, i.e., not much different from the identity matrix. If we suppose that one may find a matrix which is a logarithm of the given matrix then we may use that to acquire this characterization since the exponential of the logarithm is of course the original quantity, but the exponential also is just the (infinite) series away from the identity which has the first term as that logarithm. The set of all possible such logarithms has the character of a Lie algebra, and from a physical point of view are often of considerable interest. Therefore, we suppose that we may first consider some specific Lorentz transformation L that is "not too far away" from the identity transformation and write it in terms of its logarithm, which we refer to as Q. We then enter than assignment into the requirements above for a Lorentz transformation, to determine which form that requirement takes when considered as a constraint on the logarithm matrix, i.e., as a constraint on the corresponding Lie algebra:

$$L = e^Q \text{ or, in terms of components, } L^\mu{}_\alpha = e^{Q^\mu{}_\alpha} \quad \Longrightarrow \quad e^{Q^T} \eta e^Q = \eta$$

$$\Longrightarrow \quad e^{Q^T} = \eta\, e^{-Q} \eta^{-1} = e^{-\eta Q \eta^{-1}} \Longrightarrow \quad Q^T = -\eta Q \eta^{-1} \quad \Longrightarrow \quad Q^T \eta = -\eta Q \tag{1.8}$$

$$\text{or, again in terms of components,} \quad Q^\mu{}_\alpha \eta_{\mu\beta} = -\eta_{\alpha\mu} Q^\mu{}_\beta \quad \Longleftrightarrow \quad Q_{\mu\alpha} = -Q_{\alpha\mu} \ .$$

This says that the matrix Q may be any matrix whatsoever that is skew-symmetric when it has had both its indices moved to the same level, i.e., when it is being considered as a tensor of type $\left[{0 \atop 2}\right]$. Such a set of matrices is a 6-dimensional vector space since one may choose 6 linearly-independent skew-symmetric matrices, as a basis set, and then take any linear combinations of them that are desired, multiplying by scalar quantities.

We begin by setting up our basis for skew-symmetric matrices; there are 6 linearly independent such matrices, so we could simply number them from 1 to 6, but a notation that is more reasonable is to label them via some skew-symmetric labeling scheme, making the skew-symmetry more immediate. Therefore, let us suppose that we pick out a set of 6 distinct, linearly-independent matrices, and label them by the symbols $\{\mathcal{J}^{\alpha\beta} = -\mathcal{J}^{\beta\alpha} \mid \alpha, \beta = 1, 2, 3, 4\}$. Each one of these is of course a matrix, and must actually be skew-symmetric; therefore, we must also describe a choice for their elements—dependent of course on the basis set for vectors that we are using. We continue with our choice of a Cartesian basis set and choose the simplest possible matrix presentation:

$$(\mathcal{J}^{\alpha\beta})_{\mu\nu} \equiv \eta_{\mu\lambda}(\mathcal{J}^{\alpha\beta})^{\lambda}{}_{\nu} \equiv -\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\right) = -2\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \ . \tag{1.9}$$

For example, the explicit presentation of two of these are the following:

$$\mathcal{J}^{12} = (((\mathcal{J}^{12})_{\mu\nu})) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \quad \mathcal{J}^{34} = (((\mathcal{J}^{34})_{\mu\nu})) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix} \ . \tag{1.10}$$

Having the basis set our general form for an arbitrary logarithm, Q, is just some arbitrary linear combination of all of these matrices. As they are labeled by their skew-symmetric pair of indices it is reasonable to also label their scalar coefficients in the same way; therefore, we choose a set of 6 scalar quantities, which are to be thought of as the 6 independent parameters that describe an arbitrary Lorentz transformation, $\{q_{\alpha\beta} = -q_{\beta\alpha} \mid \alpha, \beta = 1, 2, 3, 4\}$, such that

$$L = e^{Q} \ , \qquad Q = \tfrac{1}{2} q_{\beta\alpha}\, \mathcal{J}^{\alpha\beta} \ , \tag{1.11}$$

where the index order on the two entries has been chosen in the way that is most appropriate for matrix multiplication, while the factor 1/2 is simply because the double sum counts things twice because both sets of quantities are skew-symmetric in their indices.

While the method above is quite useful, at least in part because it is explicitly 4-dimensional at every possible place, we have better intuition about a breakdown of the 4 dimensions into 3 spatial ones and 1 temporal one. Therefore it is useful to divide these 6 basis matrices in a different way. We define $\{\mathfrak{J}_i\}_{i=1}^3$ and $\{\mathfrak{K}^j\}_{j=1}^3$ so that the following statements are true:

$$\mathfrak{J}^{ij} = \eta^{ijk}\mathfrak{J}_k = \eta^{ijk4}\mathfrak{J}_k \; , \qquad \mathfrak{J}^{4i} = \mathfrak{K}^i \; . \tag{1.11}$$

The following are a couple of examples, in fact the same examples as shown above in Eqs. (1.10), but where we have raised the matrix index for the row index—the left one—so that we have the form of an actual matrix that will act on 4-vectors:

$$\mathfrak{J}_z = (((\mathfrak{J}_z)^\mu{}_\beta)) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; , \quad \mathfrak{K}^3 = \left(((\mathfrak{K}^3)^\mu{}_\beta)\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; . \tag{1.12}$$

In this form we may then choose the parameters themselves, $q_{\alpha\beta}$, so that their physical meaning is clear. Working from Eqs. (1.11) we have the form of the most general logarithm matrix, Q, in terms of the 6 parameters that describe an arbitrary Lorentz transformation:

$$Q = \tfrac{1}{2} q_{\beta\alpha}\mathfrak{J}^{\alpha\beta} = \theta\hat{n}\cdot\vec{\mathfrak{J}} - \lambda\hat{v}\cdot\vec{\mathfrak{K}} \; , \qquad q_{ij} = -\theta\,\eta_{ijk}\hat{n}^k \; , \; q_{4i} = -\lambda\,\hat{v}_i \; . \tag{1.13}$$

It is worthwhile now to take this attitude back to the entire Poincaré group. For the moment the simplest approach is to think of it in terms of the 5×5 matrices, so that we may create a set of generators also for the translations. In terms of those 5×5 matrices, we define basis matrices for translations as

$$\mathfrak{P}^1 \Longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \mathfrak{P}^2 \Longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \mathfrak{P}^3 \Longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \mathfrak{P}^4 \Longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \tag{1.14}$$

which allows us to think of some matrix representation—not yet described—of the Poincaré group in terms of its generators:

$$\{L, \widetilde{a}\} \implies \boldsymbol{\mathcal{M}} = e^W \ , \quad W = \tfrac{1}{2} q_{\beta\alpha} \boldsymbol{\mathcal{J}}^{\alpha\beta} + a_\gamma \boldsymbol{\mathcal{P}}^\gamma \ . \tag{1.15}$$

II. Representations for the Groups, and their Generators

We now want a way to determine such a matrix representation of this affine group. As it turns out, a useful way to begin this quest is to first consider the action of the group on itself. In order to determine that we first suppose that, yes, we do have a matrix representation of the group on some space of objects and we want to determine how that representation will treat linear operators, i.e., matrices, on that same space of objects. Obviously, since these objects have matrices acting on them, they will be vectors, in a vector space of some dimension; therefore, we will first consider given two arbitrary such vectors, $\widetilde{Z}$ and $\widetilde{A}$, related by some matrix X. We then write this relationship as observed in our two standard frames, $\mathcal{O}'$and $\mathcal{O}$, where these two frames are related by one of our Poincaré transformations, where we label its matrix representation, for the moment, as $U(L, \widetilde{a})$:

$$\begin{aligned} \widetilde{Z} = X\widetilde{A} \ , \qquad \widetilde{Z}' = X'\widetilde{A}' \ , \\ \widetilde{Z}' = U\widetilde{Z} \ , \qquad \widetilde{A}' = U\widetilde{A} \ . \end{aligned} \tag{2.1}$$

We now put all this algebra together, intending to determine the relationship between the two observations of the operator, i.e., between X and X':

$$\begin{aligned} UX\widetilde{A} = U\widetilde{Z} = \widetilde{Z}' \ = X'\widetilde{A}' = X'U\widetilde{A} \\ \implies \quad X' = UXU^{-1} \quad \text{or} \quad X'^{\mu}{}_{\nu} = U^{\mu}{}_{\alpha} X^{\alpha}{}_{\beta} U^{-1\beta}{}_{\nu} \ , \end{aligned} \tag{2.2}$$

which answers the question as to how linear operators on the space of vectors transform themselves, i.e., with a transformation that is often referred to as a *"similarity transformation."*

With that information in hand, we now consider the particular case where we are considering the action of the group on itself, by taking W to be a representation of some arbitrary element of the Poincaré group, i.e, $U(L,\widetilde{a})$, and $U \equiv U(L_0,\widetilde{a}_0)$ as another such element. Then this tells us that the transformation of $U(L,\widetilde{a})$ under the action of $U(L_0,\widetilde{a}_0)$ on the underlying vectors is given by $U(L_0,\widetilde{a}_0)U(L,\widetilde{a})U^{-1}(L_0,\widetilde{a}_0)$. However, we already know how to multiply the group elements themselves; by definition of a representation, that tells us that we know how to calculate this particular product, using Eqs. (1.4):

$$\begin{aligned} U(L_0,\widetilde{a}_0)U(L,\widetilde{a})U^{-1}(L_0,\widetilde{a}_0) &= U[\{L_0,\widetilde{a}_0\}\{L,\widetilde{a}\}\{L_0^{-1}, -L_0^{-1}\widetilde{a}_0\}] \\ &= U(L_0LL_0^{-1}, \widetilde{a}_0 + L_0\widetilde{a} - L_0LL_0^{-1}\widetilde{a}_0)\ . \end{aligned} \tag{2.3}$$

Since this is a similarity transformation acting on $U(L,\widetilde{a})$, we can simply move it to act on the representation of its logarithm, i.e., the representation of W above, given in Eq. (1.15). To denote the representations of the original generating elements that lie in the Lie algebra, we put an "overbar" on the symbols for them:

$$\begin{aligned} \tfrac{1}{2}q_{\beta\alpha}U(L_0,\widetilde{a}_0)\overline{\boldsymbol{\mathfrak{J}}}^{\alpha\beta}U^{-1}(L_0,\widetilde{a}_0) \ &+ a_\gamma U(L_0,\widetilde{a}_0)\overline{\boldsymbol{\mathfrak{P}}}^{\gamma}U^{-1}(L_0,\widetilde{a}_0) \\ &= \tfrac{1}{2}(L_0QL_0^{-1})_{\beta\alpha}\overline{\boldsymbol{\mathfrak{J}}}^{\alpha\beta} + (\widetilde{a}_0 + L_0\widetilde{a} - L_0LL_0^{-1}\widetilde{a}_0)_\beta\overline{\boldsymbol{\mathfrak{P}}}^{\beta}\ . \end{aligned} \tag{2.4}$$

Since $q_{\alpha\beta} = -q_{\beta\alpha}$ and a_γ are arbitrary, we may separately compare their coefficients on both sides of the equality. We begin with those for a_γ, using the following index substitution scheme to pull out a_γ on the right-hand side of the equation, which is given first, below, with the desired result on the second line:

$$\begin{aligned} (L_0\widetilde{a})_\beta\overline{\boldsymbol{\mathfrak{P}}}^{\beta} &= (L_0)^{\delta}{}_{\gamma}a^{\gamma}\eta_{\delta\beta}\overline{\boldsymbol{\mathfrak{P}}}^{\beta} = (L_0)_\beta{}^{\gamma}a_\gamma\overline{\boldsymbol{\mathfrak{P}}}^{\beta}\ , \\ \Longrightarrow \quad U(L_0,\widetilde{a}_0)\overline{\boldsymbol{\mathfrak{P}}}^{\gamma}U^{-1}(L_0,\widetilde{a}_0) &= (L_0)_\beta{}^{\gamma}\overline{\boldsymbol{\mathfrak{P}}}^{\beta} = (L_0^{-1})^{\gamma}{}_{\beta}\overline{\boldsymbol{\mathfrak{P}}}^{\beta}\ . \end{aligned} \tag{2.5}$$

This relationship can, perhaps, be seen in a "cleaner" way if we multiply it back, now, by a_γ, treating a_γ multiplied by $\boldsymbol{\mathfrak{P}}^\gamma$ as a (4-dimensional) dot product between the two vectors, one just the scalar components of a 4-vector, but the other where each of the 4 components is a linear operator in the representation space:

$$U_0\left(\widetilde{a}\cdot\overline{\widetilde{\boldsymbol{\mathfrak{P}}}}\right)U_0^{-1} = (L\widetilde{a})\cdot\overline{\widetilde{\boldsymbol{\mathfrak{P}}}}\ , \tag{2.6}$$

where we have simplified the notation for $U(L_0,\widetilde{a}_0)$ to simply U_0.

We can understand this form of the representation statement very nicely: Since $\widetilde{\mathfrak{P}}$ is both a 4-vector and a matrix operator, the statement on the left-hand side tells us how the components transform as matrix operators, while the statement on the right-hand side tells us how the components transform as elements of a 4-vector, while the equality says that these two different transformation properties are required to be consistent.

We now go back to Eq. (2.5) and pull out the coefficients of $q_{\alpha\beta}$, remembering that they are skew-symmetric, so that only the skew-symmetric portion of their coefficients is relevant. It is simplest to do this by simply re-writing Eq. (2.5) with $\widetilde{a}$ set equal to zero—since we have already determined what is implied by its existence there—and re-formulating the right-hand side until the correct form of the coefficients for $q_{\alpha\beta}$ are obvious. However, as there is an explicit L on the right-hand side, which contains q, let us first re-formulate just that part:

$$(\widetilde{a}_0 - L_0 L L_0^{-1}\widetilde{a}_0)_\gamma = (\widetilde{a}_0 - L_0(I + Q + \ldots)L_0^{-1}\widetilde{a}_0)_\gamma = -(L_0 Q L_0^{-1}\widetilde{a}_0)_\gamma + \ldots \ . \tag{2.7}$$

Inserting this immediately into our transcription, we now have the following:

$$\begin{aligned} &\tfrac{1}{2} q_{\beta\alpha} U_0\, \overline{\boldsymbol{\mathfrak{J}}}^{\alpha\beta} U_0^{-1} = \tfrac{1}{2}(L_0\, Q\, L_0^{-1})_{\delta\gamma} \left(\overline{\boldsymbol{\mathfrak{J}}}^{\gamma\delta} - 2a_0^\delta \overline{\boldsymbol{\mathfrak{P}}}^\gamma\right) \\ &\quad = \tfrac{1}{2}\eta_{\delta\sigma}(L_0)^\sigma{}_\beta q^\beta{}_\alpha (L_0)_\gamma{}^\alpha \left(\overline{\boldsymbol{\mathfrak{J}}}^{\gamma\delta} - 2a_0^\delta \overline{\boldsymbol{\mathfrak{P}}}^\gamma\right) = \tfrac{1}{2} q_{\beta\alpha}(L_0)_\delta{}^\beta (L_0)_\gamma{}^\alpha \left(\overline{\boldsymbol{\mathfrak{J}}}^{\gamma\delta} + 2\, a_0^{[\gamma} \overline{\boldsymbol{\mathfrak{P}}}^{\delta]}\right) \ . \end{aligned} \tag{2.8}$$

We may now pick off the coefficients of $q_{\beta\alpha}$, and use the form for the inverse Lorentz transform, to acquire the desired transformation for the representation of the generators of the Lorentz transform, which, perhaps unexpectedly, contains the generators for translations, which actually must occur here since the $\boldsymbol{\mathfrak{J}}^{\alpha\beta}$ are set up to organize their activities around the origin, so that when the origin is moved there are changes:

$$U_0\, \overline{\boldsymbol{\mathfrak{J}}}^{\alpha\beta} U_0^{-1} = (L_0^{-1})^\alpha{}_\gamma (L_0^{-1})^\beta{}_\delta \left(\overline{\boldsymbol{\mathfrak{J}}}^{\gamma\delta} + a_0^\gamma \overline{\boldsymbol{\mathfrak{P}}}^\delta - a_0^\delta \overline{\boldsymbol{\mathfrak{P}}}^\gamma\right) \ . \tag{2.9}$$

III. Commutators of the Generators

We have now determined the action of an arbitrary element of the Poincaré group on its various generators. However, we may use that information further, in a very general sort of way, to determine the commutators of the generators, which are the structural identities for the Lie algebra itself. We do this by now re-considering Eqs. (2.5) and Eq. (2.9) for the case when we expand out U_0 around the identity and pull off only the lowest-order, interesting portions. Therefore we set $U_0 = I + \frac{1}{2}(q_0)_{\alpha\beta}\overline{\mathfrak{J}}^{\alpha\beta} + (\widetilde{a}_0)_\gamma \overline{\mathfrak{P}}^\gamma + \ \ldots$ and insert this form into those equations above, beginning with Eq. (2.5):

$$\begin{aligned}\bigl(I + \tfrac{1}{2}(q_0)_{\beta\alpha}\overline{\mathfrak{J}}^{\alpha\beta} + (\widetilde{a}_0)_\gamma \overline{\mathfrak{P}}^\gamma\bigr)\overline{\mathfrak{P}}^\delta\bigl(I - \tfrac{1}{2}(q_0)_{\beta\alpha}\overline{\mathfrak{J}}^{\alpha\beta} - (\widetilde{a}_0)_\gamma \overline{\mathfrak{P}}^\gamma\bigr) &= (I+Q_0)_\beta{}^\delta \overline{\mathfrak{P}}^\beta \\ = \overline{\mathfrak{P}}^\delta + (q_0)_{\beta\alpha}\eta^{\alpha\delta}\overline{\mathfrak{P}}^\beta = \overline{\mathfrak{P}}^\delta + \tfrac{1}{2}(q_0)_{\beta\alpha}&\left(\eta^{\alpha\delta}\overline{\mathfrak{P}}^\beta - \eta^{\beta\delta}\overline{\mathfrak{P}}^\alpha\right) \ .\end{aligned} \tag{3.1}$$

We may now pick off the coefficients of $(a_0)_\gamma$ and $(q_0)_{\alpha\beta}$ in this expression, giving us two of the sets of desired commutators:

$$[\overline{\mathfrak{P}}^\gamma, \overline{\mathfrak{P}}^\delta] = 0 \ , \qquad [\overline{\mathfrak{J}}^{\alpha\beta}, \overline{\mathfrak{P}}^\delta] = \eta^{\alpha\delta}\overline{\mathfrak{P}}^\beta - \eta^{\beta\delta}\overline{\mathfrak{P}}^\alpha = 2\eta^{[\alpha\delta}\overline{\mathfrak{P}}^{\beta]} \ . \tag{3.2}$$

The next task is to make the same sort of operations on Eq. (2.9), setting first $\widetilde{a}_0 = \widetilde{0}$ since we already have the information from there:

$$\bigl(I + \tfrac{1}{2}(q_0)_{\beta\alpha}\overline{\mathfrak{J}}^{\alpha\beta}\bigr)\,\overline{\mathfrak{J}}^{\gamma\delta}\,\bigl(I - \tfrac{1}{2}(q_0)_{\beta\alpha}\overline{\mathfrak{J}}^{\alpha\beta}\bigr) = (I+Q_0)_\rho{}^\gamma (I+Q_0)_\sigma{}^\delta\,\overline{\mathfrak{J}}^{\rho\sigma} \ , \tag{3.3}$$

which allows us to write

$$\tfrac{1}{2}(q_0)_{\beta\alpha}[\overline{\mathfrak{J}}^{\alpha\beta}, \overline{\mathfrak{J}}^{\gamma\delta}] = (q_0)_{\rho\tau}\eta^{\tau\gamma}\overline{\mathfrak{J}}^{\rho\delta} + (q_0)_{\sigma\tau}\eta^{\tau\delta}\overline{\mathfrak{J}}^{\gamma\sigma} = (q_0)_{\rho\tau}\eta^{[\tau\gamma}\overline{\mathfrak{J}}^{\rho]\delta} + (q_0)_{\sigma\tau}\eta^{[\tau\delta}\overline{\mathfrak{J}}^{\gamma\sigma]} \ , \tag{3.4}$$

from which we may pick off the desired commutator relationships:

$$[\overline{\mathfrak{J}}^{\alpha\beta}, \overline{\mathfrak{J}}^{\gamma\delta}] = \eta^{\alpha\gamma}\overline{\mathfrak{J}}^{\beta\delta} - \eta^{\beta\gamma}\overline{\mathfrak{J}}^{\alpha\delta} + \eta^{\beta\delta}\overline{\mathfrak{J}}^{\alpha\gamma} - \eta^{\alpha\delta}\overline{\mathfrak{J}}^{\beta\gamma} = 2\eta^{\alpha[\gamma}\overline{\mathfrak{J}}^{\beta\delta]} - 2\eta^{\beta[\gamma}\overline{\mathfrak{J}}^{\alpha\delta]} \ . \tag{3.5}$$

While these are in fact all the (desired) commutators, it is nonetheless useful to go ahead and revert the $\overline{\mathfrak{J}}^{\alpha\beta}$ back to their 3-dimensional presentations, using Eqs. (1.12), and also the division of the 4-vector generator for translations into its 3-dimensional parts:

$$\overline{\widetilde{\mathfrak{P}}} = \begin{pmatrix} \vec{\overline{\mathfrak{P}}} \\ \overline{\mathcal{H}} \end{pmatrix} \ , \qquad \begin{cases} \overline{\mathfrak{J}}^{ab} = \epsilon^{abc}\overline{\mathfrak{J}}_c \ , \\ \overline{\mathfrak{J}}^{4i} = \overline{\mathfrak{K}}^i \ . \end{cases} \tag{3.6}$$

Beginning with the commutators involving $\overline{\mathfrak{P}}$ here, from Eqs. (3.2), we may write but the following, where when necessary we multiply by $\frac{1}{2}\epsilon_{abf}$ to move an ϵ to the other side of the equation, since $\frac{1}{2}\epsilon_{abf}\epsilon^{abc} = \delta^c_f$:

$$\begin{gathered} [\overline{\mathfrak{P}}^i, \overline{\mathfrak{P}}^j] = 0 = [\overline{\mathfrak{P}}^i, \overline{\mathfrak{H}}] \; , \\ \left.\begin{aligned} 2\delta^{[a}_d \overline{\mathfrak{P}}^{b]} &= [\overline{\mathfrak{J}}^{ab}, \overline{\mathfrak{P}}_d] = \epsilon^{abc}[\overline{\mathfrak{J}}_c, \overline{\mathfrak{P}}_d] \; , \\ 2\delta^{[a}_4 \overline{\mathfrak{P}}^{b]} &= [\overline{\mathfrak{J}}^{ab}, \overline{\mathfrak{P}}^4] = [\overline{\mathfrak{J}}^{ab}, \overline{\mathfrak{H}}] = 0 \; , \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} &[\overline{\mathfrak{J}}_f, \overline{\mathfrak{P}}_d] = \epsilon_{fda}\overline{\mathfrak{P}}^a \; , \\ &[\overline{\mathfrak{J}}_f, \overline{\mathfrak{H}}] = 0 \qquad , \end{aligned}\right. \\ 2\eta^{[4d}\overline{\mathfrak{P}}^{b]} = [\overline{\mathfrak{J}}^{4b}, \overline{\mathfrak{P}}^d] = [\overline{\mathfrak{K}}^b, \overline{\mathfrak{P}}^d] = -\eta^{bd}\overline{\mathfrak{H}} \; , \\ 2\eta^{[44}\overline{\mathfrak{P}}^{b]} = [\overline{\mathfrak{J}}^{4b}, \overline{\mathfrak{H}}] = [\overline{\mathfrak{K}}^b, \overline{\mathfrak{H}}] = -\overline{\mathfrak{P}}^b \; , \end{gathered} \tag{3.7}$$

At least some of the reductions of the commutators involving only the Lorentz group generators, from Eqs. (3.5), are slightly more complicated, so we will present their reductions one at a time. We begin with the purely spatial ones:

$$\begin{gathered} [\overline{\mathfrak{J}}^{ab}, \overline{\mathfrak{J}}^{cd}] = 2\eta^{a[c}\overline{\mathfrak{J}}^{bd]} - 2\eta^{b[c}\overline{\mathfrak{J}}^{ad]} \; , \\ \text{or } \epsilon^{abr}\epsilon^{cds}[\overline{\mathfrak{J}}_r, \overline{\mathfrak{J}}_s] = 2\{\eta^{a[c}\epsilon^{bd]t} - 2\eta^{b[c}\epsilon^{ad]t}\}\overline{\mathfrak{J}}_t \; . \end{gathered} \tag{3.8a}$$

We then multiply both sides by $\frac{1}{2}\epsilon_{abm}\frac{1}{2}\epsilon_{cdn}$, which, on the left-hand side simply gives us the commutator we want, while we have to follow some algebra on the right-hand side:

$$\begin{aligned} [\overline{\mathfrak{J}}_m, \overline{\mathfrak{J}}_n] &= \tfrac{1}{2}\epsilon_{abm}\epsilon_{cdn}\{\eta^{ac}\epsilon^{bdt} - \eta^{bc}\epsilon^{adt}\}\overline{\mathfrak{J}}_t = \epsilon_{abm}\eta^{ac}\{\delta^t_n\delta^b_c - \delta^t_c\delta^b_n\}\overline{\mathfrak{J}}_t \\ &= \epsilon_{acm}\eta^{ac}\overline{\mathfrak{J}}_n - \epsilon_{anm}\eta^{ac}\overline{\mathfrak{J}}_c = +\epsilon^{mnc}\overline{\mathfrak{J}}_c \; . \end{aligned} \tag{3.8b}$$

Next we can consider one spatial and one temporal generator:

$$\epsilon^{abr}[\overline{\mathfrak{J}}_r, \overline{\mathfrak{K}}^d] = [\overline{\mathfrak{J}}^{ab}, \overline{\mathfrak{J}}^{4d}] = 2\eta^{a[4]}\overline{\mathfrak{J}}^{bd]} - 2\eta^{b[4}\overline{\mathfrak{J}}^{ad]} = \eta^{ad}\overline{\mathfrak{K}}^b - \eta^{bd}\overline{\mathfrak{K}}^a \; . \tag{3.9a}$$

Again multiplying by $\frac{1}{2}\epsilon_{abs}$, and also multiplying by η_{td} to lower the index on the left-hand side $\overline{\mathfrak{K}}^d$, we have

$$[\overline{\mathfrak{J}}_s, \overline{\mathfrak{K}}_t] = \tfrac{1}{2}\epsilon_{abs}\{\delta^a_t\overline{\mathfrak{K}}^b - \delta^b_t\overline{\mathfrak{K}}^a) = \epsilon_{stb}\overline{\mathfrak{K}}^b \; . \tag{3.9b}$$

Lastly we consider the purely temporal version of the commutator:

$$[\overline{\mathfrak{K}}^b, \overline{\mathfrak{K}}^d] = [\overline{\mathfrak{J}}^{4b}, \overline{\mathfrak{J}}^{4d}] = \eta^{4[4}\overline{\mathfrak{J}}^{bd]} - \eta^{b[4}\overline{\mathfrak{J}}^{4d]} = -\overline{\mathfrak{J}}^{bd} = -\epsilon^{bdf}\overline{\mathfrak{J}}_f \; . \tag{3.10}$$

IV. Invariants of the Groups

Beginning with the Lorentz group itself, it is actually slightly more useful to retreat back to the rotation subgroup for a moment, which is generated just by the triplet of angular momentum operators. We show that the (3-dimensional) square of that vector is an invariant under the action of the group, in that it commutes with all the generators, and therefore with every group element:

$$\begin{aligned}[\vec{\mathcal{J}}^{\,2}, \mathcal{J}^k] = [\mathcal{J}^m \mathcal{J}_m, \mathcal{J}^k] = \mathcal{J}^m [\mathcal{J}_m, \mathcal{J}^k] + [\mathcal{J}^m, \mathcal{J}^k]\mathcal{J}_m = \mathcal{J}_m[\mathcal{J}^m, \mathcal{J}^k] + [\mathcal{J}^m, \mathcal{J}^k]\mathcal{J}_m \\ = \mathcal{J}_m \epsilon^{mkr}\mathcal{J}_r + \epsilon^{mkr}\mathcal{J}_r\mathcal{J}_m = \epsilon^{mkr}(\mathcal{J}_m\mathcal{J}_r + \mathcal{J}_r\mathcal{J}_m) = 0 \,,\end{aligned} \tag{4.1}$$

since the quantity in the parentheses is symmetric in the summation indices m and r, while it is multiplied, and summed, by the Levi-Civita symbol, which is skew-symmetric in those indices. Because this quantity is an invariant under actions by the group, any particular, irreducible representation will have some (scalar) value for it, which we may use to label the representation. In the case of the rotation group it may be shown that there is an irreducible representation via matrices of every possible dimension, i.e., of dimensions, $1, 2, 3, \ldots$. It is usual to label these representations by using a label j, where $2j$ may be any positive integer, arranged so that $2j + 1$ is the dimension of the vector space on which the representation, by means of matrices, acts. With this notation one may show that the action of $\vec{\mathcal{J}}^{\,2}$ on the vector space is just the same as the action of the identity matrix on that space, multiplied by the scalar $j(j + 1)$. Therefore, it is usual to refer to the representation, in this vector space, of some rotation, say $R(\theta; \hat{n})$ by the symbol $D^{(j)}[R(\theta; \hat{n})]$.

To proceed further, going on to the Lorentz group, with its two triplets of generators, we first define two complex-conjugated sums of them:

$$\mathcal{F}^j \equiv \frac{1}{2}(\mathcal{J}^j + i\mathcal{K}^j) \,, \qquad \mathcal{G}^k \equiv \frac{1}{2}(\mathcal{J}^k - i\mathcal{K}^k) \,. \tag{4.2}$$

We now show below that the commutators of each of these triplets has exactly the same structure as the $\mathfrak{J}^m$ themselves:

$$
\begin{aligned}
[\mathfrak{F}^j, \mathfrak{F}^k] &= \tfrac{1}{4}[\mathfrak{J}^j + i\mathfrak{K}^j, \mathfrak{J}^k + i\mathfrak{K}^k] = \tfrac{1}{4}\epsilon_{jkm}\{\mathfrak{J}_m + i\mathfrak{K}_m - (-i\mathfrak{K}_m) + \mathfrak{J}_m\} = \epsilon^{jkm}\mathfrak{F}_m \ , \\
[\mathfrak{G}^j, \mathfrak{G}^k] &= \tfrac{1}{4}[\mathfrak{J}^j - i\mathfrak{K}^j, \mathfrak{J}^k - i\mathfrak{K}^k] = \tfrac{1}{4}\epsilon_{jkm}\{\mathfrak{J}_m - i\mathfrak{K}_m - (+i\mathfrak{K}_m) + \mathfrak{J}_m\} = \epsilon^{jkm}\mathfrak{G}_m \ , \\
[\mathfrak{F}^j, \mathfrak{G}^k] &= \tfrac{1}{4}[\mathfrak{J}^j + i\mathfrak{K}^j, \mathfrak{J}^k - i\mathfrak{K}^k] = \tfrac{1}{4}\epsilon_{jkm}\{\mathfrak{J}_m - i\mathfrak{K}_m - (-i\mathfrak{K}_m) - \mathfrak{J}_m\} = 0 \ ,
\end{aligned}
\tag{4.3}
$$

where the last line shows, additionally, that the two triplets commute between themselves. Therefore we may now consider irreducible representations of these two independent copies of a "rotation" group, labeling them by f and g, where each of $2f$ and $2g$ are allowed to take on all non-zero integer values. Since we may resolve the original equations backwards, for $\mathfrak{J}^i$ and $\mathfrak{K}^j$, as follows:

$$
\mathfrak{J}^m = \mathfrak{F}^m + \mathfrak{G}^m \ , \qquad \mathfrak{K}^m = i(\mathfrak{G}^m - \mathfrak{F}^m) \ , \tag{4.4}
$$

we may then determine their representations, as direct sums of the two representations, so that it is reasonable to label an irreducible representation of the Lorentz group by the two half-integers, f and g: $D^{(f,g)}[L(\lambda;\hat{v})]$, for example.

Not pursuing those representation questions any more at this time and place, let us go on to determine the invariants for the Poincaré group. We begin with a rather reasonable suggestion, namely the square of the 4-vector translation operator, $\overline{\overline{\mathfrak{P}}}^{\,2}_{\sim}$, which can be shown to be invariant by using Eqs. (2.5) above:

$$
\begin{aligned}
U_0\overline{\mathfrak{P}}^\nu\overline{\mathfrak{P}}_\nu U_0^{-1} &= U_0\overline{\mathfrak{P}}^\nu U_0^{-1}U_0\overline{\mathfrak{P}}_\nu U_0^{-1} = (L_0)_\alpha{}^\nu\overline{\mathfrak{P}}^\alpha(L_0)_\gamma{}^\beta\eta_{\beta\nu}\eta^{\gamma\delta}\overline{\mathfrak{P}}_\delta \\
&= \overline{\mathfrak{P}}^\alpha[(L_0)_\alpha{}^\nu(L_0)_\gamma{}^\beta\eta_{\beta\nu}]\eta^{\gamma\delta}\overline{\mathfrak{P}}_\delta = \overline{\mathfrak{P}}^\alpha\eta_{\gamma\alpha}\eta^{\gamma\delta}\overline{\mathfrak{P}}_\delta = \overline{\mathfrak{P}}^\alpha\overline{\mathfrak{P}}_\alpha \ ,
\end{aligned}
\tag{4.5}
$$

where we have used the fact that Lorentz transformations preserve the metric.

There are in fact 2 invariants for the Poincaré group, but the second one is not quite so obvious. It is often, at least, referred to as the Pauli-Ljubanski 4-vector, and is some "sort" of "cross-product" of the angular momentum and boost generators with the translation generators:

$$
\mathfrak{W}^\mu \equiv \frac{1}{2}\eta^{\mu\nu\lambda\eta}\mathfrak{J}_{\nu\lambda}\mathfrak{P}_\eta \quad \Longrightarrow \quad
\begin{cases}
\mathfrak{W}^i = \tfrac{1}{2}\eta^{i\nu\lambda\eta}\mathfrak{J}_{\nu\lambda}\mathfrak{P}_\eta = -\tfrac{1}{2}\eta^{ijk4}\mathfrak{J}_{jk}\mathfrak{H} + 2\tfrac{1}{2}\eta^{in4e}\mathfrak{J}_{n4}\mathfrak{P}_e \\
\qquad = -\{\mathfrak{H}\vec{\mathfrak{J}} + \vec{\mathfrak{K}} \times \vec{\mathfrak{P}}\}^i \ , \\
\mathfrak{W}^4 = \tfrac{1}{2}\eta^{4ijk}\mathfrak{J}_{ij}\mathfrak{P}_k = -\tfrac{1}{2}\epsilon^{ijk}\epsilon_{ijm}\mathfrak{J}^m\mathfrak{P}_k = -\vec{\mathfrak{J}}\cdot\vec{\mathfrak{P}} \ .
\end{cases}
\tag{4.6}
$$

We will show shortly that the square of this 4-vector operator is also an invariant, for any given representation; however, for the moment, let us retreat just a bit and consider the fact that $\vec{\mathcal{P}}^2$ is an invariant. Since this is the case we may use its eigenvalues to label a given representation; it is customary to ascribe a parameter m to it, in such a way that

$$\widetilde{\mathcal{P}}^2 \Longrightarrow \quad -m^2\,\mathbf{I}\,, \tag{4.7}$$

where $\mathbf{I}$ is the identity operator on that representation. Thinking of m as the "mass" associated with that representation—appropriate to thinking of $\mathcal{P}^\mu$ as the components of the "momentum operator," usually associated with translational invariance—then we can think of looking at the representation associated with a particle of mass m, in its own rest frame. In that case, we would expect the states in that frame to associated with a zero eigenvalue for $\mathcal{P}^i$; therefore, in that frame we could ask about the eigenvalues associated with $\widetilde{\mathcal{W}}^2$, which should just be the same as those for $(\mathcal{H}\mathcal{J}^i)(\mathcal{H}\mathcal{J}_i)$ in that frame; however, this should clearly just be such that

$$\widetilde{\mathcal{W}}^2 \Longrightarrow \quad m^2 j(j+1)\mathbf{I} = m^2 s(s+1)\mathbf{I}\,, \tag{4.8}$$

where the last equality is because a particle in its rest frame does not have any orbital angular momentum, but only whatever angular momentum it may have "all on its own," which is usually referred to as its spin.

To now demonstrate the proof that this 4-vector square is indeed an invariant, we proceed rather as before, but with rather more steps in the algebra, and we don't bother to put a zero subscript on the transformation U_0, since it's the only one around at the moment:

$$\begin{aligned} U\overline{\mathcal{W}}^\mu\overline{\mathcal{W}}_\mu U^{-1} &= \eta^{\mu\nu\lambda\tau}\eta_{\mu\sigma}\eta^{\sigma\alpha\beta\delta}U\overline{\mathcal{J}}_{\nu\lambda}U^{-1}U\overline{\mathcal{P}}_\tau U^{-1}U\overline{\mathcal{J}}_{\alpha\beta}U^{-1}U\overline{\mathcal{P}}_\delta U^{-1} \\ &= \eta^{\mu\nu\lambda\tau}\eta_{\mu\sigma}\eta^{\sigma\alpha\beta\delta}L^\rho{}_\nu L^\epsilon{}_\lambda(\overline{\mathcal{J}}_{\rho\epsilon}+2a_{[\rho}\overline{\mathcal{P}}_{\epsilon]})L^\upsilon{}_\tau\overline{\mathcal{P}}_\upsilon L^\zeta{}_\alpha L^\gamma{}_\beta(\overline{\mathcal{J}}_{\zeta\gamma}+2a_{[\zeta}\overline{\mathcal{P}}_{\gamma]})L^\theta{}_\delta\overline{\mathcal{P}}_\theta\,. \end{aligned} \tag{4.9}$$

I now replace the $\eta_{\mu\sigma}$ in the last term above with its equality via a Lorentz transformation, namely $L^\pi{}_\mu\eta_{\pi\omega}L^\omega{}_\sigma$, and then put the various L's for the transformations together where 4 of them can be "seen" by each one of the Levi-Civita tensors, use the fact that the determinant of a Lorentz transformation is just 1, which gives me for the right-hand side the following:

$$
\begin{aligned}
&\eta_{\pi\omega}\eta^{\mu\nu\lambda\tau}L^{\pi}{}_{\mu}L^{\rho}{}_{\nu}L^{\epsilon}{}_{\lambda}L^{\upsilon}{}_{\tau}\,\eta^{\sigma\alpha\beta\delta}L^{\omega}{}_{\sigma}L^{\zeta}{}_{\alpha}L^{\gamma}{}_{\beta}L^{\theta}{}_{\delta}(\overline{\boldsymbol{\mathcal{J}}}_{\rho\epsilon}+2a_{[\rho}\overline{\boldsymbol{\mathcal{P}}}_{\epsilon]})\overline{\boldsymbol{\mathcal{P}}}_{\upsilon}(\overline{\boldsymbol{\mathcal{J}}}_{\zeta\gamma}+2a_{[\zeta}\overline{\boldsymbol{\mathcal{P}}}_{\gamma]})\overline{\boldsymbol{\mathcal{P}}}_{\theta}\\
&\qquad=\eta_{\pi\omega}\eta^{\pi\rho\epsilon\upsilon}\eta^{\omega\zeta\gamma\theta}\overline{\boldsymbol{\mathcal{J}}}_{\rho\epsilon}\overline{\boldsymbol{\mathcal{P}}}_{\upsilon}\overline{\boldsymbol{\mathcal{J}}}_{\zeta\gamma}\overline{\boldsymbol{\mathcal{P}}}_{\theta}=\eta_{\pi\omega}\overline{\boldsymbol{\mathcal{W}}}^{\pi}\overline{\boldsymbol{\mathcal{W}}}^{\omega}=\widetilde{\overline{\boldsymbol{\mathcal{W}}}}^{2}\;,
\end{aligned}
\tag{4.10}
$$

where we have noted that a Levi-Civita tensor summed into two copies of the same quantity—in this case into two copies of components of $\overline{\boldsymbol{\mathcal{P}}}_{\mu}$—must vanish.

As a last aside at this point we should note that all the irreducible, non-trivial representations of the Poincaré group are infinite dimensional, i.e., they involve functions/vectors in a Hilbert space. As they are labeled by the invariants discussed above, they are most easily thought of in terms of spinor-valued functions of 3-momentum, $\vec{p}$. We will not begin to say more about them here, although one should look at the notes for functional representations of the rotation group, which are important, but rather simpler.