Physics 495

Introduction to Spinors 19 November, 2001

O. Initial Relation to Spacetime

We begin with the bundle of tangent vectors, or 1-forms, over the standard (4-dimensional) spacetime. Each section of this bundle, i.e., a (C^∞) choice of a tangent vector, or a 1-form, in the vector space over each point (of some neighborhood) of that spacetime, has 4 (real) degrees of freedom. We have been presenting those degrees of freedom via the 4 elements of a real, 4×1 (column) matrix—or, alternatively, via the 4 elements of a real 1×4 (row) matrix. Yet a different alternative is to use the 4 real degrees of freedom embodied in the elements of a complex-valued, 2×2, Hermitean matrix. If, for instance, we begin with a very simple 4-vector for coordinates, $\widetilde{x}$, then the following is a presentation as such a matrix:

$$(x)^\mu \equiv x^\mu = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \Longleftarrow \quad \widetilde{x} \Longrightarrow \quad \begin{pmatrix} z-t & x-iy \\ x+iy & -z-t \end{pmatrix} \equiv \mathbf{X} \, . \tag{0.1}$$

As this is quite a different way of presenting these 4 degrees of freedom, we anticipate that this method will open up new ways of looking at them. We begin a search for such new ways by first looking at the (squared) "length," of our 4-vector. As it is given in Cartesian/Minkowski coordinates, the spacetime interval associated with it is just $\widetilde{x}^2 \equiv x^2 + y^2 + z^2 - t^2$, which turns out to be the same as the negative of the determinant of the matrix presentation:

$$x^T H x = x^\mu \, \eta_{\mu\nu} \, x^\nu \equiv \boldsymbol{\eta}(\widetilde{x}, \widetilde{x}) \equiv \widetilde{x} \cdot \widetilde{x} = -\det \mathbf{X} \, . \tag{0.2}$$

A second thing we notice is the appearance of the form of the standard 3 Pauli σ-matrices. By appending to them the 2×2 identity matrix, I_2, we can create a set of 4 (generalized) Pauli matrices, appropriate for 4-dimensional spacetime:

$$\begin{gathered} \sigma^\mu \equiv \begin{pmatrix} \vec{\sigma} \\ I_2 \end{pmatrix} \equiv \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ I_2 \end{pmatrix} , \quad \mathbf{X} \equiv \widetilde{x} \cdot \widetilde{\boldsymbol{\sigma}} = x^\mu \sigma_\mu = \begin{pmatrix} z-t & x-iy \\ x+iy & -z-t \end{pmatrix} , \\ \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} , \quad \sigma_z \equiv \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \mathbf{I}_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \end{gathered} \tag{0.3}$$

We can also use this approach to create a matrix form of the 4 basis 1-forms, say, for spacetime, or the basis tangent vectors. Using the current approach, with Cartesian coordinates, we may simply differentiate the earlier equation for $\mathbf{X}$ to create a basis for 1-forms, and use partial derivatives with respect to those variables to create the reciprocal basis:

$$\begin{aligned} \underset{\approx}{\boldsymbol{\omega}} \equiv d\mathbf{X} = \sigma_\mu \, dx^\mu &= \begin{pmatrix} dz - dt & dx - i\,dy \\ dx + i\,dy & -dz - dt \end{pmatrix} , \\ \Longrightarrow \quad \eta_{\mu\nu} \, \underset{\approx}{\boldsymbol{\omega}}^\mu \otimes \underset{\approx}{\boldsymbol{\omega}}^\nu = \boldsymbol{\eta} &= \; - \det(\underset{\approx}{\boldsymbol{\omega}}) , \\ \widetilde{\mathbf{E}} \equiv \sigma^\mu \, \partial_{x^\mu} \equiv \sigma^\mu \, \partial_\mu &= \begin{pmatrix} \partial_z + \partial_t & \partial_x - i\,\partial_y \\ \partial_x + i\,\partial_y & -\partial_z + \partial_t \end{pmatrix} , \end{aligned} \tag{0.5}$$

where, even though we are in fact considering only the coordinate basis sets for each of these, I use the symbols $\underset{\approx}{\boldsymbol{\omega}}$ and $\widetilde{\mathbf{E}}$ simply to remind us of what sort of objects are being described, and also where the product used in this determinant is the symmetric part of the tensor product of two 1-forms, appropriate to create the metric tensor which is a symmetric, type [0,2] tensor.

We also need a reverse mechanism: Given an arbitrary Hermitean, 2×2 matrix, how do we find the associated (real) 4-vector? Although in principle one may clearly do it "by hand," 'tis better to have an algorithmic approach, which we create by using the properties of the traces of the Pauli matrices:

$$\tfrac{1}{2}\,\mathrm{tr}\;\sigma^\mu = \delta^\mu_4 \,, \; \tfrac{1}{2}\,\mathrm{tr}\,(\sigma_i \sigma^j) = \delta^j_i \quad \Longrightarrow \quad x_4 = -x^4 = \tfrac{1}{2}\,\mathrm{tr}\;\mathbf{X} \,, \quad \vec{x} = \tfrac{1}{2}\,\mathrm{tr}\,(\vec{\boldsymbol{\sigma}}\mathbf{X}) \,. \tag{0.6}$$

One of our most important interests in our study of tensorial objects over spacetime is their behavior under Lorentz transformations. Therefore, we must determine how the action of a Lorentz transformation acting on $\widetilde{x}$ influences the form of its 2×2 matrix presentation, $\mathbf{X}$. One might easily suppose that the action on a matrix would be by some matrix transformation equation, i.e., if $\widetilde{x}' = L\widetilde{x}$ we might expect that there exist some 2×2 matrix, A, dependent on the choice of L, such that $\mathbf{X}' = A\mathbf{X}A^\dagger$, where we have inserted the matrix $A^\dagger$ on the side opposite to A since this is necessary to preserve the Hermitean character of $\mathbf{X}$. Moreover, since the (4-dimensional) interval of $\widetilde{x}$ is proportional to the determinant of $\mathbf{X}$, and Lorentz transformations preserve that interval, we must preserve the determinant of $\mathbf{X}$ under this transformation, i.e., we must insist that $\det(AA^\dagger) = +1$. In principle this allows the determinant of A to be some phase factor of modulus unity; however, as that extra phase factor would not appear in the form of the transformed matrix, it is not very useful, and we throw it away and summarize by saying

$$\widetilde{x}\,' = L\,\widetilde{x} \quad \Longrightarrow \quad \mathbf{X}' = A\mathbf{X}A^\dagger \;, \quad \text{and} \quad \det A = +1 \;. \tag{0.7}$$

Notice that an arbitrary 2×2, complex matrix has 8 (real) degrees of freedom, and the condition that its determinant should equal $+1$ is 2 real constraints on these quantities, leaving us with 6 degrees of freedom, as is desirable for the 6 degrees of freedom of the Lorentz group. [These groups of matrices have standard names. The group of matrices of the form A, above, are referred to by the name $\boldsymbol{SL}(2,\mathbb{C})$, which stands for the set of all 2×2 complex, invertible matrices with determinant equal to $+1$—this requirement on the determinant being the meaning of the $\boldsymbol{S}$ in front of the rest of the name. On the other hand, in this notation the (usual) 3-dimensional, proper rotation group is referred to by the name $\boldsymbol{SO}(3,\mathbb{R})$, while the Lorentz group is often called $\boldsymbol{SO}(3,1)$, since there are 3 spacelike directions and 1 timelike direction. It is also quite common **not to write** $\mathbb{R}$ in the rotation group, that option being the default. This is even more true in the case of the Lorentz group, because the distinction between spacelike and timelike vectors is one that is lost when you allow general complex-valued transformations.]

Now, we may identify those 6 degrees of freedom much more precisely by proceeding, as usual, to look at the properties of the Lie algebra of generators of those elements of the group that are not too far from the identity transformation. Writing out the matrix $A = e^Q$ in terms of its generators, and remembering the requirement that $+1 = \det A = e^{\operatorname{tr} Q}$, we see that the only requirement on Q is that it should be traceless, i.e., have trace equal to zero. The standard Pauli matrices form a basis for all 2×2 traceless matrices; i.e., we may write any Q in the form of $\vec{\alpha} \cdot \vec{\sigma}$, where $\vec{\alpha}$ is a 3-dimensional vector, with complex components. In order to calculate this exponential we need to know how to multiply the Pauli matrices: the product of two arbitrary Pauli matrices may be written in the form

$$
\begin{aligned}
(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) &= (\vec{a}\cdot\vec{b})\mathbf{I}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \\
\Longrightarrow \quad (\vec{\alpha}\cdot\vec{\sigma})^2 = \alpha^2\,\mathbf{I}_2 \quad &\text{and} \quad \vec{\sigma}\,(\vec{b}\cdot\vec{\sigma}) = \vec{b}\,\mathbf{I}_2 + i\,\vec{b}\times\vec{\sigma}\ .
\end{aligned}
\tag{0.8}
$$

Therefore a transformation matrix not too far from the identity would look like

$$
A = e^{Q} = e^{\vec{\alpha}\cdot\vec{\sigma}} = (\cosh\alpha)\,\mathbf{I}_2 + (\sinh\alpha)\,\hat{\alpha}\cdot\vec{\sigma}\ , \tag{0.9}
$$

and the actual transformed Hermitean matrix would have the form

$$
\mathbf{X}' = |\cosh\alpha|^2\,\mathbf{X} + \sinh\alpha\,(\hat{\alpha}\cdot\vec{\sigma})\mathbf{X} + (\sinh\alpha)^*\,\mathbf{X}(\hat{\alpha}^*\cdot\vec{\sigma}) + |\sinh\alpha|^2(\hat{\alpha}\cdot\vec{\sigma})\mathbf{X}(\hat{\alpha}^*\cdot\vec{\sigma})\ . \tag{0.10}
$$

We may now take the trace of both sides of the expression, keeping only terms to lowest order in $|\alpha|$, which will give us, using Eq. (0.6):

$$
t' = -\tfrac{1}{2}\,\mathrm{tr}\ \mathbf{X}' = t - (\vec{\alpha}+\vec{\alpha}^*)\cdot\vec{x} + O^2(|\alpha|)\ . \tag{0.11}
$$

Knowing that a Lorentz boost would mix t and $\hat{v}\cdot\vec{x}$ so that $t' = t - \vec{v}\cdot\vec{x}$, to lowest order in v, while a rotation would leave t invariant, we may see that twice the real part of $\vec{\alpha}$ should correspond to $\vec{v}$, while the imaginary part must correspond to rotations since it would leave t invariant. On the other hand if we now first multiply Eq. (0.10) by $\vec{\sigma}$ and then take half the trace, and keep terms only to lowest order in $|\alpha|$, we will acquire

$$
\begin{aligned}
\vec{x}\,' = \tfrac{1}{2}\,\mathrm{tr}\,(\vec{\sigma}\,\mathbf{X}') &= \vec{x} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\sigma}\,(\vec{\alpha}\cdot\vec{\sigma})\mathbf{X}\} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\sigma}\,\mathbf{X}(\vec{\alpha}^*\cdot\vec{\sigma})\} + O^2(|\alpha|) \\
&= \vec{x} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\sigma}\,(\vec{\alpha}\cdot\vec{\sigma})\mathbf{X}\} + \tfrac{1}{2}\,\mathrm{tr}\,\{\mathbf{X}(\vec{\alpha}^*\cdot\vec{\sigma})\,\vec{\sigma}\} + O^2(|\alpha|) \\
&= \vec{x} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\alpha}\,\mathbf{X} + i\,(\vec{\alpha}\times\vec{\sigma})\mathbf{X} + \vec{\alpha}^*\,\mathbf{X} - i\,\mathbf{X}(\vec{\alpha}^*\times\vec{\sigma})\} + O^2(|\alpha|) \\
&= \vec{x} - (\vec{\alpha}+\vec{\alpha}^*)t + i(\vec{\alpha}-\vec{\alpha}^*)\times\vec{x} + O^2(|\alpha|)\ .
\end{aligned}
\tag{0.12}
$$

Again we see that it is the real part of $\vec{\alpha}$ that generates mixing of $\vec{x}$ and t, i.e., that is an infinitesimal part of a boost, while, now, we see that the imaginary part generates rotations.

More specifically, if we decompose $\vec{\alpha}$ into its real and imaginary parts we may identify things as follows: $\vec{\alpha} \equiv \vec{a} + i\vec{b} = \frac{1}{2}\{\lambda\hat{v} - i\theta\hat{e}\}$ so that the transformation is parametrized by

$$\frac{\lambda}{2}\hat{v}\cdot\vec{\sigma} - i\,\frac{\theta}{2}\hat{e}\cdot\vec{\sigma} \quad\longleftrightarrow\quad \lambda\hat{v}\cdot\vec{\mathfrak{K}} + \theta\hat{e}\cdot\vec{\mathfrak{J}}\,. \tag{0.13}$$

We now consider what happens when we perform two such transformations in a row: let A_i correspond to L_i so that

$$\widetilde{x}'' = L_1\,\widetilde{x}' = L_1 L_2\,\widetilde{x} \quad\Longleftrightarrow\quad \mathbf{X}'' = A_1\mathbf{X}'A_1^\dagger = A_1(A_2\mathbf{X}A_2^\dagger)A_1^\dagger = (A_1A_2)\mathbf{X}(A_1A_2)^\dagger\ , \tag{0.14}$$

which tells us that A_1A_2 corresponds to L_1L_2. This last requirement is what is needed to satisfy the definition for a *representation* of a group of transformations. Therefore we may conclude that the set of all the 2×2 matrices $\{A = e^{\vec{\alpha}\cdot\vec{\sigma}}\}$ constitutes a representation of the (proper, orthochronous) Lorentz group of transformations, which we could make more definite by using a symbol D for this mapping; i.e.,

$$\begin{aligned} R(\theta;\hat{e})\ &= e^{\theta\,\hat{e}\cdot\mathfrak{J}} \quad \text{and} \quad D[R(\theta;\hat{e})] = e^{-i(\theta/2)\hat{e}\cdot\vec{\sigma}}\ , \\ B(\vec{v})\ &= e^{\lambda\,\hat{v}\cdot\mathfrak{K}} \quad \text{and} \quad D[B(\vec{v})] = e^{(\lambda/2)\hat{v}\cdot\vec{\sigma}}\ . \end{aligned} \tag{0.15}$$

This representation has some peculiarities that should immediately be pointed out. They arise for two, related reasons. The first property is that the action of the matrices $D[L]$ is quadratic so that the action of $+D[L]$ on $\mathbf{X}$ is exactly the same as the action of $-D[L]$; this means that the representation in question is **not faithful**, but, rather, two to one: for every Lorentz transformation, L, there are two independent 2×2 matrices, $\pm D[L]$, which correspond to it. We would of course prefer to "normalize" this question by choosing the 2×2 identity matrix, $\mathbf{I}_2$, to be the representation of the identity Lorentz transformation. Then, as θ and/or λ vary, continuously, away from zero, we would continue to choose that positive sign, at least as long as possible. What we will see below is that this does lead to some trouble.

Secondly, because of the "extra" factor of one half in the representation, we can immediately see that the representation of a rotation by angle 2π is not quite what we would have expected:

$$D[R(2\pi;\hat{e})] = e^{-i\pi\hat{e}\cdot\vec{\sigma}} = \cos\pi\,\mathbf{I}_2 - i\sin\pi\,\hat{e}\cdot\vec{\sigma} = -\mathbf{I}_2\ . \tag{0.16}$$

Therefore if we begin at the identity Lorentz transformation, and choose the 2×2 identity matrix as its representation, then the representation of a complete rotation "all the way around," i.e., by angle 2π, about any direction, is represented by the negative of the identity matrix. Of course, it is obviously true that if one rotated by angle 4π he would indeed come back to the identity matrix again. This is the way that this $2 \to 1$ mapping works so as to include all the elements on both sides: there are actually twice as many elements in the group $\boldsymbol{SL}(2,\mathbb{C})$ as there are in the Lorentz group, $\boldsymbol{SO}(3,1)$! This of course does not cause a problem with the actual action, on spacetime viewed as components of 2×2, Hermitean matrices, as described above, because it is a quadratic action. Nonetheless, when we consider linear actions, on the underlying 2-dimensional, complex vectors on which these representation matrices could be allowed to act, they will have the property that a complete rotation, by angle 2π, of the physical system—if any—which they represent would nonetheless change the sign of the representing vector!

I. Fundamental Spinor Spaces

Since we now have an action on 2×2, Hermitean matrices, which is quadratic in the fundamental representation matrices, we wonder if there is any meaning to the underlying, 2-dimensional, complex vector space on which these various 2×2 matrices "live" as **linear** operators. We take the basic 2-dimensional vectors in this space, say ξ, as quantities on which the action of the matrices $D(L)$ would take them from being viewed in, say, a frame S' to how they would be viewed in the frame S, those two being related as usual by the Lorentz transformation L, with $\xi = D(L)\,\xi'$. These objects are called *spinors.* As mentioned above, if there are any physical systems which would have its properties described by such objects, then when that system is rotated all the way around, and so returns to its original state, the 2-dimensional, complex vector, i.e., the spinor, representing its properties will have its sign changed.

It is customary to describe a spinor in terms of its components with respect to some chosen basis, that basis often never being discussed further. Therefore, we refer to the space of all spinors by the symbol V^2 and we denote the components of an arbitrary element of this space by a symbol such as $\{\xi^A \mid A = 1, 2\}$. [Sometimes the basis is described by giving names to a particular pair of non-parallel spinors.] Since these components are complex-valued, we suppose that the basis in question is real, and create a "complex-conjugated space," $\overline{V}^2$, with elements described by $\{\psi^{\dot{A}} \mid A = 1, 2\}$. For a given, particular spinor, these two sets of numbers are complex conjugates of one another, i.e., $\overline{\xi^A} = \xi^{\dot{A}}$, with the overline meaning complex conjugation as usual.

Since the spinor spaces are certainly examples of vector spaces, it is reasonable that there should be associated with (each of) them dual spaces, which we label V^2_* and $\overline{V}_*{}^2$. In order to preserve the Einstein summation convention, that causes implied sums when there is a repeated upper and lower index, we use the notation ξ_A and $\xi_{\dot{A}}$ to refer to elements of these spaces. However, since in ordinary spacetime the existence of the metric tensor allows us to create a mapping that maps tangent vectors and 1-forms—dual to one another—into each other, we might wonder if there is not also some appropriate mapping in the spinor space that allows this. Just as in an arbitrary manifold, we do not really expect this to happen in all cases; nonetheless, we know that the important quantity, that describes lengths in the case of Hermitean matrices over these spaces, is the determinant. In just 2 dimensions the Levi-Civita symbol, ϵ_{AB}, which creates determinants, has just 2 indices, so that in fact it could also be used as a mapping sending ξ^A into ξ_A. Since we have restricted our transformation matrices, acting on V^2 and on $\overline{V}_*{}^2$, to have determinant $+1$, such a map would preserve the values of determinants involved.

It is therefore usual to suppose the existence in each of our 4 (2-dimensional) vector spaces, of a numerical, index quantity, a Levi-Civita symbol, each with exactly the same numerical values for their components, namely $+1$ when the components take the (ordered) values $1, 2$, -1 for the values $2, 1$, and 0 otherwise, i.e., 0 along the diagonal:

$$\begin{gathered} \epsilon_{AB} \equiv \epsilon_{\dot{A}\dot{B}} \Longrightarrow \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \Longleftarrow \epsilon^{AB} \equiv \epsilon^{\dot{A}\dot{B}} , \\ \Longrightarrow \epsilon_{AB}\epsilon^{AB} = \delta^A_B \Longrightarrow (\epsilon_{AB})^{-1} = -\epsilon^{AB} . \end{gathered} \tag{1.1}$$

We then use these matrices as a metric in that they map spinors to the corresponding dual spinor, and vice versa. Since we intend to consider only transformations with determinant equal to +1 in these spaces, those transformations always preserve these Levi-Civita symbols, so that they play a distinguished role that is very similar to that of the standard spacetime (Lorentz) metric. However, because this symbol is skew-symmetric, rather than symmetric, it is important to be very careful when using it to "raise" and "lower" indices; the convention for these rules that I use is the following one:

$$\begin{gathered} \xi_A \equiv \epsilon_{AB}\,\xi^B\ , \quad \xi^C \equiv \xi_D\,\epsilon^{DC}\ , \\ \xi^A = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_A = \begin{pmatrix} \xi^2 \\ -\xi^1 \end{pmatrix}\ , \end{gathered} \tag{1.2}$$

along with the same rules for "dotted indices." [Some people use the opposite convention!]

It is therefore true that the invariant sum of the components of a spinor and its dual, i.e., $F_A\,F^A$*, is always identically zero, while for two, non-parallel spinors we have* $F_A\,G^A = -F^A\,G_A$*.*

The action of the elements of the group $\boldsymbol{SL}(2,\mathbb{C})$ on our spinors is then given simply by

$$\xi'^{\,R} = A^R{}_C\,\xi^C\ . \tag{1.3}$$

However, there are now apparently two different ways to anticipate the correct behavior under transformations of the dual spinor, coming from its two "different" roles. Since the dual spinor should map a spinor into a complex number, i.e., a scalar, we should anticipate that it would transform according to the inverse transformation matrix. On the other hand, since we have insisted that we may use the Levi-Civita symbol to map spinors to dual spinors, it should be that we can create the appropriate transformation behavior by using "lowering operators," i.e., appropriate (2-dimensional) Levi-Civita symbols on both sides of the previous equation, Eq. (1.3). It turns out that the constraint that the transformation have determinant of +1 is the same as the statement that these two approaches give exactly the same answer:

$$\begin{gathered} \epsilon_{RS}\,A^R{}_E\,A^S{}_F = \epsilon_{EF}(\det A) = (+1)\epsilon_{EF} = \epsilon_{EF} \\ \Longrightarrow \epsilon_{RS}\,A^R{}_E = \epsilon_{EF}\,(A^{-1})^F{}_S \quad \Longrightarrow \quad \epsilon_{RS}\,A^R{}_E\,\epsilon^{EG} = (A^{-1})^G{}_S = -A_S{}^G\ . \end{gathered} \tag{1.4}$$

But, now, we begin with Eq. (1.3) and multiply both sides by ϵ_{SR}:

$$\xi'{}_S = \epsilon_{SR}\, A^R{}_B\, \xi^B = \epsilon_{SR}\, A^R{}_B\, \epsilon^{CB}\, \xi_C = \epsilon_{RS}\, A^R{}_B\, \epsilon^{BC}\, \xi_C = (A^{-1})^C{}_S\, \xi_C \;. \tag{1.5}$$

The same sorts of statements are of course true for the "dotted" spinors, using the "dotted" transformation equations:

$$\xi'{}^{\dot R} = A^{\dot R}{}_{\dot C}\, \xi^{\dot C} \;, \quad \xi'{}_{\dot R} = (A^{-1})^{\dot C}{}_{\dot R}\, \xi_{\dot C} \quad . \tag{1.6}$$

II. Spinors with multiple indices

Having these 4 sorts of spinor spaces, we may then create many sorts of tensor products, which would contain quantities with components such as F^{AB}, $G^A{}_B$, C_{ABCD}, $R_{AB\dot C\dot D}$, and, of course the Hermitean tensors we have already considered, $X^{A\dot B}$, where we now re-consider further the mapping between Hermitean, 2nd-rank spinors and 4-vectors, or 1-forms, in space-time, from §0. (We will see that several of these other sorts also can play important roles in studies of physical phenomena.)

Beginning with the individual spinor spaces, there are several different ways to create an Hermitean spinor from a simple, 1-index spinor. We first attempt to further understand the meaning of the indices on the (Hermitean) tensor above, $X^{A\dot B}$, by beginning with k^A as an arbitrary element of V^2. Using the corresponding complex-conjugated, or "dotted," spinor, $k^{\dot A} \equiv \overline{k^A}$, we may immediately create the following Hermitean matrix, i.e., 2nd-rank spinor, which we immediately note is Hermitean, but the real 4-vector to which it corresponds is null:

$$\begin{aligned} K^{A\dot B} &\equiv k^A k^{\dot B} = k^A \overline{k^B} = \overline{\overline{k^{\dot A}} k^B} = \overline{k^B k^{\dot A}} = \overline{K^{B\dot A}} \;, \\ \Longrightarrow \quad \det K &= \epsilon_{AC} K^{A\dot 1} K^{C\dot 2} \;= \epsilon_{AC} k^A k^{\dot 1} k^C k^{\dot 2} = 0 \;, \end{aligned} \tag{2.1}$$

and the last equality is because ϵ_{AC}, which is skew symmetric, is being multiplied by the same quantity on each of its two indices, i.e., $\epsilon_{AC} k^A k^C = 0$. Therefore the corresponding 4-vector is of zero length, i.e., a tangent vector to the trajectory of a light ray! In order to create a more general sort of Hermitean spinor from 1-index spinors, we will have to have two of them that are not parallel. Therefore, suppose that k^A and ℓ^B are two non-parallel spinors, where the requirement of not being parallel is the same as the mathematical constraint that $k^A\ell_A \neq 0$. Then, there are a couple of simple ways to create an Hermitean spinor from this pair. The first method begins with a non-Hermitean, 2nd-rank spinor, such as $k^A\ell^{\dot{B}}$, and takes its Hermitean part. As the spinor is complex, and therefore the associated 4-vector is also complex, we may imagine this process as being quite the same as taking either the real or the imaginary part of that 4-vector. This is completely analogous to beginning with an arbitrary complex number, z, and taking either the real part, $(z+\overline{z})/2$, or the imaginary part, $(z-\overline{z})/(2i)$. To describe this for such a matrix it is useful to have a simple scalar to play either the role of $+1$, for the real part, or $+i$, for the imaginary part; therefore, we define the quantity ε such that $\varepsilon^2 = \pm 1$, and then consider the following:

$$
\begin{aligned}
& X^{A\dot{B}} \equiv \varepsilon(k^A\ell^{\dot{B}} \pm \ell^A k^{\dot{B}}) \;, \\
\Longrightarrow \; -\det X = & -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}X^{A\dot{B}}X^{C\dot{D}} = -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}\,\varepsilon(k^A\ell^{\dot{B}} \pm \ell^A k^{\dot{B}})\varepsilon(k^C\ell^{\dot{D}} \pm \ell^C k^{\dot{D}}) \\
& = -\tfrac{1}{2}\varepsilon^2\{\pm(k^A\ell_A)(\ell^{\dot{B}}k_{\dot{B}}) \pm (\ell^A k_A)(k^{\dot{B}}\ell_{\dot{B}})\} = +|k^A\ell_A|^2 \;.
\end{aligned}
\tag{2.2a}
$$

The calculation above shows us that the (associated) length of the 4-vector, $\widetilde{x}$, associated with the Hermitean spinor, $\mathbf{X}$, is real and also positive. This tells us that 4-vectors created in this way from a pair of 2-spinors will always be spacelike; therefore, there must surely be another way to create such Hermitean spinors. Of course there is; instead of looking for real and imaginary parts of complex-valued 4-vectors, instead, we take linear combinations of null vectors, directly:

$$
\begin{aligned}
& X^{A\dot{B}} \equiv k^A k^{\dot{B}} \pm \ell^A\ell^{\dot{B}} \;, \\
\Longrightarrow \; -\det X = & -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}X^{A\dot{B}}X^{C\dot{D}} = -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}(k^A k^{\dot{B}} \pm \ell^A\ell^{\dot{B}})(k^C k^{\dot{D}} \pm \ell^C\ell^{\dot{D}}) \\
& = \mp\tfrac{1}{2}\{|k^A\ell_A|^2 + |\ell^A k_A|^2\} = \mp|(k^A\ell_A|^2 \;,
\end{aligned}
\tag{2.2b}
$$

so that this method allows us the option of creating either timelike or spacelike 4-vectors.

We may therefore begin with two arbitrary, non-parallel 2-spinors, such as k^A and ℓ^B, with $k^A\ell_A \neq 0$, and create the 4 basis vectors needed for tangent vectors, or 1-forms, in spacetime.

Such an approach is often very useful when considering eigenvector problems! Although they are in principle arbitrary, there is no obvious reason why we should not normalize them in some convenient way; therefore, we choose to normalize them so that $k^A\ell_A = +1$. We may then immediately create a linearly-independent set of basis 1-forms, all of which are null:

$$\underset{\sim}{k} \equiv -\tfrac{1}{\sqrt{2}} k_A k_{\dot{B}}\, \underset{\sim}{\boldsymbol{\omega}}^{A\dot{B}}\ ,\ \underset{\sim}{\ell} \equiv \tfrac{1}{\sqrt{2}} \ell_A \ell_{\dot{B}}\, \underset{\sim}{\boldsymbol{\omega}}^{A\dot{B}}\ ,\ \underset{\sim}{m} \equiv \tfrac{1}{\sqrt{2}} k_A \ell_{\dot{B}}\, \underset{\sim}{\boldsymbol{\omega}}^{A\dot{B}}\ ,\ \overline{\underset{\sim}{m}} \equiv \tfrac{1}{\sqrt{2}} \ell_A k_{\dot{B}}\, \underset{\sim}{\boldsymbol{\omega}}^{A\dot{B}}\ , \tag{2.3}$$

where the minus sign in the definition of $\underset{\sim}{k}$ makes later signs much more convenient. However, since the matrix, $k^A\ell^{\dot{B}}$ is not Hermitean, we see that $\underset{\sim}{m}$ is complex; this is of course expected, since it is impossible to find 4, real-valued, linearly-independent null vectors in our spacetime, i.e., a 4-dimensional space with 3 spacelike directions and 1 timelike one. [There are various naming conventions here: some other people use $\underset{\sim}{\ell}$, $\underset{\sim}{n}$ with $\underset{\sim}{m}$ as the complex one, while yet some others follow the original names created by Sachs, which are $\underset{\sim}{k}$, $\underset{\sim}{m}$, with $\underset{\sim}{t}$ as the complex one.]

Since this is an all-null basis set, it is desirable to relate to a more standard, "orthonormal" choice of basis set for 1-forms, made of 3 orthogonal, spacelike 1-forms and a single timelike 1-form also orthogonal to the spacelike ones. The calculations just above, showing how to make spacelike and timelike vectors from the spinorial forms of null vectors, can allow us to do that, which we will do just below. It should, however, be pointed out again that there are very many conventions on exactly how to do that; I follow one originally chosen by Debever and Plebański:

$$dx = \tfrac{1}{\sqrt{2}}(\underset{\sim}{m} + \overline{\underset{\sim}{m}})\ ,\ dy = \tfrac{-i}{\sqrt{2}}(\underset{\sim}{m} - \overline{\underset{\sim}{m}})\ ,\ dz = \tfrac{1}{\sqrt{2}}(\underset{\sim}{\ell} + \underset{\sim}{k})\ ,\ dt = \tfrac{1}{\sqrt{2}}(\underset{\sim}{k} - \underset{\sim}{\ell})\ . \tag{2.4}$$

These definitions may be put into the Hermitean matrix in Eq. (0.5) describing the basis of 1-forms in terms of an Hermitean matrix, and we may also put them into the definition of the metric, to give its form directly in terms of null vectors:

$$
\begin{aligned}
\underset{\sim}{\boldsymbol{\omega}} = \sqrt{2}\begin{pmatrix} \underset{\sim}{\ell} & \overline{\underset{\sim}{m}} \\ \underset{\sim}{m} & -\underset{\sim}{k} \end{pmatrix} &\equiv \sqrt{2}\begin{pmatrix} \underset{\sim}{\boldsymbol{\sigma}}^4 & \underset{\sim}{\boldsymbol{\sigma}}^2 \\ \underset{\sim}{\boldsymbol{\sigma}}^1 & -\underset{\sim}{\boldsymbol{\sigma}}^3 \end{pmatrix} \\
\Longrightarrow \qquad -\det \underset{\sim}{\boldsymbol{\omega}} = \mathbf{g} = \nu_{\alpha\beta}\, \underset{\sim}{\boldsymbol{\sigma}}^\alpha \otimes \underset{\sim}{\boldsymbol{\sigma}}^\beta &= \underset{\sim}{m} \otimes \overline{\underset{\sim}{m}} + \overline{\underset{\sim}{m}} \otimes \underset{\sim}{m} + \underset{\sim}{k} \otimes \underset{\sim}{\ell} + \underset{\sim}{\ell} \otimes \underset{\sim}{k}\,, \\
\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix} &= \nu_{\alpha\beta} = \eta_{\mu\nu} (L^{-1})^\mu{}_\alpha (L^{-1})^\nu{}_\beta \;,
\end{aligned}
\tag{2.6}
$$

where, as before, the product used in taking the determinant is the symmetric tensor product. Any particular choice of 4 basis 1-forms (or basis tangent vectors) is often referred to as a *tetrad*; in the one case a Minkowski tetrad—when the metric is presented by $\eta_{\mu\nu}$—and a null tetrad in this other case when all the elements have 0 length.

Because of the non-diagonal nature of the metric in this null basis, raising and lowering indices requires some little care. As a simple example, if $\underset{\sim}{W}$ is a particular 1-form, say $\underset{\sim}{W} = \alpha\, \underset{\sim}{k} + \beta\, \underset{\sim}{m} \equiv \alpha\, \underset{\sim}{\boldsymbol{\sigma}}^3 + \beta\, \underset{\sim}{\boldsymbol{\sigma}}^1 \equiv W_\mu\, \underset{\sim}{\boldsymbol{\sigma}}^\mu$, then the corresponding tangent vector, $\widetilde{W} \equiv W^\nu\, \widetilde{\boldsymbol{\sigma}}_\mu = \alpha\, \widetilde{\boldsymbol{\sigma}}_4 + \beta\, \widetilde{\boldsymbol{\sigma}}_2$, where we have used the symbols $\widetilde{\boldsymbol{\sigma}}_\mu$ to indicate the reciprocal basis to $\underset{\sim}{\boldsymbol{\sigma}}^\nu$.

One may write the transformation above may in the form of a transformation matrix, which is sometimes helpful:

$$
\begin{aligned}
\underset{\sim}{\boldsymbol{\sigma}}^\alpha = L^\alpha{}_\mu\, dx^\mu \;, \quad \text{i.e.,} \quad \begin{pmatrix} \underset{\sim}{\boldsymbol{\sigma}}^1 \\ \underset{\sim}{\boldsymbol{\sigma}}^2 \\ \underset{\sim}{\boldsymbol{\sigma}}^3 \\ \underset{\sim}{\boldsymbol{\sigma}}^4 \end{pmatrix} \equiv \begin{pmatrix} \underset{\sim}{m} \\ \overline{\underset{\sim}{m}} \\ \underset{\sim}{k} \\ \underset{\sim}{\ell} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +i & 0 & 0 \\ +1 & -i & 0 & 0 \\ 0 & 0 & +1 & +1 \\ 0 & 0 & +1 & -1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} \;, \\
\text{or} \;\; dx^\mu = (L^{-1})^\mu{}_\alpha\, \underset{\sim}{\boldsymbol{\sigma}}^\alpha \;, \quad \text{with} \quad (L^{-1})^\mu{}_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 & 0 & 0 \\ -i & +i & 0 & 0 \\ 0 & 0 & +1 & +1 \\ 0 & 0 & +1 & -1 \end{pmatrix} \;.
\end{aligned}
\tag{2.5}
$$

The determinant of this transformation matrix is given by $\det(L^{-1}) = -i$. This fact is of course needed to determine Hodge duals of various p-forms when presented relative to this (null) basis set. In particular, self-dual, and anti-self-dual, 2-forms are useful physical quantities; therefore, we now show that such 2-forms are represented by symmetric, 2nd-rank spinors. To see this, we begin by considering wedge products of our basic matrix-valued 1-forms. This should give us 2×2 matrices, with entries which are individual 2-forms, with the intent of creating basis sets for 2-forms:

$$\begin{aligned} \underset{\sim}{S}^{AB} &\equiv \tfrac{1}{2}\underset{\sim}{\boldsymbol{\omega}}^{A\dot{C}} \wedge \underset{\sim}{\boldsymbol{\omega}}^{B}{}_{\dot{C}} \equiv \tfrac{1}{2} S^{AB}{}_{\alpha\beta}\, \underset{\sim}{\boldsymbol{\omega}}^{\alpha} \wedge \underset{\sim}{\boldsymbol{\omega}}^{\beta} \in \Lambda^2 \quad , \\ \underset{\sim}{S}^{\dot{A}\dot{B}} &\equiv \tfrac{1}{2}\underset{\sim}{\boldsymbol{\omega}}^{C\dot{A}} \wedge \underset{\sim}{\boldsymbol{\omega}}_{C}{}^{\dot{B}} \equiv \tfrac{1}{2} S^{\dot{A}\dot{B}}{}_{\alpha\beta}\, \underset{\sim}{\boldsymbol{\omega}}^{\alpha} \wedge \underset{\sim}{\boldsymbol{\omega}}^{\beta} \in \Lambda^2 \quad . \end{aligned} \tag{2.6}$$

We write them out more explicitly, but use the null basis 1-forms for this presentation, since the matrices are much simpler then:

$$\begin{aligned} \underset{\sim}{S}^{AB} &= \begin{pmatrix} 2\underset{\sim}{\boldsymbol{\sigma}}^4 \wedge \underset{\sim}{\boldsymbol{\sigma}}^2 & \underset{\sim}{\boldsymbol{\sigma}}^1 \wedge \underset{\sim}{\boldsymbol{\sigma}}^2 + \underset{\sim}{\boldsymbol{\sigma}}^3 \wedge \underset{\sim}{\boldsymbol{\sigma}}^4 \\ \underset{\sim}{\boldsymbol{\sigma}}^1 \wedge \underset{\sim}{\boldsymbol{\sigma}}^2 + \underset{\sim}{\boldsymbol{\sigma}}^3 \wedge \underset{\sim}{\boldsymbol{\sigma}}^4 & 2\underset{\sim}{\boldsymbol{\sigma}}^3 \wedge \underset{\sim}{\boldsymbol{\sigma}}^1 \end{pmatrix} \quad , \\ \underset{\sim}{S}^{\dot{A}\dot{B}} &= \begin{pmatrix} 2\underset{\sim}{\boldsymbol{\sigma}}^4 \wedge \underset{\sim}{\boldsymbol{\sigma}}^1 & -\underset{\sim}{\boldsymbol{\sigma}}^1 \wedge \underset{\sim}{\boldsymbol{\sigma}}^2 + \underset{\sim}{\boldsymbol{\sigma}}^3 \wedge \underset{\sim}{\boldsymbol{\sigma}}^4 \\ -\underset{\sim}{\boldsymbol{\sigma}}^1 \wedge \underset{\sim}{\boldsymbol{\sigma}}^2 + \underset{\sim}{\boldsymbol{\sigma}}^3 \wedge \underset{\sim}{\boldsymbol{\sigma}}^4 & 2\underset{\sim}{\boldsymbol{\sigma}}^3 \wedge \underset{\sim}{\boldsymbol{\sigma}}^2 \end{pmatrix} = \overline{\underset{\sim}{S}^{AB}} \quad , \end{aligned} \tag{2.7}$$

where the statement concerning complex conjugation requires us to recall that $\underset{\sim}{\boldsymbol{\sigma}}^3$ and $\underset{\sim}{\boldsymbol{\sigma}}^4$ are real, while $\underset{\sim}{\boldsymbol{\sigma}}^2$ is the complex conjugate of $\underset{\sim}{\boldsymbol{\sigma}}^1$. One immediately sees that the matrices, $\underset{\sim}{S}^{AB}$ and $\underset{\sim}{S}^{\dot{A}\dot{B}}$ are symmetric in their spinor indices. Therefore, each of them has exactly **three** linearly-independent 2-forms between them, and there is no overlap between the two sets, so that between the two of them they contain the entirety of a basis for 2-forms over M^4, i.e., for Λ^2, analogous to the way that the $\underset{\sim}{\boldsymbol{\sigma}}^{\alpha}$ form a basis for Λ^1. There is in fact an important extra feature to this splitting of the 6 basis 2-forms into two groups of 3. Working through the definition of the Hodge dual for 2-forms, in this null basis, one can in fact show that

$$\begin{aligned} &\text{(a)} \qquad * \underset{\sim}{S}^{AB} = \underset{\sim}{S}^{AB} \quad , \text{ so that those 3 are self-dual,} \\ &\text{(b)} \qquad * \underset{\sim}{S}^{\dot{A}\dot{B}} = -\underset{\sim}{S}^{\dot{A}\dot{B}} \quad , \text{ so that those 3 are anti-self-dual.} \end{aligned} \tag{2.8}$$

Since the two (matrix) sets of 2-forms contain all 6 of the basis 2-forms, we are not surprised when told that the "converse" of these statements is most simply written as the following equality:

$$\underset{\sim}{\omega}^{A\dot{B}} \wedge \underset{\sim}{\omega}^{C\dot{D}} = \epsilon^{\dot{B}\dot{D}} \underset{\sim}{S}^{AC} + \epsilon^{AC} \underset{\sim}{S}^{\dot{B}\dot{D}} \quad . \tag{2.9}$$

which allows us to split an arbitrary 2-form into two parts as follows, its self-dual part and its anti-self-dual part, respectively:

$$\underset{\sim}{F} = \tfrac{1}{2} F_{\alpha\beta}\, \underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{\beta} = F_{AB}\, \underset{\sim}{S}^{AB} + F_{\dot{A}\dot{B}}\, \underset{\sim}{S}^{\dot{A}\dot{B}} \quad , \tag{2.10}$$

where

$$F_{AB} = \tfrac{1}{8} S^{\alpha\beta}{}_{AB}\, F_{\alpha\beta} = \overline{F_{\dot{A}\dot{B}}} \quad , \tag{2.11}$$

so that

$$\begin{aligned} &F_{AB}\, \underset{\sim}{S}^{AB} \quad \text{is a self-dual 2-form,} \\ &F_{\dot{A}\dot{B}}\, \underset{\sim}{S}^{\dot{A}\dot{B}} \quad \text{is an anti-self-dual 2-form.} \end{aligned} \tag{2.12}$$

This gives us the promised "explanation" of the physical meaning of symmetric, 2nd-rank spinors, namely that those in $V^2 \otimes V^2$ are the image of the (3 independent degrees of freedom of the) self-dual parts of 2-forms over spacetime, while those in $\overline{V^2} \otimes \overline{V^2}$, i.e., those with 2 dotted indices, are the image of the anti-self-dual parts.

A choice for an explicit labelling for the 3 degrees of freedom of any such symmetric, 2nd-rank spinor, say F_{AB}, is as follows:

$$F_{AB} = \frac{1}{4} \begin{pmatrix} C^{+} & -C^{0} \\ -C^{3}0 & -C^{-} \end{pmatrix} , \qquad C^{\pm} \equiv C^{x} \pm i C^{y} \ , \quad C^{0} \equiv C^{z} \ . \tag{2.13}$$

Since these are only 2×2 matrices, one may always decompose a symmetric, 2nd-rank spinor, such as F_{AB} in terms of two 1-spinors, k_A and ℓ_B, and a scalar λ, which amounts to a magnitude:

$$F_{AB} = 2\lambda\, k_{(A}\ell_{B)}, \ \text{along with} \ \begin{cases} k^{A}\ell_{A} = 1\, , & \text{so that } \det(F_{AB}) \neq 0\, , \\ k^{A}\ell_{A} = 0\, , & \text{so that } \det(F_{AB}) = 0\, . \end{cases} \tag{2.14}$$

These quantities may be thought of as "eigenspinors," with eigenvalues $\pm\lambda$, since one has the following obvious consequences:

$$F_{AB}\, k^{B} = \lambda\, k_{A} \quad , \qquad F_{AB}\, \ell^{B} = -\lambda\, \ell_{A} \quad . \tag{2.15}$$

Having such a decomposition of our self-dual 2-form, for example, we may use these two eigenspinors to create a specially-oriented tetrad back in spacetime, following the model given in Eqs. (2.2), which puts the original 2-form in its optimally-simple form.

III. Indicial Approach to the Spinor Transformation Equations

Having now more details about the place(s) where the various combinations of spinors exist, we may now return to our original understandings of the matrices forming the spinor transformations and put more matrix indices into those equations. We began with Eqs. (0.7), relating the transformations of 1-forms and of Hermitean, 2×2 matrices over spacetime. We now propose to rewrite those equations from an "earlier" point of view, with the fundamental proposition that the *unimodular* matrices A, used in §0, are the same as the basic transformation matrices for our fundamental spinor space, V^2, as described in Eq. (1.3): Therefore, our Hermitean matrices would transform as follows, where, for example, we pick a (null) 1-form, $\mathbf{Y}$, created from an arbitrary $k^A \in V^2$:

$$\begin{aligned} Y'^{\,R\dot{S}} = k'^{\,R} k'^{\,\dot{S}} = A^R{}_B\, \overline{A^S{}_E}\, k^B k^{\dot{E}} = A^R{}_B\, A^{\dot{S}}{}_{\dot{E}}\, Y^{B\dot{E}} \\ \text{or, in matrix form} \quad Y' = A\, Y\, \overline{A^T} = A\, Y\, A^\dagger \ . \end{aligned} \tag{3.1}$$

At this point a minor aside is of some value, relative to the equations describing the Lie algebra for these matrices, A, involving representations in terms of the Pauli sigma matrices, as noted at Eqs. (0.9) and (0.13), and the equations describing the use of the Pauli sigma matrices to form a basis set for 1-forms, already at Eqs. (0.3). In principle these two sets of objects are quite different:

a. the basis 1-forms are objects from the product spinor space, $V^2 \otimes \overline{V}^2$, and we used the (standard) Pauli sigma matrices to easily say what their presentation was in a particular frame, while

b. the identification of the generators of the Lie algebra, $\theta\hat{e} \cdot \boldsymbol{\mathcal{J}} + \lambda\hat{v} \cdot \boldsymbol{\mathcal{K}}$, must clearly have their presentation in terms of objects that live in the product spinor space, $V^2 \otimes V^2_*$, so that the use of the actual matrices $\widetilde{\alpha} \cdot \widetilde{\sigma}$ to describe them is again simply a reasonable way to present the desired form of the matrices in terms of a standard, well-defined set of matrices.

This means that there are no particular spinor indices associated to the actual Pauli matrices; rather, we must know the physical intent/content of the objects being considered in order to know how they should transform. A different point of view is given by noting that although both the elements of $\underset{\sim}{\boldsymbol{\omega}}^{A\dot{B}}$ and of $\left(\theta\hat{e}\cdot\boldsymbol{\mathfrak{J}}+\lambda\hat{v}\cdot\boldsymbol{\mathfrak{K}}\right)^{A}{}_{B}$ are given in terms of the Pauli sigma matrices in a particular choice of basis, if we were to transform both of these quantities to some different frame they would transform differently so that they would no longer appear to use the *same* set of matrices for their presentations.

Returning, now, to Eqs. (0.15), which gives the explicit formulation of the representations, within $\boldsymbol{SL}(2,\mathbb{C})$, of rotations and boosts, the description above suggests that if the set of all unimodular matrices, A, constitutes a set of transformation equations for V^2, while the set of all of the complex conjugates of these matrices constitutes a set of transformation equations for the complex-conjugate spinor space, $\overline{V}^2$, these being in fact two different representations of the Lorentz group. Choosing a convention and agreeing that the representation usually labelled as $D(0,\frac{1}{2})$ is the one given in Eqs. (0.13) and (0.15), the alternative one is the complex conjugate representation, $D(\frac{1}{2},0)$:

$$\begin{gathered}
D(0,\tfrac{1}{2}) \implies \begin{cases} \vec{\boldsymbol{\mathfrak{J}}} \Longrightarrow & -\frac{i}{2}\vec{\sigma} \\ \vec{\boldsymbol{\mathfrak{K}}} \Longrightarrow & +\frac{1}{2}\vec{\sigma} \end{cases} \implies \begin{cases} D^{A}{}_{B}[R(\theta;\hat{e})] = (e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}})^{A}{}_{B} \\ D^{A}{}_{B}[B(\lambda;\hat{v})] = (e^{+(\lambda/2)\hat{v}\cdot\vec{\sigma}})^{A}{}_{B} \end{cases}, \\
D(\tfrac{1}{2},0) \implies \begin{cases} \vec{\boldsymbol{\mathfrak{J}}} \Longrightarrow & -\frac{i}{2}\vec{\sigma} \\ \vec{\boldsymbol{\mathfrak{K}}} \Longrightarrow & -\frac{1}{2}\vec{\sigma} \end{cases} \implies \begin{cases} D^{A}{}_{B}[R(\theta;\hat{e})] = (e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}})^{A}{}_{B} \\ D^{A}{}_{B}[B(\lambda;\hat{v})] = (e^{-(\lambda/2)\hat{v}\cdot\vec{\sigma}})^{A}{}_{B} \end{cases},
\end{gathered} \tag{3.2}$$

where it is true that if one needed to do so, for clarity of notation, one could have written $(D^{(0,1/2)})^{A}{}_{B}$ and $(D^{(1/2,0)})^{A}{}_{B}$, respectively. On the other hand, if you do believe that one obtains the matrices for the representation $D(1/2,0)$ from those for $D(0,1/2)$ by complex conjugation, as is indeed stated above, you might wonder. This wonderment occurs because σ_y is imaginary while the other two are real, so that complex conjugation actually does strange things to some quantity such as $\hat{v}\cdot\vec{\sigma}$; however, this strangeness may be cleaned up by an appropriate change of basis. More precisely, notice that a change of basis that amounts to a rotation about the $\hat{y}$-axis by angle $\theta=\pi$ would change the sign of $\hat{x}$ and $\hat{z}$ but leave invariant the sign of $\hat{y}$; coupled with a complex-conjugation this would change the sign of the entire vector, $\vec{\sigma}$, making it appear much less strange. Using the generic expansion given in Eq. (0.9) and the formulation for $D(0,1/2)$ given in Eqs. (0.15), we see that such a rotation would be given simply by the matrix $i\sigma_y$, with inverse $-i\sigma_y$; therefore, we may quickly write that if we first complex conjugate the matrix elements, and then perform this transformation we would have the following effects on the generators, and on the matrix elements:

$$\sigma_y \overline{\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}} \sigma_y = - \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} , \quad \Longrightarrow \quad \begin{cases} \sigma_y \, (\overline{e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}}})\sigma_y = e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}} \\ \sigma_y \, (\overline{e^{+(\lambda/2)\,\hat{v}\cdot\vec{\sigma}}})\sigma_y = e^{-(\lambda/2)\,\hat{v}\cdot\vec{\sigma}} \end{cases} \tag{3.3}$$

We are now ready to generalize our discussion of spinorial representations of the Lorentz group beyond the simple, 2-dimensional cases of $D(0,1/2)$ and $D(1/2,0)$. It is worthwhile to begin by re-writing some of the other equations in §0 with explicit spinorial indices. As the matrix presentation of the components of an ordinary 4-vector, or 4-dimensional 1-form, was made in terms of an Hermitean matrix, we anticipate that the spinorial indices on that matrix should indeed have that character. Therefore we rewrite Eqs. (0.3) and (0.5) as follows:

$$\begin{gathered} \mathbf{X}^{A\dot{B}} \equiv x^\mu \, \sigma_\mu^{A\dot{B}} = \utilde{\boldsymbol{\omega}}^{A\dot{B}}(\widetilde{x}) \, , \\ \utilde{\boldsymbol{\omega}}^{A\dot{B}} \equiv \sigma_\mu^{A\dot{B}} \, \utilde{\boldsymbol{\omega}}^\mu = \begin{pmatrix} dz - dt & dx - i\,dy \\ dx + i\,dy & -dz - dt \end{pmatrix} = \sqrt{2} \begin{pmatrix} \utilde{\boldsymbol{\sigma}}^4 & \utilde{\boldsymbol{\sigma}}^2 \\ \utilde{\boldsymbol{\sigma}}^1 & -\utilde{\boldsymbol{\sigma}}^3 \end{pmatrix} \, . \end{gathered} \tag{3.4}$$

We want now to have a simple, analytical, indicial form for the inverse of this equation, which gives us $\utilde{x}$, having been given $\mathbf{X}$, in terms of the components of $\widetilde{x}$. To do this we begin by writing down some equalities relating determinant-like forms of the basic σ^μ matrices:

$$\begin{gathered} +\epsilon_{\dot{R}\dot{S}}\sigma^{\alpha A\dot{R}}\,\sigma^{\beta B\dot{S}} = -\eta^{\alpha\beta}\epsilon_{AB} + S^{\alpha\beta AB} \, , \quad \utilde{S}^{AB} \equiv \tfrac{1}{2} S^{AB}_{\alpha\beta}\,\utilde{\boldsymbol{\omega}}^\alpha \wedge \utilde{\boldsymbol{\omega}}^\beta \;\Rightarrow\; \sigma^{\alpha A\dot{R}}\sigma_{\beta A\dot{R}} = -2\delta^\alpha_\beta \, , \\ -\tfrac{1}{2}\sigma^{\alpha A\dot{B}}\,\sigma_{\alpha C\dot{D}} = \delta^A_C\,\delta^{\dot{B}\dot{D}} \, , \; S^{AB}_{\alpha\beta}\,S^{\alpha\beta}_{CD} = 4\delta^A_{(C}\delta^B_{D)} \, , \; S^{\dot{A}\dot{B}}_{\alpha\beta}\,S^{\alpha\beta}_{\dot{C}\dot{D}} = 4\delta^{\dot{A}}_{(\dot{C}}\,\delta^{\dot{B}}_{\dot{D})} \, , \; S^{\alpha\beta}_{AB}\,S^{\dot{C}\dot{D}}_{\alpha\beta} = 0 \, , \\ \tfrac{1}{2} S^{\alpha\beta}_{AB}\,S^{AB}_{\gamma\delta} = \delta^\alpha_{[\gamma}\delta^\beta_{\delta]} + \eta^{\alpha\beta\sigma\tau} g_{\gamma\sigma}g_{\delta\tau} \, , \quad S^{\alpha\beta A}{}_R\,S_{\gamma\delta}{}^R{}_B = -2\,\delta^{\alpha\beta\tau}_{\gamma\delta\sigma}\,S^\sigma{}_\tau{}^A{}_B \, . \end{gathered} \tag{3.5}$$

where the basis forms for self-dual 2-forms, $\utilde{S}^{AB}$, are described in the material following Eqs. (2.6), the components of the metric tensor $\mathbf{g}$, used above may be either an orthonormal or null basis, as used for the basic 1-forms, and the shorthand notations with round and square brackets around the indices indicate anti-commutators and commutators, respectively, i.e., $X_{(a}Y_{b)} \equiv X_aY_b + X_bY_a$ and $X_{[a}Y_{b]} \equiv X_aY_b - X_bY_a$. These equalities, and some algebra, allow us to invert the defining formula, Eq. (3.4), as was originally desired, for an arbitrary 1-form, $\underset{\sim}{Y}$ and its associated tangent vector, $\widetilde{Y}$:

$$Y_\mu \underset{\approx}{\boldsymbol{\omega}}^\mu \equiv \underset{\sim}{Y} = -\tfrac{1}{2}\mathbf{Y}_{A\dot{B}}\,\underset{\approx}{\boldsymbol{\omega}}^{A\dot{B}} = -\tfrac{1}{2}\mathbf{Y}_{A\dot{B}}\,\sigma_\mu^{A\dot{B}}\underset{\approx}{\boldsymbol{\omega}}^\mu\ , \qquad \mathbf{Y}^{A\dot{B}} = \underset{\approx}{\boldsymbol{\omega}}^{A\dot{B}}(\widetilde{Y}) = \sigma_\mu^{A\dot{B}}\,\underset{\approx}{\boldsymbol{\omega}}^\mu(\widetilde{Y}) = \sigma_\mu^{A\dot{B}}\,Y^\mu\ . \tag{3.6}$$

We may now describe in somewhat more detail the relation between the matrix $\mathbf{A} \in SL(2,\mathcal{C})$ and the matrix $L \in SO(3,1)$. We first set down the defining equations for an arbitrary tangent vector, $\widetilde{Y}$, the associated 1-form, $\underset{\sim}{Y}$, and for their Hermitean matrix presentations with either upper or lower indices:

$$\begin{gathered}
Y'^{\,\mu} = L^\mu{}_\alpha\, Y^\alpha\ , \quad Y'{}_\nu = (L^{-1})^\alpha{}_\nu\, Y_\alpha = L_\nu{}^\alpha\, Y_\alpha\ , \qquad g_{\alpha\beta} = L^\mu{}_\alpha\, L^\nu{}_\beta\, g_{\mu\nu}\ , \\
Y'^{\,R\dot{S}} = A^R{}_C\, A^{\dot{S}}{}_{\dot{E}}\, Y^{C\dot{E}}\ , \quad Y'{}_{R\dot{S}} = (A^{-1})^C{}_R\,(A^{-1})^{\dot{E}}{}_{\dot{S}}\, Y_{C\dot{E}} = A_R{}^C\, A_{\dot{S}}{}^{\dot{E}}\, Y_{C\dot{E}}\ , \\
\eta_{\mu\nu}\, L^\mu{}_\alpha\, L^\nu{}_\beta = \eta_{\alpha\beta} \quad \Longrightarrow \quad L_\kappa{}^\alpha = (L^{-1})^\alpha{}_\kappa \ \text{ or } \ (HLH^{-1})^T = L^{-1}\ , \\
+1 = \det A \quad \Longrightarrow \ \epsilon_{AB}\, A^A{}_M\, A^B{}_N = \epsilon_{MN} \ \Longrightarrow \ -A_B{}^M = +(A^{-1})^M{}_B \ \text{ or } \ (\epsilon A \epsilon^T)^T = A^{-1}\ .
\end{gathered} \tag{3.7}$$

These relations allow us to "factor out" the components of the 1-form itself, from which we may infer the equality of the action of the Lorentz transformation on the spacetime indices of the sigma matrices and the action of the spinorial transformation matrices on the matrix indices of those same sigma matrices:

$$L^\mu{}_\alpha\, \sigma_\mu^{R\dot{S}} = A^R{}_C\, A^{\dot{S}}{}_{\dot{E}}\, \sigma_\alpha^{C\dot{E}} \ \text{ or } \ L^\mu{}_\alpha\, \sigma_\mu = A\, \sigma_\alpha\, \overline{A}^T = A\, \sigma_\alpha\, A^\dagger \quad , \tag{3.8}$$

Using now the equation for the trace of the product of two sigma matrices with different spacetime indices, from the first line of Eqs. (3.5), we may resolve this equality for the matrix elements of the Lorentz transformation desired:

$$L^\mu{}_\beta = -\tfrac{1}{2} A^R{}_C\, \sigma_\beta^{C\dot{E}}\, A^{\dot{S}}{}_{\dot{E}}\, \sigma^\mu_{R\dot{S}}\ , \qquad \text{or} \quad L^\mu{}_\beta = -\tfrac{1}{2}\mathrm{tr}(A\sigma_\beta A^\dagger \sigma^{\mu T}) \quad . \tag{3.9}$$

We had two earlier discussions using two basis spinors, k^A and ℓ^B. We could use them to create a null basis in spacetime, as discussed following Eqs. (2.3), and, also, we could use them to determine a complete set of eigenspinors for an arbitrary symmetric, second-rank spinor, as in the discussion near Eqs. (2.14). We may now use that same thought to create a somewhat different parametrization for the Lorentz transformations. In both those cases there is an ambiguity that arises in their choice: Having chosen some such pair of spinors, one still has the freedom to change to a new pair, k'_A and/or ℓ'_B, related to the original ones by any one, *or more*, of the following transformations:

$$A: \quad \begin{cases} A_\sigma : \begin{cases} {k_A}' = e^\sigma k_A \\ {l_A}' = e^{-\sigma} {l_A}' \end{cases} & , \\ A_\rho : \begin{cases} {k_A}' = k_A \\ {l_A}' = l_A - \rho k_A \end{cases} & , \\ A_\eta : \begin{cases} {k_A}' = k_A - \eta l_A \\ {l_A}' = l_A \end{cases} & . \end{cases} \tag{3.10}$$

Treating each of these 3 parameters as a complex number, this gives us a parametrization of a 6-real-parameter group, identifying the ambiguity with which we could have chosen this pair of spinors so as to constitute a spinorial basis, and, thereby, a null basis for spacetime. Therefore they constitute 6 degrees of freedom for the choice of such a basis, which corresponds to degrees of freedom for Lorentz transformations, since they preserve the character of a basis set. Each of these transformations has the property that it preserves some 4-vector of zero length, while the standard "rotations" and "boosts" preserve each preserve two vectors, one of which is spacelike and one timelike—for rotations—while for boosts both of them are spacelike. If we choose the unprimed versions of these two spinors as a basis for spinorial matrices, then we may write down the 2×2 matrices associated with these 3 sorts of Lorentz transformations as follows:

$$A: \quad \begin{cases} A_\sigma = \begin{pmatrix} e^\sigma & 0 \\ 0 & e^{-\sigma} \end{pmatrix} & , \quad \sigma\text{-transformations, for } \sigma \in \mathcal{C} \\ A_\rho = \begin{pmatrix} 1 & -\rho \\ 0 & 1 \end{pmatrix} & , \quad \rho\text{-transformations, for } \rho \in \mathcal{C} \\ A_\eta = \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix} & , \quad \eta\text{-transformations, for } \eta \in \mathcal{C} \end{cases} \tag{3.11}$$

Of course we can use, for instance, Eqs. (2.3) and (2.6) to move these transformations into spacetime. Perhaps the simplest way to do this is to write them in a matrix format that makes changes on the null basis sets for 1-forms, $\underset{\sim}{\boldsymbol{\sigma}}^\alpha$:

$$\underset{\sim}{\boldsymbol{\sigma}}'^\alpha \equiv L^\alpha{}_\nu \, \underset{\sim}{\boldsymbol{\sigma}}^\nu \ : \qquad \begin{cases} L_\sigma = \begin{pmatrix} e^{2i\,\mathrm{Im}(\sigma)} & 0 & 0 & 0 \\ 0 & e^{-2i\,\mathrm{Im}(\sigma)} & 0 & 0 \\ 0 & 0 & e^{2\,\mathrm{Re}(\sigma)} & 0 \\ 0 & 0 & 0 & e^{-2\,\mathrm{Re}(\sigma)} \end{pmatrix} \\ L_\rho = \begin{pmatrix} 1 & 0 & -\overline{\rho} & 0 \\ 0 & 1 & -\rho & 0 \\ 0 & 0 & 1 & 0 \\ \rho & \overline{\rho} & -\rho\overline{\rho} & 1 \end{pmatrix} \\ L_\eta = \begin{pmatrix} 1 & 0 & 0 & -\eta \\ 0 & 1 & 0 & -\overline{\eta} \\ \overline{\eta} & \eta & -\eta\overline{\eta} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{cases} \tag{3.12}$$

One sees that the σ-transformations amount to re-scalings of the (affine) parameters along the null directions, while the other transformations are of the form as to preserve one of the real null-directions while "rotating" all the other directions "around" that one.

Another interesting comment concerns the generators of these transformations. Hopefully it is straightforward to see that if we write any one of these transformations as the exponential of its logarithm, i.e., we write $L_i \equiv e^{Q_i}$, then we have

$$\begin{aligned} Q_\sigma &= \begin{pmatrix} \sigma - \overline{\sigma} & 0 & 0 & 0 \\ 0 & -\sigma + \overline{\sigma} & 0 & 0 \\ 0 & 0 & \sigma + \overline{\sigma} & 0 \\ 0 & 0 & 0 & -\sigma - \overline{\sigma} \end{pmatrix}, \\ Q_\rho &= \begin{pmatrix} 0 & 0 & -\overline{\rho} & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & 0 \\ \rho & \overline{\rho} & 0 & 0 \end{pmatrix}, \\ Q_\eta &= \begin{pmatrix} 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & -\overline{\eta} \\ 0 & 0 & 0 & 0 \\ \overline{\eta} & \eta & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.13}$$

The re-scaling transformation has a generator which is diagonal, in this null basis, while the other two are "parabolic" in the sense that the exponential series terminates after the second-order term since the cube of the generator vanishes.

IV. Higher-order Representations of $\boldsymbol{SL}(2,\mathbb{C})$

A different way to create the original, 4×4 matrix, transformations on spacetime from the 2×2 spinorial transformations is to start looking at higher-order spinor transformations. We have already described carefully the two basic representations, $D(0,1/2)$—corresponding to transformations on V^2—and $D(1/2,0)$—corresponding to transformations on $\overline{V}^2$. We may therefore begin to take various products of them. The representations $D(0,n/2)$, for any integer n, correspond to transformations acting on spinorial tensors with n indices, completely symmetric under interchange of any of them, such as, say, $F_{A_1A_2\ldots A_n}$. These spinors must be symmetric because any pair of skew-symmetric indices in spinor space can only really take on one value, either $1,2$, or $2,1$, and, therefore, must be proportional to the Levi-Civita symbol there. We can demonstrate this easily, using, first, a second-rank spinorial quantity:

$$Z_{AB}=\tfrac{1}{2}\left(Z_{AB}+Z_{BA}\right)+\tfrac{1}{2}\left(Z_{AB}-Z_{BA}\right)=\tfrac{1}{2}\left(Z_{AB}+Z_{BA}\right)+\tfrac{1}{2}\epsilon_{AB}\,\epsilon^{CD}\,Z_{CD}\ . \tag{4.1}$$

This tells us that that portion of a tensor that has a pair of indices for which the tensor is skew symmetric must be proportional to just the Levi-Civita symbol, with the proportionality actually being the trace, $\epsilon^{AB}\,Z_{AB}$. As well, it is clear that, in a 2-dimensional space, a non-zero tensor cannot be skew-symmetric on an entire triplet of indices; therefore, we can always reduce an arbitrary n-th rank spinorial tensor to one which is totally symmetric in all n indices, plus some others which are products of some number of Levi-Civita symbols multiplied by tensors of lower order, i.e., of order $n-2$, $n-4$, and so forth.

An entirely other way to approach this question comes from thinking about the representation theory for our transformation group. Since all the matrices have determinant one, the Levi-Civita symbol is actually an invariant tensor, and therefore may be said to correspond to a 1-dimensional representation, $D(0,0)$, where every such transformation in the original group, $\boldsymbol{SL}(2,\mathbb{C})$ is represented by just the scalar $+1$. It is also obvious, I suggest, that the summed out trace, $Z^{AB}\,Z_{AB}$, is also an invariant, and therefore corresponds to transformation by the representation $D(0,0)$. However, we now consider the product of $D(0,1/2)\otimes D(0,1/2)$, which must equal the direct sum, $D(0,1) \oplus D(0,0)$. In geometrical language this statement about representations says that an arbitrary tensor with two indices of the same kind, i.e., an element of the direct product space, $V^2 \otimes V^2$, should decompose into a subspace with dimension 3, corresponding to the representation $D(0,1)$, and a subspace with dimension 1, corresponding to the representation $D(0,0)$. The subspace of dimension 3 is of course the subspace symmetric on the pair of indices, as described in Eq. (4.1) above, and the subspace of dimension 1 is the skew-symmetric part, made up of the invariant Levi-Civita tensor multiplied by some scalar quantity, the "trace" of that second-rank tensor. Likewise consideration of the decomposition of the product $D(0,1) \otimes D(0,1/2) = D(0,3/2) \oplus D(0,1/2)$ should start us with $F_{AB} W_C$, where F_{AB} is symmetric, and end us with a 3-index, symmetric tensor and a Levi-Civita symbol picking up two indices and an additional one left over, for the $D(0,1/2)$.

Let us now go on to more complicated forms. We of course begin with the representation $D(1/2,1/2) = D(0,1/2) \otimes D(1/2,0)$, appropriate for the matrix presentation of real 4-vectors (or 4-dimensional 1-forms). To see the details of the representation in more detail, we should try to create the 4-dimensional generators, beginning of course from the presentations given in Eqs. (3.2) for the two 2-dimensional representations. To do this, we will have to "back up" slightly, and consider what happens to the generators when considering a product representation. We understand that the matrices representing the group elements of a product representation correspond to direct products of the individual representations of that group element. However, the following is the right approach for the generators, for any Lie group, G, with Lie algebra, $\mathcal{G}$:

a given $g \in G$ such that there exists $Q \in \mathcal{G}$ such that $G = e^Q$;

b there are two distinct representations of G, and we label the two representations of $g \in G$ by $D_1(g)$ and $D_2(g)$;

c we label the representations of the generator Q, for convenience, simply by $Q^{(1)}$ and $Q^{(2)}$, so that

$$D_1(g) = e^{Q^{(1)}} \;, \quad D_2(g) = e^{Q^{(2)}} \;. \tag{4.2}$$

Then the product representation is generated in the following way:

$$\{D_1 \otimes D_2\}(g) \equiv [D_1(g)] \otimes [D_2(g)] = e^{Q^{(12)}} \ , \quad Q^{(12)} \equiv Q^{(1)} \otimes I_2 + I_1 \otimes Q^{(2)} \ , \tag{4.3}$$

where I_i indicates the matrix which is the identity matrix in representation D_i. To verify this we consider the following, setting, first $T \equiv Q^{(1)}$:

$$\begin{aligned} e^{T\otimes I_2} = I_{1\otimes 2} + T \otimes I_2 + \frac{1}{2!}(T\otimes I_2)(T\otimes I_2) + \frac{1}{3!}(T\otimes I_2)(T\otimes I_2)(T\otimes I_2) + \ \ldots \\ = \left\{ I_1 + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \ \ldots \right\} \otimes I_2 = D_1(g) \otimes I_2 \ . \end{aligned} \tag{4.3a}$$

We have an entirely analogous construction for the exponential of $I_1 \otimes Q^{(2)}$, so that we may finally write

$$\begin{aligned} e^{Q^{(1)}\otimes I_2 + I_1 \otimes Q^{(2)}} = e^{Q^{(1)}\otimes I_2}\, e^{I_1\otimes Q^{(2)}} = \{D_1(g)\otimes I_2\}\,\{I_1 \otimes D_2(g)\} \\ = D_1(g) \otimes D_2(g) \equiv \{D_1 \otimes D_2\}(g) \ , \end{aligned} \tag{4.3b}$$

where we have used the "obvious" fact that $Q^{(1)} \otimes I_2$ commutes with $I_1 \otimes Q^{(2)}$.

With that notion in hand, we may immediately use Eq. (3.2) to write down the generators for the representation $D(1/2, 1/2)$:

$$\vec{\mathfrak{J}} \Longrightarrow \quad -\tfrac{i}{2}\{\vec{\sigma}\otimes I_2 + I_2 \otimes \vec{\sigma}\} \ , \qquad \vec{\mathfrak{K}} \Longrightarrow \quad \tfrac{1}{2}\{\vec{\sigma}\otimes I_2 - I_2 \otimes \vec{\sigma}\}$$

$$\begin{gathered} \mathfrak{J}^x \Longrightarrow -\frac{i}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ \mathfrak{J}^y \Longrightarrow +\frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ \mathfrak{J}^z \Longrightarrow i\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathfrak{K}^x \Longrightarrow \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \ \mathfrak{K}^y \Longrightarrow +\frac{i}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \ \mathfrak{K}^z \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{4.4}$$

You should now immediately point out that we already know what the generators $\vec{\mathfrak{J}}$ and $\vec{\mathfrak{K}}$ look like in 4 dimensions, as they act on tangent vectors in spacetime, say, and the ones above are not those at all. This is certainly true; however, as usual, the problem here is that these are given with respect to a different set of basis vectors. With some algebra it is not too hard to show the following:

$$
\begin{aligned}
& W\,\vec{\mathfrak{M}}^{(12)}\,W^{-1} = \vec{\mathfrak{M}}^{(3+1)}\,, \\
& W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & +1 & -1 & 0 \end{pmatrix}\,,
\end{aligned}
\tag{4.5}
$$

where $\mathfrak{M}$ is meant to indicate that it is true for both $\mathfrak{J}$ and $\mathfrak{K}$, and the representation labelled by $(3+1)$ is our original $\{x, y, z, t\}$ representation for these generators. This tells us, as expected, that the standard matrices for $D(1/2, 1/2)$ are simply given with respect to a different basis, built on $(x \pm iy)/\sqrt{2}$ and $(\pm z - t)/\sqrt{2}$, which is not too surprising.