Physics 495

Midterm Examination

Wednesday, 21 October, 2009

Part A: Please answer 3, and only 3 of the questions in this part. [18 pts each]

1. Please make a good drawing of a (standard, only 2-dimensional) Minkowski diagram, as seen by an observer $\mathcal{O}$, with his x and t-axes shown at 90° to one another. On this diagram, please show the following things:

a.) the world line of a clock that $\mathcal{O}$ maintains at the location $x = 1$ m;

b.) the trajectories of two light rays sent out by $\mathcal{O}$ at his time $t = 0$, into the $+\hat{x}$-direction and the $-\hat{x}$-direction;

c.) the world line of a distinct inertial observer, $\mathcal{O}'$, that $\mathcal{O}$ measures to be moving with speed $v = 0.25$, and which was coincident with $\mathcal{O}$ at their mutual values of $t = 0 = t'$;

d.) the line of events that $\mathcal{O}'$ measures to be occurring at her time $t' = 3$ meters;

e.) the world line of an observer at rest with respect to $\mathcal{O}'$, but who was at the location $x = -2$ m at the time $t = 0$ m;

f.) the locus of all events which, relative to the origin, have $(\Delta s)^2 = -9$ meters^2;

g.) the locus of all events which, relative to the origin, have $(\Delta s)^2 = +4$ meters^2, and which also occur at $t \geq 0$.

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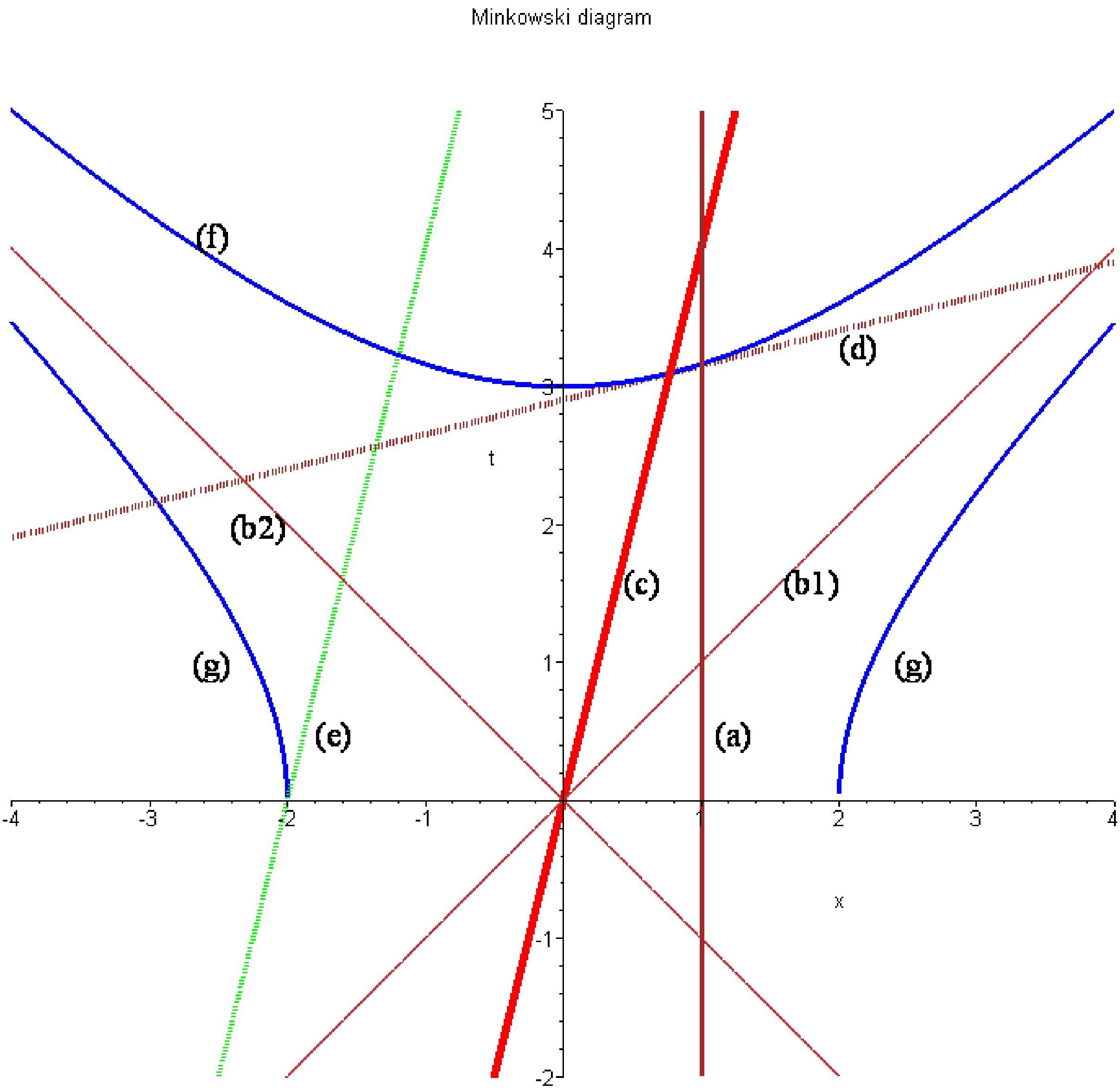

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2. Our standard observer $\mathcal{O}$ says that $\mathcal{O}'$ is moving with velocity $\vec{v} = v\hat{x}$, and also says that $\mathcal{O}'$ has just sent out a light ray that is moving at an angle of θ relative to the direction of the velocity of $\mathcal{O}'$. At what angle, θ', does $\mathcal{O}'$ measure that light ray to be moving relative to his $\hat{x}'$-direction, which is, of course parallel to that of $\mathcal{O}$. (A relation between $\tan\theta'$ and $\tan\theta$ is what is wanted. It is helpful to consider the 4-momentum of the light ray in the two frames, and to have the light ray moving in the x, y-plane.)

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An approach suggested by the hint is to note that the 4-momentum of the photon, as measured by $\mathcal{O}$, is just $(p\cos\theta, p\sin\theta, 0, E)$, where we have chosen the $\hat{y}$-axis so that the photon moves in the x, y-plane. Since a photon has $p = E$, we could also rewrite this 4-vector as $E(\cos\theta, \sin\theta, 0, 1)$. Now we may transform this to the reference frame $\mathcal{O}'$ by using (an inverse) boost to velocity $v\hat{x}$:

$$E'\begin{pmatrix}\cos\theta'\\ \sin\theta'\\ 0\\ 1\end{pmatrix} = \widetilde{p}' = B(-v\hat{x})\widetilde{p} = \begin{pmatrix}\gamma & 0 & 0 & -\gamma v\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\gamma v & 0 & 0 & \gamma\end{pmatrix} E\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\\ 1\end{pmatrix} = E\begin{pmatrix}\gamma(\cos\theta - v)\\ \sin\theta\\ 0\\ \gamma(-v\cos\theta + 1)\end{pmatrix}$$

By dividing the x-component into the y-component on both sides we obtain

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - v)} = \frac{\tan\theta}{\gamma(1 - v\sec\theta)}\ .$$

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3. The standard observer $\mathcal{O}$ is measuring data for a particle of rest mass 4, in some appropriate units, and 4-momentum $\widetilde{p} \Longrightarrow (0, 2, 4, E)^T$, where E is its energy. Also he notices a different observer, $\mathcal{O}'$, whom he measures to have 4-velocity $\widetilde{u} \Longrightarrow (0, 4/3, 0, 5/3)^T$.

a. What is the value of the energy E of the moving particle?

b. What is the value of the 3-velocity of the observer $\mathcal{O}'$?

c. What is the value of the energy E' of the moving particle as measured by $\mathcal{O}'$?

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a. As $\mathcal{O}$ has measured $\widetilde{p}$ we may simply use the fact that its square is the negative of the square of its rest mass:

$$-16 = -(4)^2 = -m^2 = \widetilde{p}\cdot\widetilde{p} = 0^2 + 2^2 + 4^2 - E^2 = 20 - E^2 \quad\Longrightarrow\quad E = \sqrt{36} = 6\ .$$

b. The 3-velocity of $\mathcal{O}'$ is just given by the following, using its given 4-velocity:

$$\vec{v} = \frac{\frac{4}{3}\hat{y}}{\frac{5}{3}} = \frac{4}{5}\hat{y}\ .$$

c. The energy E' is just $-\widetilde{u}' \cdot \widetilde{p}'$, which, however, is an invariant, so we calculate it with the numbers that $\mathcal{O}$gave us:

$$E' = -\widetilde{u} \cdot \widetilde{p} = -[0(0) + 2(4/3) + 4(0) - 6(5/3)] = \tfrac{22}{3} = 7.333 \,.$$

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4. Two photons of energy E move toward one another, one in the $+\hat{x}$-direction, and one in the $+\hat{y}$-direction.

a. What is the velocity of the barycentric frame for these two photons, i.e., the frame in which their center of mass is at rest?

b. Write down the explicit 4×4 matrix for the Lorentz boost that will take one from the barycentric frame to the original (laboratory) frame.

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The total 3-momentum of the two photons is just $\vec{P} = E(\hat{x} + \hat{y})$, since the magnitude of the 3-momentum of a photon is the same as its energy. On the other hand, the total energy of the two photons is $2E$; therefore, the velocity of its barycenter is given by

$$\vec{V} = \tfrac{1}{2}(\hat{x} + \hat{y}) = \frac{1}{\sqrt{2}}\left(\frac{\hat{x} + \hat{y}}{\sqrt{2}}\right) ,$$

where the quantity in the parentheses is a unit vector so that the quantity in front, namely $1/\sqrt{2}$, is the magnitude of this velocity.

Since $V = 1/\sqrt{2}$, the factor $\gamma_V = \sqrt{2}$ and the 3×3 matrix

$$\hat{V}\hat{V}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

so that we may construct quickly the boost matrix that takes one from the barycenter to the lab frame:

$$B(-\vec{V}) = \begin{pmatrix} I_3 + (\sqrt{2} - 1)\hat{V}\hat{V}^T & \gamma V \\ \gamma V^T & \gamma \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}+1}{2} & \frac{\sqrt{2}-1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}-1}{2} & \frac{\sqrt{2}+1}{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \sqrt{2} \end{pmatrix} .$$

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5. A particular electromagnetic field is described by the following Faraday 2-form, with its components, relative to a Cartesian basis set, presented in terms of the following matrix:

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu \ , \quad \mathcal{F}_{\mu\nu} \Longrightarrow \quad f(x,y,z,t)\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

a. What are the electric and magnetic fields as viewed in the particular reference frame in which this is the correct presentation of the components?

b. What are the (two) invariants of this electromagnetic field?

c. What is the 4-momentum density for this field?

d. Can you give a physical interpretation of this field?

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We begin by recalling both the equation for the components of the Faraday 2-form:

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & E^x \\ -B^z & 0 & B^x & E^y \\ B^y & -B^x & 0 & E^z \\ -E^x & -E^y & -E^z & 0 \end{pmatrix} .$$

and also the components of its associated energy-momentum tensor:

$$4\pi M^{\mu\nu} \equiv \mathcal{F}^{\mu}{}_{\lambda}\mathcal{F}^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} \ .$$

a. The electric and magnetic fields are easily pulled out of the matrix:

$$\vec{E} = -f\,\hat{x} \ , \quad \vec{B} = +f\,\hat{y} \ .$$

b. The two invariants must, easily, be both zero:

$$\vec{E}\cdot\vec{B} = 0 \ , \qquad \vec{B}^2 - \vec{E}^2 = f^2 - f^2 = 0 \ .$$

c. The 4-momentum density if the 4th column (or row) of the energy-momentum tensor, which we must therefore calculate. It is simpler than it might be for we recall that the coefficient of the second term is simply one of the invariants, and is therefore zero. We now calculate explicitly the other term, where we have to use the metric matrix η to move the indices around appropriately:

$$4\pi M^{\mu\nu} \implies f\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} f\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = f^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} .$$

This tells us that the 4-momentum density is given by either the 4th column or the 4th row, which are identical:

$$p^\nu = M^{4\nu} = \frac{f^2}{4\pi}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} .$$

We recall that this says that the Poynting vector, i.e., the 3-momentum per unit volume is just $-(f^2/4\pi)\hat{z}$, consistent with our earlier values for $\vec{E}$ and $\vec{B}$, while the energy per unit volume is $+f^2/4\pi$, again consistently.

Part B: Please answer 2, only 2, of the problems in this part. [23 pts each]

6. Two spaceships, piloted by Jill and Joe, leave Earth in exactly opposite directions, each traveling at the speed $v = 0.8c$.

a. After Earth says they have each traveled for 1 year, how far apart does Earth believe them to be ? Let the event that corresponds to this time and distance be A for Jill, and B for Joe.

b. At that time on Earth, i.e., at event A, how much older is Jill than when she left?

c. At the time and place of event A, Jill measures how far away is Joe. What answer does she get?

d. Jill's measurement in part (c) is not the same as Earth's measurement in part (a). Please explain why, and include a spacetime diagram in your answer.

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a. After traveling for 1 year at v=0.8c, each person will have traveled a distance of 0.8 light-years. As they have been traveling in opposite directions, the total distance between them, as measured on Earth, will be $\mathbf{D = 1.6}$ **light-years.**

b. As the relativistic factor $\gamma = 5/3$ for this speed, Jill will have aged less than 1 year, by a multiplicative factor of $1/\gamma$, i.e., while $t_A = 1$ year (and $x_A = 0.8$ light-year), we have that $\mathbf{t'_A} = \mathbf{0.6}$ year.

c. There are several different ways to determine how far away Jill believes Joe to be. One of the most straightforward is to simply move immediately into Jill's reference frame from the beginning. In that frame, Joe is moving directly away from her at a speed

$$u' = (u - v)/(1 - uv) = (-.8 - .8)/[1 - (-.8)(.8)] = -1.6/(1.64) = -\frac{40}{41} = -0.9756 \ .$$

He has been traveling at this speed for a time, as she measures it, of $t' = 0.6$ year, so that he has separated from her by a distance of

$$d' = |u'|t' = \frac{40}{41}\left(\frac{3}{5}\right) = \frac{24}{41} = 0.5854 \text{ light-year.}$$

This is the correct answer; however, below we explain why it is so different from what one might, perhaps, have expected.

d. If we translated the Earth distance to Jill's frame by simply multiplying, or dividing, by γ, then we would have $D' = D/\gamma = 1.6(0.6) = 0.96$ light-years, or, perhaps $D'' = \gamma D = 1.6/0.6 = 2.67$ light-years. However, neither is correct since lengths must be between two events that are simultaneous in the frame in question.

We may best explain this by first noting that event B, on Joe's rocketship, at $t = 1$ year (Earth time), is simultaneous in Earth's reference frame with event A, on Jill's rocketship, at $t = 1$ year (Earth time). However, they are surely not simultaneous in any moving observer's frame, and the "distance" that Jill measures in her frame is along some surface of simultaneity **as she defines it**. Therefore, the distance measured in part (c) above is the (spatial) distance between her event A and some other event Q, on Joe's worldline, that has the time $t' = 0.6$ years, i.e., simultaneous with her becoming 0.6 years older. That event occurs earlier on Joe's worldline than does event B. We may check this by determining the Earth coordinates of the intersection of her line of simultaneity with Joe's worldline:

$$\text{Joe's worldline:} \;\; x = v_{\rm Joe} t = -.8t \; ,$$

$$\text{Jill's line of simultaneity with event A:} \;\; 0.6 = t' = \gamma_v (t - v_{\rm Jill} x) = (5/3)(t - .8x) \; .$$

The solution of these two equations for x and t, i.e., the intersection of Joe's worldline and Jill's definition of $t' = 0.6$, gives us the following Earth coordinates for Q:

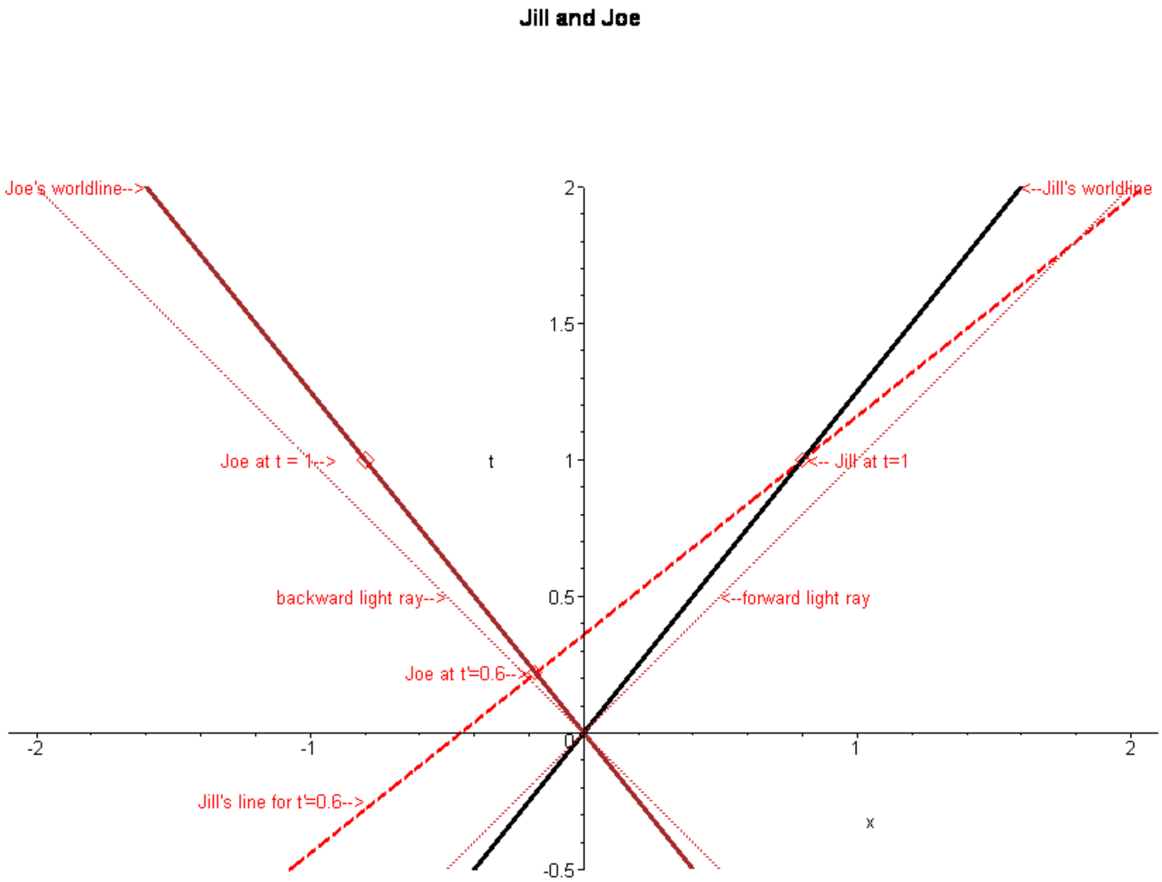
$$t = .36/(1.64) = \tfrac{9}{41} = .2195 \; , \quad x = -.8(.36/1.64) = -\tfrac{36}{205} = -0.1756 \; .$$

Thus the event Q is much earlier than the event C which has $t = 1$ and $x = -.8$. Moreover we may use the Earth coordinates of the event Q above to determine Jill's coordinates for it, giving us a second, more complicated, method of determining her measurement of the distance to Joe:

$$t'_Q = \tfrac{5}{3}[\tfrac{9}{41} + \tfrac{4}{5}\left(\tfrac{36}{205}\right)] = \tfrac{3}{5} = 0.6 \text{ years,}$$
$$x'_Q = \tfrac{5}{3}[-\tfrac{36}{205} - \tfrac{4}{5}\left(\tfrac{9}{41}\right)] = -\tfrac{24}{41} \; ,$$

which is the same answer we acquired above, except for the minus sign, which simply shows us that Joe is in her $-\hat{x}$ direction.

We insert below the spacetime diagram, with these three different events noted, as well as the appropriate lines of simultaneity.



7. A pair of linearly-independent 1-forms are given:

$$\underset{\sim}{\omega}^1 \equiv \frac{x\,dy - y\,dx}{x^2+y^2}\;, \qquad \underset{\sim}{\omega}^2 \equiv x\,dx + y\,dy\;.$$

a. Show that in fact they are linearly independent, considered as basis vectors for a vector space over a 2-dimensional space with coordinates x and y.

b. Are they equivalent to differentials of some **other pair** of coordinates? If so, what are they? If not, show why not, and compute the connection coefficients for them?

c. What is the corresponding dual basis for tangent vectors, i.e., $\widetilde{e}_1$ and $\widetilde{e}_2$, in terms of ∂_x and ∂_y? Duality here means that $\underset{\sim}{\omega}^i(\widetilde{e}_j) = \delta^i_j$.

d. A particular 2-vector is given by $\widetilde{A} = y\partial_x + x\partial_y$. As functions of x and y, what are its components with respect to the dual basis determined in part (c)?

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a. Linear independence of basis vectors is most easily shown by calculating the determinant of their coefficients, determined with respect to a basis that is known to be linearly independent. In this case the pair known to be linearly independent is just $\{dx, dy\}$. Therefore we have

$$|J| \equiv \left| \begin{pmatrix} -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \\ x & y \end{pmatrix} \right| = -1 ,$$

which is certainly not zero.

b. Again the simplest method, but not necessarily the only one, to show that 1-forms are exact, i.e., differentials of scalars, is to show that they are closed, i.e., to show that their exterior derivatives vanish:

$$\begin{aligned} d\underset{\sim}{\omega}^1 &= \frac{dx \wedge dy - dy \wedge dx}{x^2+y^2} - 2\frac{x\,dx + y\,dy}{(x^2+y^2)^2} \wedge (x\,dy - y\,dx) \\ &= 2\frac{dx \wedge dy}{x^2+y^2} - 2\frac{x^2\,dx \wedge dy - y^2\,dy \wedge dx}{x^2+dy^2} = 0\ ; \quad d\underset{\sim}{\omega}^2 = dx \wedge dx + dy \wedge dy = 0\ . \end{aligned}$$

Therefore we see that there should be indeed scalars that these are the exterior differentials for. The second one is trivial; we define

$$v \equiv \tfrac{1}{2}(x^2+y^2) \quad \Longrightarrow \quad dv = x\,dx + y\,dy\ .$$

For the other one, we guess that it is something like an angle; therefore, let us first try

$$w \equiv \tan\theta = \frac{y}{x} \quad \Longrightarrow \quad dw = \sec^2\theta\,d\theta = \frac{x\,dy - y\,dx}{x^2}\ .$$

We see that dw is not quite right, although the numerator is what we want. Therefore, let's go ahead for the angle: We have

$$d\theta = \frac{x\,dy - y\,dx}{x^2\sec^2\theta} = \frac{x\,dy - y\,dx}{x^2(1+\tan^2\theta)} = \frac{x\,dy - y\,dx}{x^2(1+(y/x)^2)} = \frac{x\,dy - y\,dx}{x^2+y^2}\ ,$$

which is what was wanted. So our new coordinates are $\{\theta, v\}$.

c. The dual basis is obtained,by setting up an arbitrary pair of tangent vectors and solving the four equations they must satisfy:

$$\begin{aligned}
&\widetilde{e}_1 \equiv a\partial_x + b\partial_y \ , \quad \widetilde{e}_2 = \alpha\partial_x + \beta\partial_y \ ; \\
&\underset{\sim}{\omega}^1(\widetilde{e}_1) = \frac{-ya + bx}{x^2+y^2} = 1 \ , \qquad \underset{\sim}{\omega}^2(\widetilde{e}_1) = xa + yb = 0 \ , \\
&\underset{\sim}{\omega}^1(\widetilde{e}_2) = -y\alpha + x\beta = 0 \ , \qquad \underset{\sim}{\omega}^2(\widetilde{e}_2) = x\alpha + y\beta = 1 \ ; \\
&\Longrightarrow \quad \widetilde{e}_1 = -y\partial_x + x\partial_y \ , \quad \widetilde{e}_2 = \left(\frac{1}{x^2+y^2}\right)[x\partial_x + y\partial_y] \ .
\end{aligned}$$

An alternative approach is to write down the Jacobian matrix for the transformation from $\{x, y\}$ to $\{\theta, v\}$ and then invert it:

$$\begin{aligned}
\underset{\sim}{\omega}^i = J^i{}_a \, dx^a \ , \qquad &\begin{pmatrix} \theta_{,x} & \theta_{,y} \\ v_{,x} & v_{,y} \end{pmatrix} = J = \begin{pmatrix} \frac{-y}{x^2+y^2} & \frac{+x}{x^2+y^2} \\ x & y \end{pmatrix} \ , \\
\widetilde{e}_i = (J^{-1})^b{}_i \partial_{x^b} \ , \qquad &\begin{pmatrix} x_{,\theta} & x_{,v} \\ y_{,\theta} & y_{,v} \end{pmatrix} = J^{-1} = \begin{pmatrix} -y & \frac{x}{x^2+y^2} \\ x & \frac{y}{x^2+y^2} \end{pmatrix} \ .
\end{aligned}$$

which is the same calculation, but sounds "more abstract or general," which it is.

d. Again there are alternative approaches. The simplest one is to use the chain rule:

$$\begin{aligned}
\partial_x \equiv \frac{\partial}{\partial x} &= \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = -\frac{y}{x^2+y^2}\partial_\theta + x\partial_v \ , \\
\partial_y \equiv \frac{\partial}{\partial y} &= \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = +\frac{x}{x^2+y^2}\partial_\theta + y\partial_v \ .
\end{aligned}$$

Having that information we may insert this "change of basis vectors" into the definition of $\widetilde{A}$, which gives

$$\widetilde{A} = y[-\frac{y}{x^2+y^2}\partial_\theta + x\partial_v] + x[\frac{x}{x^2+y^2}\partial_\theta + y\partial_v] = \frac{x^2-y^2}{x^2+y^2}\partial_\theta + 2xy\partial_v \ .$$

However, the alternative approach, this time, is probably considerably simpler since we have already noticed the two transformations written above in part (c) for upper and lower indices between the frames; therefore, we may simply write

$$A^i = J^i{}_a \, A^a = \begin{pmatrix} \frac{-y}{x^2+y^2} & \frac{+x}{x^2+y^2} \\ x & y \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} \\ 2xy \end{pmatrix} \ .$$

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8. We have discussed the energy-momentum tensor for a perfect fluid,

$$\mathbf{T} \equiv (\rho + P)\widetilde{u} \otimes \widetilde{u} + P\mathbf{g} \ .$$

Please consider the general, special-relativistic case where the 4-velocity of the fluid is given by $\widetilde{u} = \gamma_v(0, 0, v, 1)^T$, and write out in detail the symmetric 4×4 matrix which presents the components of this tensor in a frame that observes the fluid moving with 3-velocity $v\hat{z}$.
Do notice that under such conditions this tensor has off-diagonal terms which constitute tangential stresses on the system. Point out explicitly, from this calculation, the four components which constitute the 4-momentum density of the moving fluid, and make comments concerning how they differ from what one might intuitively have expected.

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We simply insert the given 4-velocity into our tensor, which then has components that may be represented by the following matrix:

$$\begin{aligned}
\mathbf{T} \Longrightarrow \quad & (\rho + P)\gamma_v^2 \begin{pmatrix} 0 \\ 0 \\ v \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & v & 1 \end{pmatrix} + P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
& = (\rho + P)\gamma_v^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v^2 & v \\ 0 & 0 & v & 1 \end{pmatrix} + P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
& = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \gamma_v^2(P + v^2\rho) & \gamma_v(\rho + P)\gamma_v v \\ 0 & 0 & \gamma_v(\rho + P)\gamma_v v & \gamma_v^2(\rho + v^2 P) \end{pmatrix} .
\end{aligned}$$

The 4-momentum density is the last column, or row:

$$\begin{pmatrix} 0 & 0 & \gamma_v(\rho + P)\gamma_v v & \gamma_v^2(\rho + v^2 P) \end{pmatrix} \ ,$$

We might, perhaps, have expected that the 3-vector part of the momentum density would be $\rho\vec{v} = \rho v\hat{z}$, or, perhaps $\gamma_v \rho\vec{v}$. In fact we see two differences from this:

a.) there is actually an overall factor of γ_v^2 instead of just γ_v. The additional factor of γ_v comes from the Lorentz transformation behavior of the 3-volume.

b.) One might also be surprised that the 3-vector part of the momentum density has a contribution caused by the additional term due to the pressure, which enters in the same way as the mass density, although it is important to remember that it is really ρc^2 which has the same units as the pressure P which is involved, or if you prefer one must compare ρ with P/c^2. The pressure contributes to the overall energy density in the material.

c.) Coming to the fourth component, which one would perhaps surmise would just be the energy density we again find an additional contribution due to the pressure, multiplied of course by $(v/c)^2$.

d.) It is interesting to compare the $3,3$-term and the $4,4$-term:

   i. The $T^{3,3}$ term says that because we are moving in the $\hat{z}$-direction the entry that should be a pressure differs from the pressure in the other two directions by an extra term of the form $\rho(v/c)^2$;

   ii. the $T^{4,4}$ term, which we think of as energy density also has an extra term, of the form $P(v/c)^2$.
