Physics 495

Homework No. 1 **Solutions:** due Wednesday, 2 September, 2009

1. Convert the following from natural units, i.e., with $c = 1$, to SI units.

a. A velocity $v = 10^{-2}$.

b. Pressure, 10^{19} kg/m^3.

c. Time, $t = 10^{18}$ m.

d. Energy density, $u = 1$ kg/m^3.

e. Acceleration, 10 /m .

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a. for a velocity, we simply multiply by c, which, in SI units, we take to be $c = 3 \times 10^8$ m/sec:

$$v = 10^{-2} = 10^{-2}c = 10^{-2}\,[3 \times 10^8 \text{ m/sec }] = 3 \times 10^6 \text{ m/sec.}$$

b. for a pressure, which should have units of force/area, or energy/volume, since energy has units of force times distance, we multiply by c^2, because we know that $E = mc^2$, and we already have units of kg in the given quantity:

$$P = 10^{19}\,\text{kg/m}^3 = 10^{19}c^2\,\text{kg/m}^3 = 10^{19}\,\text{kg/m}^3[3 \times 10^8 \text{ m/sec}]^2 = 9 \times 10^{35} \;\; \text{Nt/m}^2 \; .$$

c. for time, we simply divide by c, which gives us

$$t = 10^{18} \text{ m } = 10^{18}c \text{ m } = (10^{18}m)/(3 \times 10^8 \text{ m/sec }) = 3.33 \times 10^9 \text{ s.}$$

d. for energy density, which of course has the same units as pressure, namely energy per unit volume, again we multiply by c^2:

$$u = 1 \text{ kg/m}^3 = c^2 \text{ kg/m}^3 = [3 \times 10^8 \text{ m/sec }]^2 \text{ kg/m}^3 = 9 \times 10^{16}\text{Joule/m}^3 \; .$$

e. for acceleration in inverse meters, again we multiply by c^2:

$$a = 10 \text{ m}^{-1} = 10c^2 \text{ m}^{-1} = 10\,[3 \times 10^8 \text{ m/sec }]^2/\text{m} = 9 \times 10^{17} \text{ m/sec}^2 \; .$$

2.

a. Use the spacetime diagram of an observer $\mathcal{O}$ to describe the following experiment performed by $\mathcal{O}$. Two bursts of particles of speed $v = 0.5$ are emitted from $x = 0$ at time $t = -2$ m, one traveling in the positive x direction and the other in the negative x direction. These encounter detectors located at $x = \pm 2$ m. After a delay of 0.5 m of time, the detectors send signals back to $x = 0$ at speed $v = 0.75$.

b. The signals arrive back at $x = 0$ at the same event. (Make sure your spacetime diagram shows this!) From this the experimenter concludes that the particle detectors did indeed send out their signals simultaneously, since he knows they are equal distances from $x = 0$. Explain why this conclusion is valid.

c. A second observer $\widetilde{\mathcal{O}}$ moves with speed $v = 0.75$ in the *negative* x direction, as measured by $\mathcal{O}$. Draw the spacetime diagram of $\widetilde{\mathcal{O}}$ and in it depict the experiment performed by $\mathcal{O}$. Does $\widetilde{\mathcal{O}}$ conclude that the two particle detectors sent out their signals simultaneously? If not, which signal was sent first?

d. Compute the interval $\mathcal{I} \equiv \Delta s^2$ between the two events at which the detectors emitted their signals, using both the coordinates of $\mathcal{O}$ and those of $\mathcal{O}'$.

Please use some computer-created graphing program, such as MATLAB or Maple or whatever is your favorite one, to create the diagrams needed for the answers to these questions, so that they are neat and accurate. They should also be labeled well enough that the grader can interpret what it is that you have presented.

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a. There are some 6 events involved here, which one can see on the figure for $\mathcal{O}$ below:

$$\text{event A, initial sending out of two signals: } x_A = 0\ ,\ t_A = -2\ ,$$

$$\text{event B, receipt of signal at } x = +2 \text{ m: } x_B = 2\ ,\ t_B = +2\ ,$$

$$\text{event C, receipt of signal at } x = -2 \text{ m: } x_C = -2\ ,\ t_C = +2\ ,$$

$$\text{event D, emission of signal from } x = +2 \text{ m: } x_D = +2\ ,\ t_D = 2.5\ ,$$

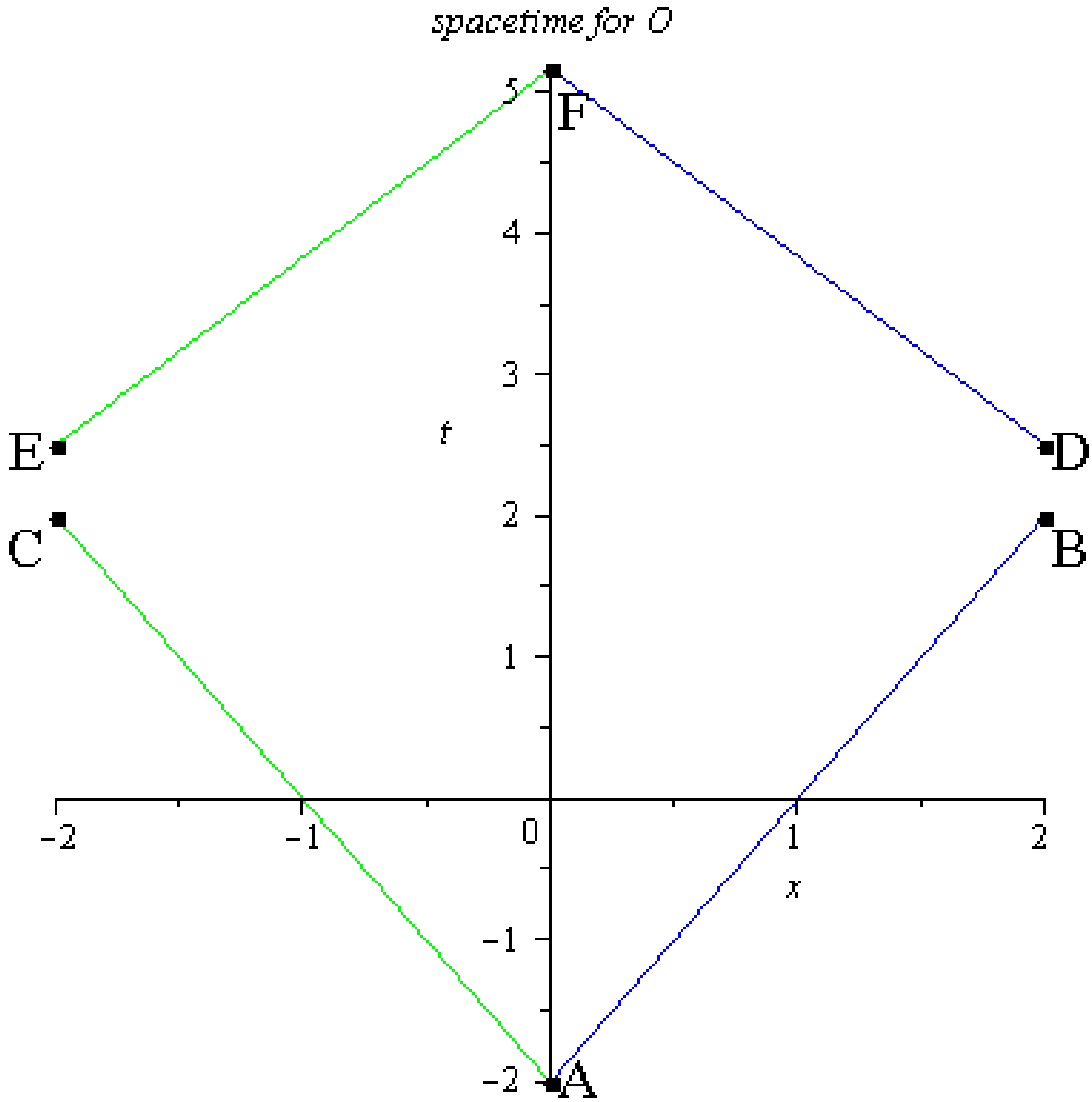
$$\text{event E, emission of signal from } x = -2 \text{ m: } x_E = -2\ ,\ t_E = 2.5\ ,$$

$$\text{event F, receipt of both signals at } x = 0 \text{ m: } x_F = 0\ ,\ t_F = 31/6\ ,$$

where the various times were computed as follows, using the given speeds:

$$t_C = t_B = t_A + x_B/(v = 0.5) = -2 + 2/(1/2) = +2 \; ,$$

$$t_F = t_D + x_D/(v = 0.75) = 2.5 + 2/0.75 = 5/2 + 8/3 = 31/6 \; .$$



b. The conclusion that they were emitted simultaneously is simply because they arrived simultaneously after having traveled the same distance at the same speed.

c. To determine coordinates of these six events as measured by $\mathcal{O}'$, we should use the Lorentz transformations between the two frames. As we are told that $\mathcal{O}'$is moving with velocity $\beta = -0.75$, which implies that $\gamma = 1/\sqrt{1-(3/4)^2} = 4/\sqrt{7}$, those equations can be written as the following:

$$x' = \gamma(x - \beta t) = \gamma(x + 3t/4) \; , \qquad t' = \gamma(t - \beta x) = \gamma(t + 3x/4) \; ,$$

$$x'_A = -(3/2)\gamma \; , \quad t'_A = -2\gamma \; ,$$

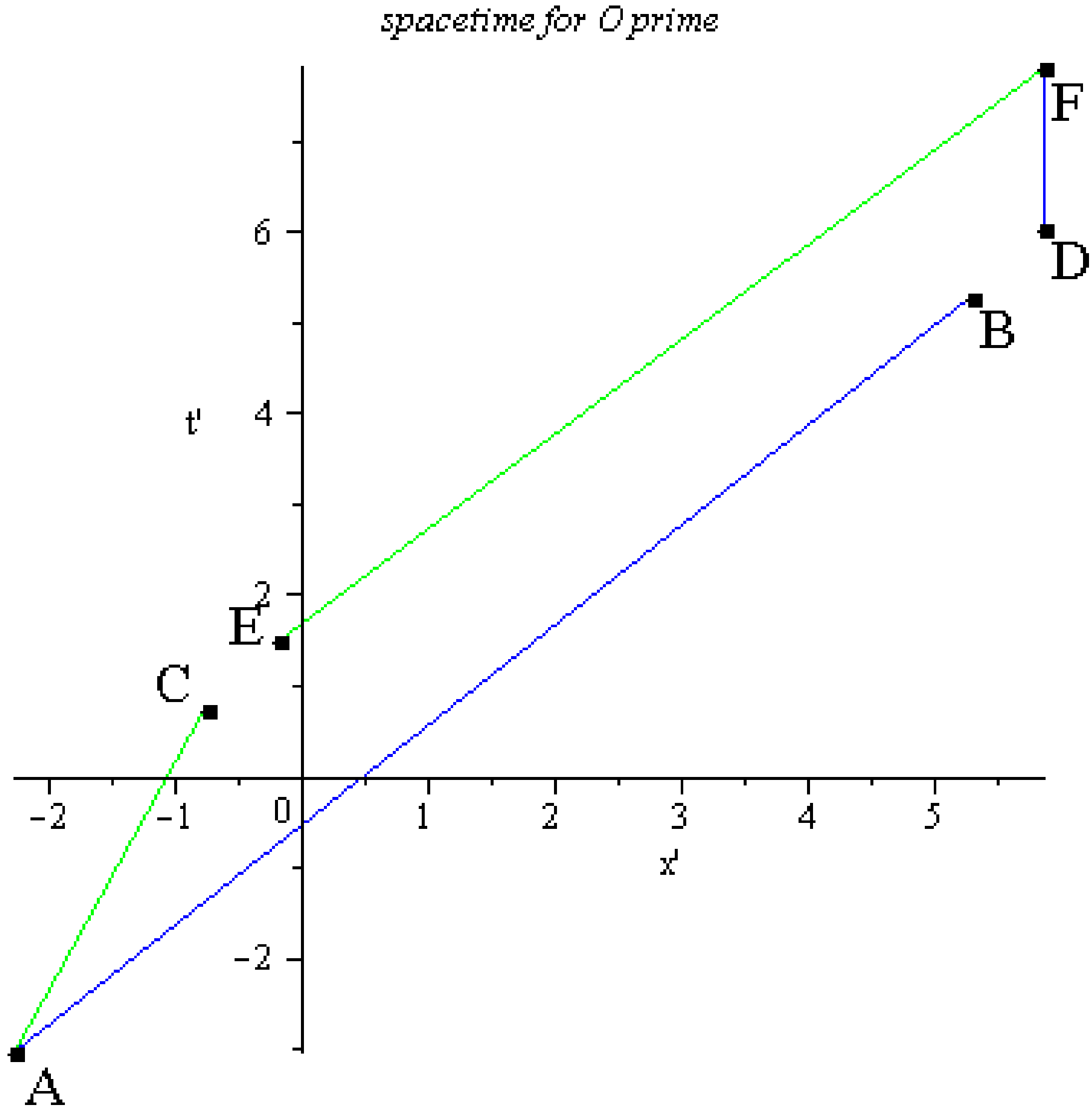
$$x'_B = +(7/2)\gamma \; , \; t'_B = +(7/2)\gamma \; ; \quad x'_C = -(1/2)\gamma \; , \; t'_C = +(1/2)\gamma \; ,$$

$$x'_D = +(31/8)\gamma \; , \; t'_D = 4\gamma \; ; \quad x'_E = -(1/8)\gamma \; , \; t'_E = 1\gamma \; ,$$

$$x'_F = (31/8)\gamma \; , \quad t'_F = (31/6)\gamma \; ,$$

where I have not inserted the actual value of γ, as given above, just because it is such a complicated-appearing "thing" that it confuses the issues, at least a little bit. A Minkowski diagram from the point of view of $\mathcal{O}'$is presented below. It is obvious from the numbers above or the diagram below that the emission event E occurred much earlier than the other emission event D, which is reasonable since our observer is traveling toward the location of event E.

Interesting note is that the line between D and F in the spacetime diagram for $\mathcal{O}'$is purely vertical, implying that the particle beam is NOT moving. This is reasonable since, back in the frame of $\mathcal{O}\,'$it is moving with velocity -0.75 while $\mathcal{O}\,'$also measures the velocity of $\mathcal{O}'$as -0.75, so that the particle beam is indeed at rest as measured by $\mathcal{O}'$.



d. To determine the values for the interval, between events D and E, we compute:

$$\Delta s^2_{DE} = [2 - (-2)]^2 - (2.5 - 2.5)^2 = 4^2 - 0^2 = 16 \ ,$$

$$\Delta s'^2_{DE} = [(31/8) - (-1/8)]^2\gamma^2 - [4 - 1]^2\gamma^2 = (4^2 - 3^2)\gamma^2 = (16 - 9)\gamma^2 = 7\frac{4^2}{7} = 16 \ ,$$

equal, as expected.

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3. A heavy plane slab moves with uniform speed v in the direction of its normal, through an inertial frame. A ball is thrown at it with velocity u, from a direction making an angle θ with its normal. Assuming that the slab has essentially infinite mass, so that there is no recoil as a result of the collision, and also that there is no dissipation of energy, use Newtonian relativity to show that the ball will leave the slab in a direction making an angle ϕ with its normal, and with a velocity w, such that

$$\frac{u}{w} = \frac{\sin\phi}{\sin\theta}\ , \qquad \frac{u\cos\theta + 2v}{u\sin\theta} = \cot\phi\ .$$

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We will choose coordinate axes so that the normal to the plane slab is the $\hat{x}$-direction, and the other axis relevant to the problem, tangent to the slab, is called the $\hat{y}$-direction. As well, we suppose the ball incoming on the side of the slab toward which the slab is moving, and that it is coming from above the point at which it strikes the slab.

The basic principles here are the following:

a.) in the reference frame where the slab is at rest, the ball will rebound with the same angle at which it struck the slab, so that the sign of the x-component of its velocity will change while that of its y-component will not, and

b.) the usual Newtonian velocity transformation between frames is such that a velocity measured in a frame where the slab is measured to be moving is the sum of the velocity of the slab and that of the object measured in the frame where the slab is at rest.

We now use these principles, supposing the following data given in the problem, but here expressed in more detail, **and** written in the reference frame that gave us that data:

$$\text{the incoming velocity is such that} \quad \begin{cases} u_x = & -\,u\cos\theta\ , \\ \\ u_y = & -\,u\sin\theta\ . \end{cases}$$

$$\text{the outgoing velocity is such that} \quad \begin{cases} w_x = & +\,w\cos\phi\ , \\ \\ w_y = & -\,w\sin\phi\ . \end{cases}$$

We now move this incoming velocity to the reference frame in which the slab is at rest, denoting measurements in that frame with a prime:

$$u'_x = -u\cos\theta - v\ , \qquad u'_y = -u\sin\theta\ .$$

Next, referring to the final velocity as w as above, we may apply the conservation of momentum and energy to the interaction of the ball with the slab:

$$w'_x = -u'_x = u\cos\theta + v\ , \qquad w'_y = u'_y = -u\sin\theta\ .$$

Now we must re-write these measured velocities in terms of measurements in the original reference frame, inserting the data we have above for the measurements made there:

$$u\cos\theta + v = w'_x = w_x - v = w\cos\phi - v\ , \qquad -u\sin\theta = w'_y = w_y = -w\sin\phi\ .$$

At this point it is then simply algebra to acquire the desired equations. The equation relating quantities in the $\hat{y}$-direction gives us

$$u\sin\theta = w\sin\phi \quad \Longrightarrow \quad \frac{u}{w} = \frac{\sin\phi}{\sin\theta}\ .$$

On the other hand the equation relating quantities in the $\hat{x}$-direction gives us

$$w\cos\phi = u\cos\theta + 2v\ ,$$

while of course

$$\cot\phi = \frac{w\cos\phi}{w\sin\phi} = \frac{u\cos\theta + 2v}{u\sin\theta}\ .$$

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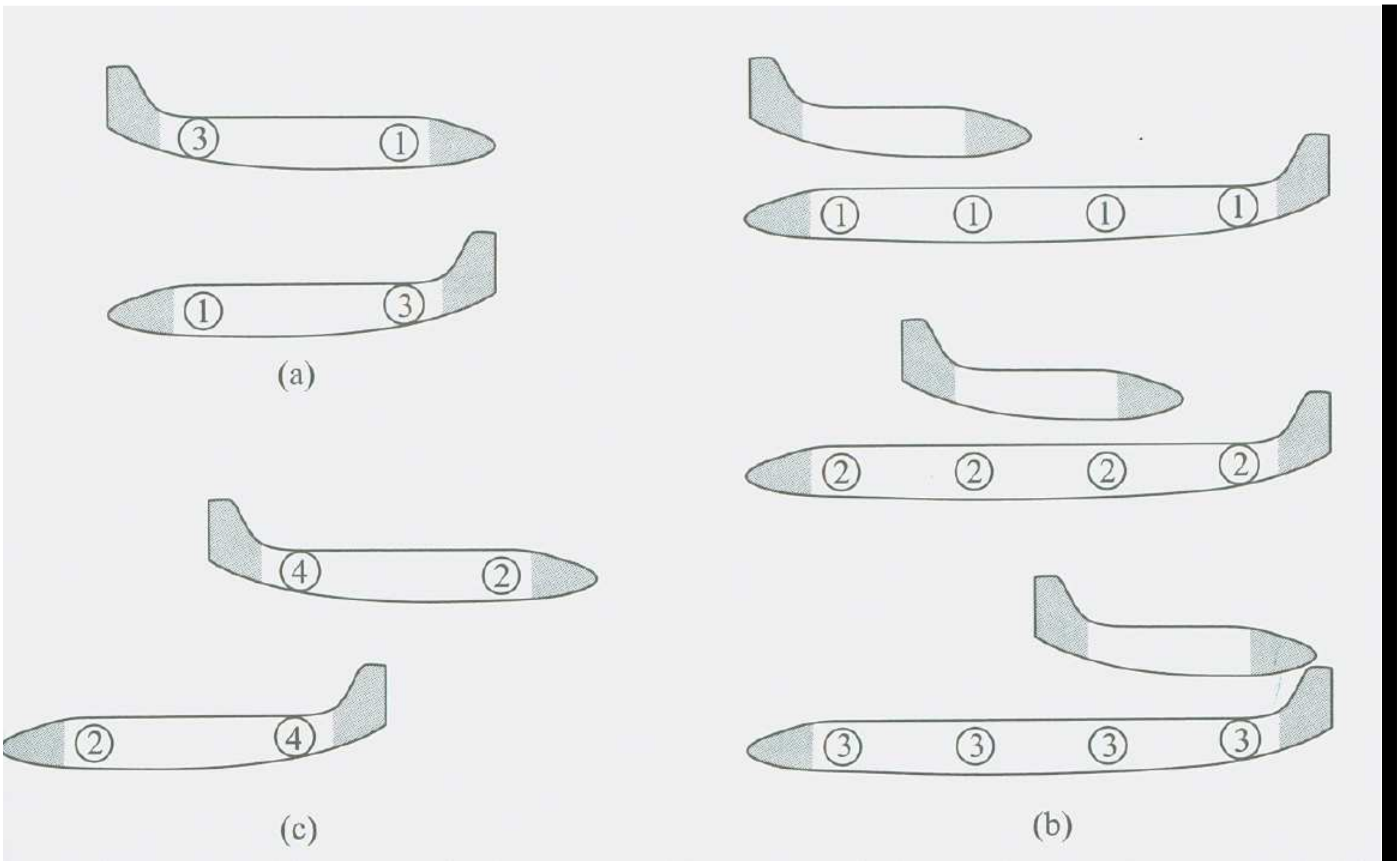
4. Looking at the figure below, where the proper length of each plane is L, deduce that the relative velocity between the two places is given by $v = L/3$ in the time units indicated by the clocks. In each of the airplanes a flash of light is sent from the exact middle of the plane toward each end, which therefore arrive at the two ends at the same time, **as measured in the plane.** In part (a) of the figure, the numbers indicate the times that we, on the ground, measure as simultaneous at the two ends of the plane when the flash reaches the rear end of the plane. In part (b) the numbers indicate the times, as measured at various points on the lower airplane, when the upper airplane is in the position shown. In addition, as can be seen, the airplanes in part (b) have been drawn with the presumption that the one airplane is 1/3 the length of the other. Lastly, the numbers on the figures in part (c) show the times when the two tails are just above one another, of course going in opposite directions (as shown).



If these units are 100 nanoseconds, prove $v = (2\sqrt{2}/3)c$ and $L = 85$ m. What is the velocity of each plane relative to the ground?

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Since we are told that, in the frame of the one airplane, the other airplane is one third the length of the first one, we know that the length-contraction factor is 1/3. Since we know that factor should be $1/\gamma = \sqrt{1-v^2}$, where v is the relative speed between the planes, i.e., the speed of the one as measured by the other, then we may solve this equation to determine that relative speed:

$$\sqrt{1-v^2} = 1/3 \quad \Longrightarrow \quad v = \sqrt{8/9} = \tfrac{2}{3}\sqrt{2} = \tfrac{2}{3}\sqrt{2}c\,.$$

In order to determine the length we also note from figures (b) and (c) that it took 3 units of time for the tail, say, of the one plane to pass totally by the other plane. Note that this is also "obvious" since we know that the one plane is measured to have one third the length of the other. Therefore, we must have the following, knowing that 1 "unit" is 100 nanoseconds:

$$L = 3v = 2\sqrt{2}\ \text{units}\ = 2\sqrt{2}c\ \text{units}\ = 2\sqrt{2}[30\ \text{m/unit}]\ \text{units}\ = 84.85\ \text{m}.$$

To determine the velocity as measured on the ground, we refer to that observer as $\mathcal{O}$ and the one on the airplane headed toward the right as $\mathcal{O}'$. As well we say that the velocity of $\mathcal{O}'$ as measured by $\mathcal{O}$ is, as usual, β, while the velocity of the airplane moving toward the left, as measured by $\mathcal{O}$, is w, while the velocity as measured by $\mathcal{O}'$ is w'. Then we have the generic formula:

$$w' = \frac{w - \beta}{1 - \beta w}\ .$$

However, for our problem we know that $w' = v$, the velocity measured above, while $\beta = -w$. Inserting this information we have

$$v = w' = \frac{w + w}{1 + w^2} \quad \Longrightarrow \quad w = \tfrac{1}{v} \pm \sqrt{(\tfrac{1}{v})^2 - 1}\ .$$

As the result must be less than 1, only the minus sign in the equation above is acceptable. We then insert our known value of v and obtain

$$w = \frac{3}{2\sqrt{2}} - \sqrt{\frac{9}{8} - 1} = \frac{3}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{c}{\sqrt{2}}\ .$$
