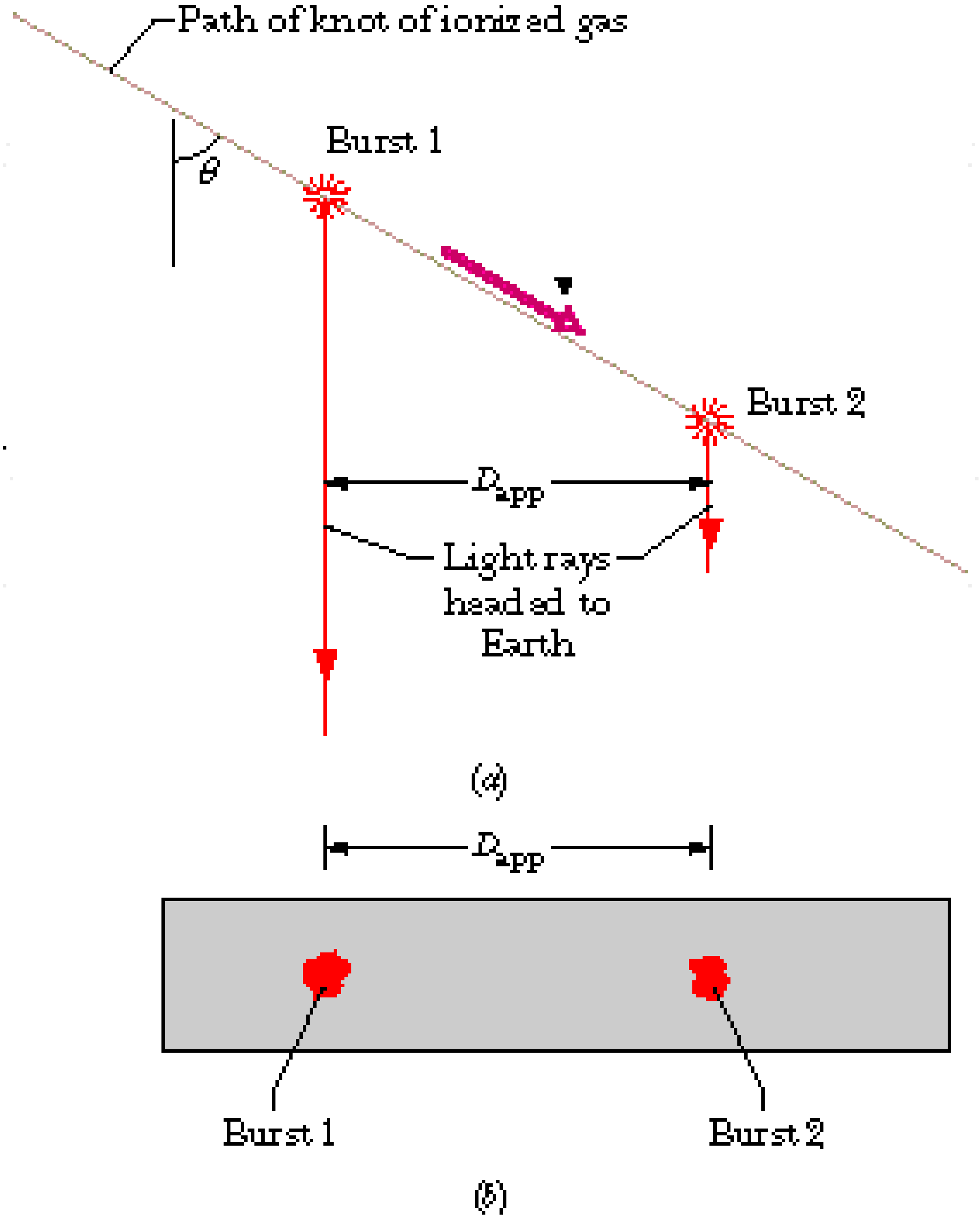
Physics 495

Homework No. 2 **Solutions:** due Wednesday, 9 September, 2009

1. A stellar object at some known large distance ejects a 'jet' at speed v towards an observer, obliquely, making an angle θ with the line of sight. To the observer, the jet appears to be ejected sideways at speed V.



Prove

$$V = v\frac{c\,\sin\theta}{c - v\cos\theta}\ ,$$

and show that this can exceed c, for example, when $\theta = 45°$. [Indeed, such apparently superluminal jets once had observers worried.]

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It is at least intimated in the problem, and we will agree with that, that the observer making these measurements is at rest relative to the stellar object in question, that is ejecting the jets. Therefore we have no particular problems with any time dilation or length contraction questions. Looking at the picture the distance traveled by the jet during the observed time t between signals received by the observer, is just given by vt, which of course is along the hypotenuse of a triangle that makes an angle θ relative to the line which the light rays take toward us. Therefore the apparent distance between the bursts, i.e., distance as projected onto our "film," is just

$$D_{\text{app}} = vt\,\sin\theta\ .$$

However, the second burst was emitted at a later time than the first one, since the jet in question was closer to the observer at that point, by a distance $vt\,\cos\theta$. Therefore, that second ray had less distance to travel, so that the actual time between the emission of the two light rays was less than the time t between their receipt here on earth:

$$T_{\text{app}} = t - (v/c)t\,\cos\theta\ .$$

Therefore the *apparent speed* is simply given by

$$V = V_{\text{app}} = \frac{D_{\text{app}}}{T_{\text{app}}} = \frac{v\,\sin\theta}{1-(v/c)\cos\theta} = v\frac{c\,\sin\theta}{c - v\cos\theta}\ .$$

To study how this varies with v and θ, we first note that for a given value of v, i.e., a given speed for the jet, the value at $\theta = 0$ is the same as at $\theta = \pi$, namely just zero. However, as it varies with observation angle there is a single maximum value between those two points, which we find by determining when the derivative vanishes:

$$\frac{\partial}{\partial\theta} v\frac{c\,\sin\theta}{c - v\cos\theta} = vc\frac{c\cos\theta - v\cos^2\theta - v\sin^2\theta}{(c - v\cos\theta)^2}\ ,$$

which then vanishes only when $\cos\theta_{\text{max}} = v/c$.

At this one maximum the apparent velocity has value

$$\max\left(V_{\rm app}\right)\Big|_{\theta\ {\rm constant}} = v\gamma_v = \frac{v}{\sqrt{1-(v/c)^2}} = c\cot\theta_{\rm max}\ .$$

It is apparent that one may choose $\theta_{\rm max}$, or, equivalently, v, so that this maximum value for V has any value we want, for fixed jet speed, as we vary θ. Alternatively, one may hold the angle fixed, and vary the speed, v. Then for $v = 0$, of course, we have $V_{\rm app} = 0$, while, when we take the partial derivative we find that there is no upper limit coming from the calculus. However, there is a maximum value because the maximum allowed value for v/c is just 1, which gives us the maximum value for V, namely

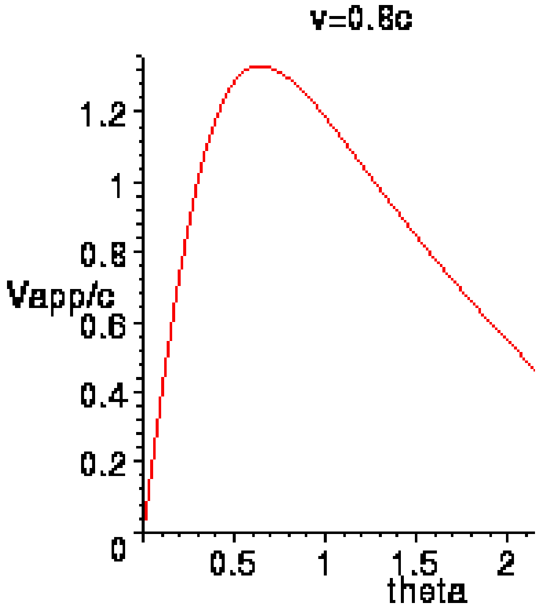
$$\max\left(V_{\rm app}\right)\Big|_{v\ {\rm constant}} = c\frac{\sin\theta}{1-\cos\theta}\ .$$

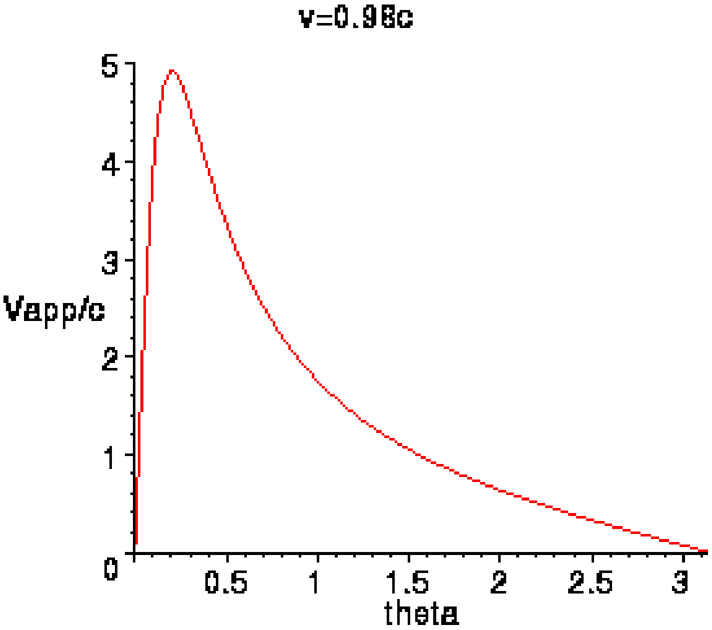
This maximum value obviously, again, can vary over a very large range, as θ varies. However, $V_{\rm app}$ has value greater than or equal to c whenever we have angles such that

$$\frac{v}{c} \geq \frac{1}{\sin\theta+\cos\theta}\ .$$

For the particular value of $45° = \pi/4$, we see that $V_{\rm app}$ is greater than c for any (local) speeds $v \geq 1/\sqrt{2} = 0.707\ldots$.

A couple of (non-required) graphs are shown below, as V varies as a function of θ, for two fixed values of v, namely 0.8 and 0.98:





2. If the twins $\widetilde{A}$ (the stay-at-home) and $\widetilde{B}$ (the traveler) in the twin-paradox experiment visually observe the regular ticking of each other's standard clocks, describe exactly what each sees as $\widetilde{B}$ moves uniformly to a distant point and back. Do draw a spacetime diagram, treating the velocity of $\widetilde{B}$ as having an instantaneous change at the halfway-point of the journey. Note that $\widetilde{B}$ receives slow ticks for half the time and fast ticks for the other half, whereas $\widetilde{A}$ receives slow ticks for more than half the time: hence $\widetilde{A}$ receives fewer ticks, hence $\widetilde{B}$ is younger when they meet again. This is one of the arguments often used to illustrate the 'non-paradoxicality' of the paradox. Why does it not lead to an age difference classically?

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a. For observations made by $\widetilde{B}$, on the first half of her journey, of uniform ticking of clocks at home, at rest with respect to $\widetilde{A}$, we suppose that they occur at events, labeled by an integer n, that have $xE_n = 0$ and $tE_n = n$. Then, as measured by Julia, these emission events have $x'E_n = -\beta\gamma n$ and $t'E_n = \gamma n$. The reception events then occur after the light ray used for the observation have traveled the distance in question, and therefore we have

$$x'R_n = 0\ , \quad t'R_n = \gamma n + \beta\gamma n = n\gamma(1+\beta) = n\sqrt{\frac{1+\beta}{1-\beta}}\ ,$$

which is a perfectly reasonable answer, indicating the usual transformation of frequency rates:

$$f_R = f_E\sqrt{\frac{1-\beta}{1+\beta}}\ ,$$

where now $-\beta$ is taken as the velocity of the emitter as measured by the receiver, and in this case the frequency of received rays is definitely smaller than the frequency of emitted ones, i.e., visually-observed ticks on the clock on earth.

On the way back, of $\widetilde{B}$, the velocity of the emitter, i.e. $\widetilde{A}$, will change sign, so that then the signals announcing the ticks will come faster, i.e., with a higher frequency.

b. On the other hand, for observations made by $\widetilde{A}$, on Earth, of the ticks of the clock on $\widetilde{B}$'s outgoing ship, we could make the same remarks, since now Earth views $\widetilde{B}$ as moving with velocity $+\beta$, so that if signals are sent out from there at $x'E_m = 0$ and $t'E_m = m$, then we have $xE_m = \beta\gamma m$ and $tE_m = \gamma m$, so that they are received when the light ray has traveled that distance, i.e., when

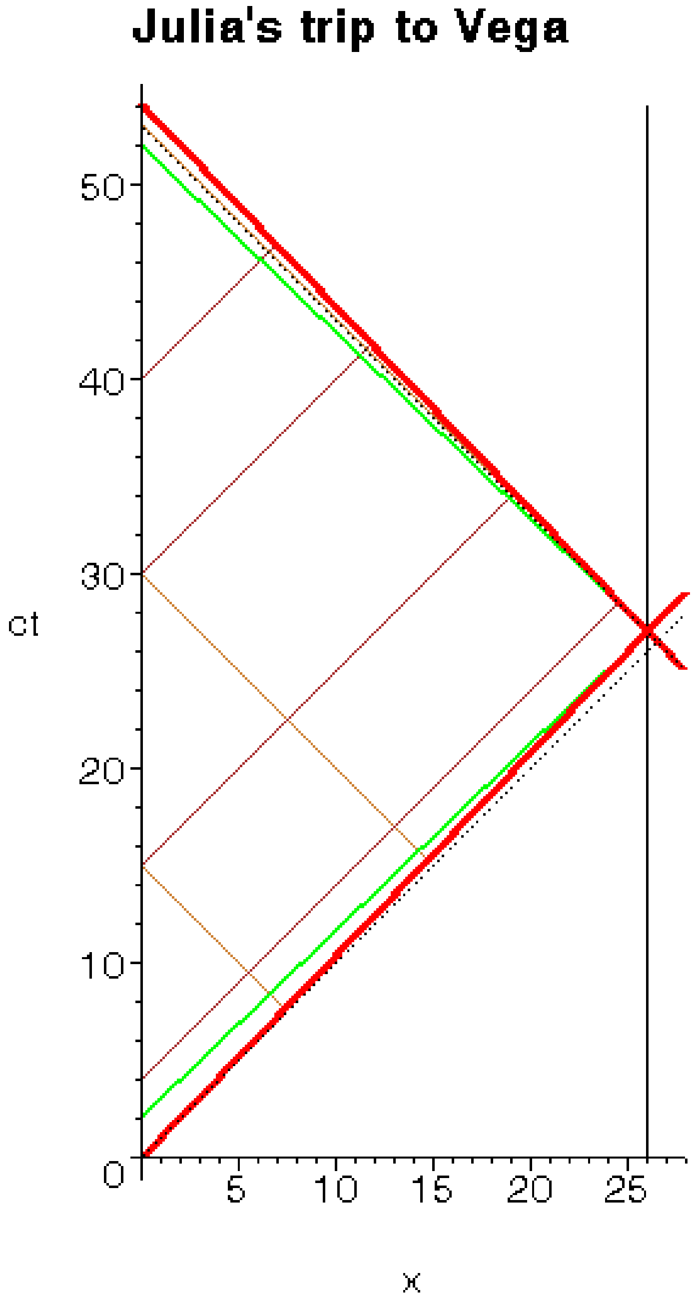
$$xR_m = 0\ , \quad tR_m = \gamma m + \beta\gamma m = \gamma m(1+\beta) = m\sqrt{\frac{1+\beta}{1-\beta}}$$

so that we have

$$f_R = f_E\sqrt{\frac{1-\beta}{1+\beta}}\ .$$

That is they occur less frequently. However, when $\widetilde{B}$ changes direction and begins returning, then her β changes sign and they occur more frequently.

A Minkowski diagram is shown below, drawn by the stay-at-home twin, and showing a few signal-observation rays from each side. Notice that for this particular, quite large, speed, none of the observations from home can be made by the outgoing twin, until she turns around and heads home again. Note also that the very last of her rays is almost not visible because it is so very close to her own world line. These rays are drawn in a brown color; the two different pieces of the worldline of $\widetilde{B}$ are drawn in red, while the green lines are her two quite different lines of simultaneity with her landing and then her instantaneously-later takeoff toward home.


Julia's trip to Vega

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3. An inertial observer $\mathcal{O}$ bounces a radar signal off an arbitrary event $\mathcal{P}$. If the signal is emitted and received by $\mathcal{O}$ at times τ_1 and τ_2, respectively, as indicated by $\mathcal{O}$'s standard clock, prove that the squared interval Δs^2 between his origin event, $\tau = 0$, and $\mathcal{P}$ is $-c^2\tau_1\tau_2$. This, in fact, constitutes a uniform method for assigning Δs^2 to any pair of events.

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Let the arbitrary point $\mathcal{P}$ have coordinates (x, t), while the emission event has coordinates $(0, \tau_1)$ and the reception event has coordinates $(0, \tau_2)$. As well, choose the positive direction of $\hat{x}$ so that $x > 0$. As the signal must propagate between these, it follows that $\tau_2 > b > \tau_1$.

a. The first signal propagates from $(0, \tau_1)$ to (x, t) and has 0 interval, which implies

$$0 = (x - 0)^2 - (t - \tau_1)^2 \quad \Longrightarrow \quad x = t - \tau_1 \ .$$

b. The second signal propagates from (x, t) to $(0, \tau_2)$ and has 0 interval, which implies

$$0 = (0 - x)^2 - (\tau_2 - t)^2 \quad \Longrightarrow \quad x = \tau_2 - t \ .$$

These are two distinct equalities for x; therefore we may write

$$\tau_2 - t = t - \tau_1 \quad \Longrightarrow \quad \tau_2 + \tau_1 = 2t \ .$$

Now we evaluate the interval between $\mathcal{P}$ and the origin:

$$\mathcal{I} = (x - 0)^2 - t^2 = (\tau_2 - t)^2 - t^2 = \tau_2(\tau_2 - 2t) = \tau_2[\tau_2 - (\tau_1 + \tau_2)] = -\tau_2\tau_1 \ ,$$

which is the result that was requested.

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4. Use the formula for a pure Lorentz transformation, as a 4 × 4 matrix, that takes measurements made by a frame $\mathcal{O}$, that measures a second frame, $\mathcal{O}'$, to be moving with velocity $\vec{\beta}$, and transforms them to measurements made by $\mathcal{O}'$. We will label such a transformation by the symbols $\Lambda(\vec{\beta})$. Use it to write down this transformation for the two cases (i), $\vec{\beta} = v\hat{x}$, and (ii) $\vec{\beta} = v\hat{y}$, i.e., to write down the matrices $\Lambda(v\hat{x})$ and $\Lambda(v\hat{y})$. Then calculate the two product matrices

$$L_{xy} \equiv \Lambda(v\hat{x})\Lambda(v\hat{y}) \ , \qquad L_{yx} \equiv \Lambda(v\hat{y})\Lambda(v\hat{x}) \ .$$

Then show, first, that the two resulting 4 × 4 matrices are in fact different, and, secondly, that neither one of them is a pure Lorentz transformation, that is a matrix of the general form $\Lambda(\vec{\beta})$ for any value of $\vec{\beta}$.

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We begin by recalling the form for a generic Lorentz boost, from measurements of $\mathcal{O}'$to those of $\mathcal{O}'$, where $\mathcal{O}'$measures $\mathcal{O}'$to be moving with velocity $\vec{\beta}$:

$$\Lambda(\vec{\beta}) = \begin{pmatrix} \mathbf{I}_3 + (\gamma-1)\hat{\beta}\hat{\beta}^T & -\gamma\vec{\beta} \\ -\gamma\vec{\beta}^T & \gamma \end{pmatrix}, \qquad \vec{\beta} \Longrightarrow \begin{pmatrix} \beta^x \\ \beta^y \\ \beta^z \end{pmatrix}$$

$$= \begin{pmatrix} 1+h(\beta^x)^2 & h\beta^x\beta^y & h\beta^x\beta^z & -\gamma\beta^x \\ h\beta^y\beta^x & 1+h(\beta^y)^2 & h\beta^y\beta^z & -\gamma\beta^y \\ h\beta^z\beta^x & h\beta^z\beta^y & 1+h(\beta^z)^2 & -\gamma\beta^z \\ -\gamma\beta^x & -\gamma\beta^y & -\gamma\beta^z & +\gamma \end{pmatrix}, \qquad h \equiv \frac{\gamma-1}{\beta^2}\,.$$

where of course the $\gamma \equiv 1/\sqrt{1-\beta^2}$, i.e., it is that dilation factor made from $\vec{\beta}$.

Noting that $\vec{\beta} = \beta\hat{x}$ means that $\vec{\beta} \Longrightarrow (\beta, 0, 0)^T$, we may easily write down

$$\Lambda(v\hat{x}) = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}, \qquad \Lambda(v\hat{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\gamma v \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma v & 0 & \gamma \end{pmatrix}.$$

We then use Maple to compute the products of these matrices, and copy down the results here:

$$\Lambda(v\hat{x})\Lambda(v\hat{y}) = \begin{pmatrix} \gamma & \gamma^2 v^2 & 0 & -\gamma^2 v \\ 0 & \gamma & 0 & -\gamma v \\ 0 & 0 & 1 & 0 \\ -\gamma v & -\gamma^2 v & 0 & \gamma^2 \end{pmatrix},$$

$$\Lambda(v\hat{y})\Lambda(v\hat{x}) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ \gamma^2 v^2 & \gamma & 0 & -\gamma^2 v \\ 0 & 0 & 1 & 0 \\ -\gamma^2 v & -\gamma v & 0 & \gamma^2 \end{pmatrix}.$$

It is surely immediately clear that the two product matrices are different; this is not too surprising, presumably, since we know that matrix multiplication is not (usually) a commutative operation. To see that neither of them is a pure Lorentz transformation, i.e., a matrix of the form $\Lambda(\vec{\beta})$ for some vector $\vec{\beta}$, we first note that the general form of such a matrix, often called a *boost matrix* between two reference frames, is a **symmetric** 4×4 matrix, whereas ours are not. That is a sufficient answer for the question asked.

It is however also an interesting fact that the two matrices we do have are the matrix transposes of each other. One might wonder if that means anything.
