Physics 495

Homework No. 3 **Solutions:** due Wednesday, 16 September, 2009

1. An incoming light ray is being observed in both $\mathcal{O}$and $\mathcal{O}'$. The direction from which it is coming is measured by $\mathcal{O}$as α and by $\mathcal{O}'$as α'. Show that

$$\tan(\alpha'/2) = \sqrt{\frac{1-\beta}{1+\beta}}\tan(\alpha/2) \; .$$

Now consider a source of light that emits isotropically in its own rest frame. When considered from a different frame in which it is moving with velocity v, show that half of the light, i.e., half of the total number of photons emitted, are emitted into a cone with half angle $\sin\theta = 1/\gamma$, so that they are somewhat narrowly focused into the forward direction.

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To determine this it is best to begin with the Lorentz transformation equations for the energy. We suppose a light ray emitted, with a 3-momentum at an angle $\pi - \alpha$ relative to the direction of motion of the emitter, which is the same as saying that the direction from which it is coming has angle α relative to the velocity, $\vec{\beta}$ of the emitter. Therefore, remembering that the magnitude of the 3-momentum of a light ray is the same as its energy we may write

$$E' = \gamma_{\beta}(E - \vec{\beta}\cdot\vec{p}) = \gamma_{\beta}E[1-\beta\cos(\pi-\alpha)] = \gamma_{\beta}E(1+\beta\cos\alpha) \; .$$

However, we could always just as easily write this Lorentz transformation equation in the other direction, i.e., the following:

$$E = \gamma_{\beta}(E' + \vec{\beta}\cdot\vec{p}') = \gamma_{\beta}E'(1-\beta\cos\alpha') \; ,$$

and the equality of the two ratios gives us an equation that may be solved for the relation between the angle, as measured by the two different observers:

$$\begin{aligned}
\gamma_{\beta}(1-\beta\cos\alpha') &= \frac{1}{\gamma_{\beta}(1+\beta\cos\alpha)} \\
\Longrightarrow \quad 1-\beta\cos\alpha' &= \frac{1-\beta^2}{1+\beta\cos\alpha} \\
\Longrightarrow \quad \cos\alpha' &= \frac{\cos\alpha+\beta}{1+\beta\cos\alpha} \\
\Longrightarrow \quad 1+\cos\alpha' &= (1+\beta)\frac{1+\cos\alpha}{1+\beta\cos\alpha} \; .
\end{aligned}$$

This is a good relation; however, we wanted a relation between tangents of half angles; therefore, we proceed onward since we know the trigonometric relationship

$$\tan(\alpha/2) = \frac{\sin\alpha}{1+\cos\alpha} \ .$$

We can find the sine of the angle by using that portion of the 3-momentum perpendicular to $\vec{\beta}$:

$$p\sin\alpha = |\vec{p}_\perp| = |\vec{p}\,'_\perp| = p'\sin\alpha'$$
$$\Longrightarrow \quad \sin\alpha' = \frac{E}{E'}\sin\alpha \ = \frac{\sqrt{1-\beta^2}\,\sin\alpha}{1+\beta\cos\alpha} \ .$$

Note to grader: the relation between $\cos\alpha'$ and $\cos\alpha$ and also the one between $\sin\alpha'$ and $\cos\alpha$ and $\sin\alpha$ were derived in class; therefore, it is legitimate for the student to begin by simply writing those down. I have inserted them here simply so as to have a "complete" derivation.

Inserting both of these relationships into the trigonometric identity above we find

$$\tan(\alpha'/2) = \frac{\sin\alpha'}{1+\cos\alpha'} = \frac{\sqrt{1-\beta^2}\,\sin\alpha}{(1+\beta)(1+\cos\alpha)} = \sqrt{\frac{1-\beta}{1+\beta}}\frac{\sin\alpha}{1+\cos\alpha} = \sqrt{\frac{1-\beta}{1+\beta}}\tan(\alpha/2) \ ,$$

as was desired.

To consider the question of intensity, we begin in the frame where radiation is being emitted isotropically. We may certainly pick out the $\hat{z}$-direction as $\vec{\beta}$, and in either frame the intensity will be independent of the azimuthal angle, **measured perpendicularly** to $\vec{\beta}$. Also notice that the angle $\theta = \pi$ corresponds to the light ray headed **directly toward** the observer, because of the way we agreed on measuring the angles, while the angle $\theta = 0$ corresponds to our definitely not seeing the light because it's headed the other way. Therefore in this frame where the light is being emitted isotropically we may safely state that half of the photons are emitted when the angle θ is between π and $\pi/2$. Considering the equation above, relating $\cos\alpha'$ to $\cos\alpha$, we see that the angle $theta = \pi$ maps to the angle $\theta' = \pi$—not truly too unexpected I would hope. [This is also true of zero.] In addition it is clear that the mapping is a continuous function; therefore, we can immediately be sure that in our frame half of all the photons emitted are emitted between

$\theta' = \pi$ and some other angle, $\theta'_{\frac{1}{2}}$, which corresponds to $\theta = \pi/2$. Returning to that equation again we may easily determine that angle, since $\cos\pi/2 = 0$. This tells us quickly that

$$0 = \frac{\beta + \cos\theta'}{1 + \beta\cos\theta'} \quad \Longrightarrow \quad \cos\theta' = -\beta \quad \Longrightarrow \quad \sin\theta' = \sqrt{1-(-\beta)^2} = 1/\gamma_\beta \ .$$

Obviously this becomes rather small for sufficiently large β, so that the "half-width" of the incoming beam is rather small, justifying the name "headlight effect."

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2. Show that the following is the transformation equation between frames $\mathcal{O}$ and $\mathcal{O}'$ for the usual 3-acceleration, $\vec{a}$, i.e., the time derivative of the velocity $\vec{v}$:

$$\vec{a} = \left(\frac{\sqrt{1-\beta^2}}{1+\vec{\beta}\cdot\vec{v}'}\right)^3 \left[\vec{a}'_{||} + \gamma_\beta\vec{a}'_\perp + \gamma_\beta\vec{\beta}\times(\vec{a}'\times\vec{v}')\right] \ .$$

. . . .

Note to grader: this transformation could in principle be determined simply by working with the acceleration as the second time-derivative of the location, and differentiating everything appropriately. I don't want to do it that way; nonetheless, it would be legitimate for the student to do so if she or he desired.

We begin by recalling that the second derivative with respect to proper time of the 4-location of an event is the 4-acceleration, which is a 4-vector so that it transforms between frames via the usual Lorentz transformation matrix:

$$\begin{pmatrix} \vec{A}_{||} \\ \vec{A}_\perp \\ A^4 \end{pmatrix} \equiv \widetilde{a} = \Lambda(-\vec{\beta})\,\widetilde{a}' = \begin{pmatrix} \gamma_\beta[\vec{A}'_{||} + A'^4\vec{\beta}] \\ \vec{A}'_\perp \\ \gamma_\beta[A'^4 + \vec{\beta}\cdot\vec{A}'] \end{pmatrix} \ ,$$

$$\widetilde{a} \equiv \frac{d^2}{d\tau^2}\widetilde{r} = \frac{d}{d\tau}\widetilde{u} = \gamma_\beta\frac{d}{dt}\gamma_\beta\begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} = \gamma_v{}^2\begin{pmatrix} \vec{a} + \gamma_v{}^2(\vec{v}\cdot\vec{a})\vec{v} \\ \gamma_v{}^2(\vec{v}\cdot\vec{a}) \end{pmatrix}$$

Next, since the form of $\widetilde{a}$ involves not only $\vec{a}$ but also $\vec{v}$, we must first determine the appropriate Lorentz transformations for $\vec{v}$ and for γ_v, which we obtain from the fact that $\widetilde{u}$ is a 4-vector:

$$\gamma_v\begin{pmatrix} \vec{v}_{||} \\ \vec{v}_\perp \\ 1 \end{pmatrix} = \widetilde{u} = \begin{pmatrix} \gamma_\beta[\gamma_{v'}\vec{v}'_{||} + \vec{\beta}\gamma_{v'}] \\ \gamma_{v'}\vec{v}'_\perp \\ \gamma_\beta[\gamma_{v'} + \vec{\beta}\cdot\gamma_{v'}\vec{v}'] \end{pmatrix} = \gamma_{v'}\begin{pmatrix} \gamma_\beta[\vec{v}'_{||} + \vec{\beta}] \\ \vec{v}_\perp \\ \gamma_\beta[1 + \vec{\beta}\cdot\vec{v}'] \end{pmatrix}$$

To use this to establish the desired transformations we first pull out the fourth component, which gives the quite important

$$\text{Lorentz transformation for } \gamma_v\text{:} \qquad \gamma_v = \gamma_\beta \gamma_{v'} (1 + \vec{\beta} \cdot \vec{v}') \ .$$

We can also of course pull out the transformations for the 3-velocity, which we have already previously determined:

$$\vec{v}_\perp = \frac{\vec{v}'_\perp}{\gamma_\beta (1 + \vec{\beta} \cdot \vec{v}')} \ , \qquad \vec{v}_\| = \frac{\vec{v}'_\| + \vec{\beta}}{1 + \vec{\beta} \cdot \vec{v}'} \ .$$

Returning to the actual project at hand, the transformation of the acceleration, again we begin with the transformation of the fourth component, which will be needed later on:

$$\begin{aligned} \gamma_v^4 (\vec{v} \cdot \vec{a}) = A^4 = \gamma_\beta [A'^4 + \vec{\beta} \cdot \vec{A}'] &= \gamma_\beta [\gamma_{v'}^4 (\vec{v}' \cdot \vec{a}') + \gamma_{v'}^2 \vec{\beta} \cdot \vec{a}' + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}') \vec{\beta} \cdot \vec{v}'] \\ &= \gamma_\beta [\gamma_{v'}^2 \vec{\beta} \cdot \vec{a}' + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}')(1 + \vec{\beta} \cdot \vec{v}')] \\ \Longrightarrow \quad \gamma_v^2 (\vec{v} \cdot \vec{a}) &= \frac{\vec{\beta} \cdot \vec{a}'}{\gamma_\beta (1 + \vec{\beta} \cdot \vec{v}')^2} + \gamma_{v'}^2 \frac{\vec{v}' \cdot \vec{a}'}{\gamma_\beta (1 + \vec{\beta} \cdot \vec{v}')} \ . \end{aligned}$$

We may now proceed onward to the transformation for the perpendicular portion:

$$\begin{aligned} &\gamma_v^2 \vec{a}_\perp + \gamma_v^4 (\vec{v} \cdot \vec{a}) \vec{v}_\perp = \vec{A}_\perp = \vec{A}'_\perp = \gamma_{v'}^2 \vec{a}'_\perp + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}') \vec{v}'_\perp \\ &\Longrightarrow \quad \gamma_v^2 \vec{a}_\perp = \gamma_{v'}^2 \vec{a}_\perp + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}') \vec{v}'_\perp - \gamma_\beta [\gamma_{v'}^2 (\vec{\beta} \cdot \vec{a}') + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}')(1 + \vec{\beta} \cdot \vec{v}')] \frac{\vec{v}'_\perp}{\gamma_\beta (1 + \vec{\beta} \cdot \vec{v}')} \\ &\qquad = \gamma_{v'}^2 \left[\vec{a}'_\perp - \frac{(\vec{\beta} \cdot \vec{a}') \vec{v}'_\perp}{1 + \vec{\beta} \cdot \vec{v}'} \right] \ , \\ &\Longrightarrow \vec{a}_\perp = \frac{\vec{a}'_\perp - \frac{(\vec{\beta} \cdot \vec{a}') \vec{v}'_\perp}{1 + \vec{\beta} \cdot \vec{v}'}}{{\gamma_\beta}^2 (1 + \vec{\beta} \cdot \vec{v}')^2} = \frac{(1 + \vec{\beta} \cdot \vec{v}') \vec{a}'_\perp - (\vec{\beta} \cdot \vec{a}') \vec{v}'_\perp}{{\gamma_\beta}^2 (1 + \vec{\beta} \cdot \vec{v}')^3} = \frac{\vec{a}'_\perp + \vec{\beta} \times (\vec{a}'_\perp \times \vec{v}')}{{\gamma_\beta}^2 (1 + \vec{\beta} \cdot \vec{v}')^3} \ . \end{aligned}$$

After that we may proceed to the parallel portion:

$$\begin{aligned} &\gamma_v^2 \vec{a}_\| + \gamma_v^4 (\vec{v} \cdot \vec{a}) \vec{v}_\| = \vec{A}_\| = \gamma_\beta [\vec{A}'_\| + A'^4 \vec{\beta}] = \gamma_\beta [\gamma_{v'}^2 \vec{a}'_\| + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}') \vec{v}'_\| + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}')(\vec{v}'_\| + \vec{\beta})] \\ &\Longrightarrow \quad \gamma_v^2 \vec{a}_\| = \gamma_\beta [\vec{A}'_\| + A'^4 \vec{\beta}] = \gamma_\beta [\gamma_{v'}^2 \vec{a}'_\| + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}') \vec{v}'_\| + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}')(\vec{v}'_\| + \vec{\beta})] \\ &\qquad\qquad - \gamma_\beta [\gamma_{v'}^2 \vec{\beta} \cdot \vec{a}' + \gamma_{v'}^4 (\vec{v}' \cdot \vec{a}')(1 + \vec{\beta} \cdot \vec{v}')] \frac{\vec{v}'_\| + \vec{\beta}}{1 + \vec{\beta} \cdot \vec{v}'} \\ &\qquad = \gamma_\beta \gamma_{v'}^2 (\hat{\beta} \cdot \vec{a}') \hat{\beta} \left[1 - \frac{\vec{\beta} \cdot \vec{v}' + \beta^2}{1 + \vec{\beta} \cdot \vec{v}'} \right] = \gamma_{v'}^2 \frac{\vec{a}'_\|}{\gamma_\beta (1 + \vec{\beta} \cdot \vec{v}')} \end{aligned}$$

This is now easily solved for the parallel portion:

$$\vec{a}_{\|} = \frac{\vec{a}'_{\|}}{[\gamma_\beta(1+\vec{\beta}\cdot\vec{v}')]^3} \ .$$

The last portion of the task is simply then to add the parallel and perpendicular portions together, which gives the result requested:

$$\vec{a} = \frac{\vec{a}'_{\|} + \gamma_\beta[\vec{a}'_\perp + \vec{\beta}\times(\vec{a}'\times\vec{v}')]}{[\gamma_\beta(1+\vec{\beta}\cdot\vec{v}')]^3} \ .$$

3.

a. Find the energy, rest mass, and 3-velocity, $\vec{v}$, of a particle whose 4-momentum has the components $(1,1,0,4)$, measured in kilograms.

b. A collision of two particles occurs. They have incoming 4-momenta

$$\widetilde{p}_1 \Longrightarrow (-1,0,0,3)\ , \qquad \widetilde{p}_2 \Longrightarrow (1,1,0,2)\ ,$$

where these are also measured in kilograms, AND we recall that Schutz' text presents 4-vectors in the order (x^0,x^1,x^2,x^3) BUT I have changed the presentations above to Finley's order which is (x^1,x^2,x^3,x^4), where of course—in these Cartesian coordinates—x^4 and x^0 are equivalent.

The collision results in the destruction of those two particles and the production of three new ones, two of which have 4-momenta

$$\widetilde{p}_3 \Longrightarrow (1,0,0,1)\ , \qquad \widetilde{p}_4 \Longrightarrow (-1/2,0,0,1)\ ,$$

measured in kilograms. Find the 4-momentum, energy, rest mass, and 3-velocity of the third particle produced. Also find the 3-velocity of the center of mass of the system.

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a. The energy associated with a single 4-momentum is just the 4-th component, namely

$$E = 4 \text{ kg } = 4c^2 \text{ J } = 3.6\times 10^{17} \text{ J}.$$

The rest mass is obtained from the invariant square of the 4-momentum:

$$m = \sqrt{-\widetilde{p}\cdot\widetilde{p}} = \sqrt{4^2 - 1^2 - 1^2 - 0^2} = \sqrt{14}\ \text{kg} \ = 3.742\ \text{kg}.$$

The 3-velocity is the ratio of the momentum to the energy:

$$\vec{v} = \frac{\vec{p}}{E} = (\,(\,)\,1/4, 1/4, 0)\ .$$

b. The total momentum of the two incoming particles is $\widetilde{P} \Longrightarrow\ (0,1,0,5)$. Since total momentum is conserved, we may immediately determine the 4-momentum of the 5th particle:

$$\widetilde{p}_5 = \widetilde{P} - \widetilde{p}_3 - \widetilde{p}_4 \Longrightarrow\ (0,1,0,5) - (1,0,0,1) - (-1/2,0,0,1) = (-1/2,1,0,3)\ .$$

Then, following the same scheme as in part (a) we find its properties:

$$E_5 = 3\ , \quad m_5 = \sqrt{3^2 - 1^2 - (-1/2)^2} = \sqrt{31/4} = 2.784\ \text{kg}.$$

c. The 3-velocity of the center of mass is just the 3-velocity of the total momentum 4-vector, namely

$$\vec{V} = \frac{\vec{P}}{P^4} \Longrightarrow\ (0, 0.2, 0)\ .$$

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4. A very long limousine has a proper length of 20 meters. It is to be driven through a garage whose proper length is only 15 meters. Given that the front door will be open when the car begins to drive through, and that the back door will not open until the rear bumper is completely inside and the door closed, this should be possible—without destroying the garage—if the limousine is driven sufficiently fast, via the perceived Lorentz contraction of the car. For example, suppose the limousine is driven at a forward speed of $v = 0.8$; then the Lorentz contraction factor should be $1/\gamma = 3/5$. Let us consider the four world lines of the front and rear of the garage and of the front and rear of the limousine. As well consider the following two events. Event 1 is when the rear bumper just clears the front door of the garage and that door is then closed. [For simplicity, put the origin of the reference frames at this event.] Event 2 is when the front bumper of the limousine reaches the back door, when that door opens (immediately) to prevent it from being destroyed. Please draw a Minkowski diagram, from the point of view of the stationary observer standing next to the stationary garage, showing all 4 world lines and these two events. Be sure to arrange things so that Event 2 occurs after Event 1.

However, we now want to consider what this entire situation looks like from the point of view of the chauffeur inside the limousine. Therefore, please create another Minkowski diagram from the chauffeur's point of view, showing all 4 world lines and the two events. In this frame, in what order do the two events occur?

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To begin we note that the Lorentz factor for $v = 0.8$ is such that

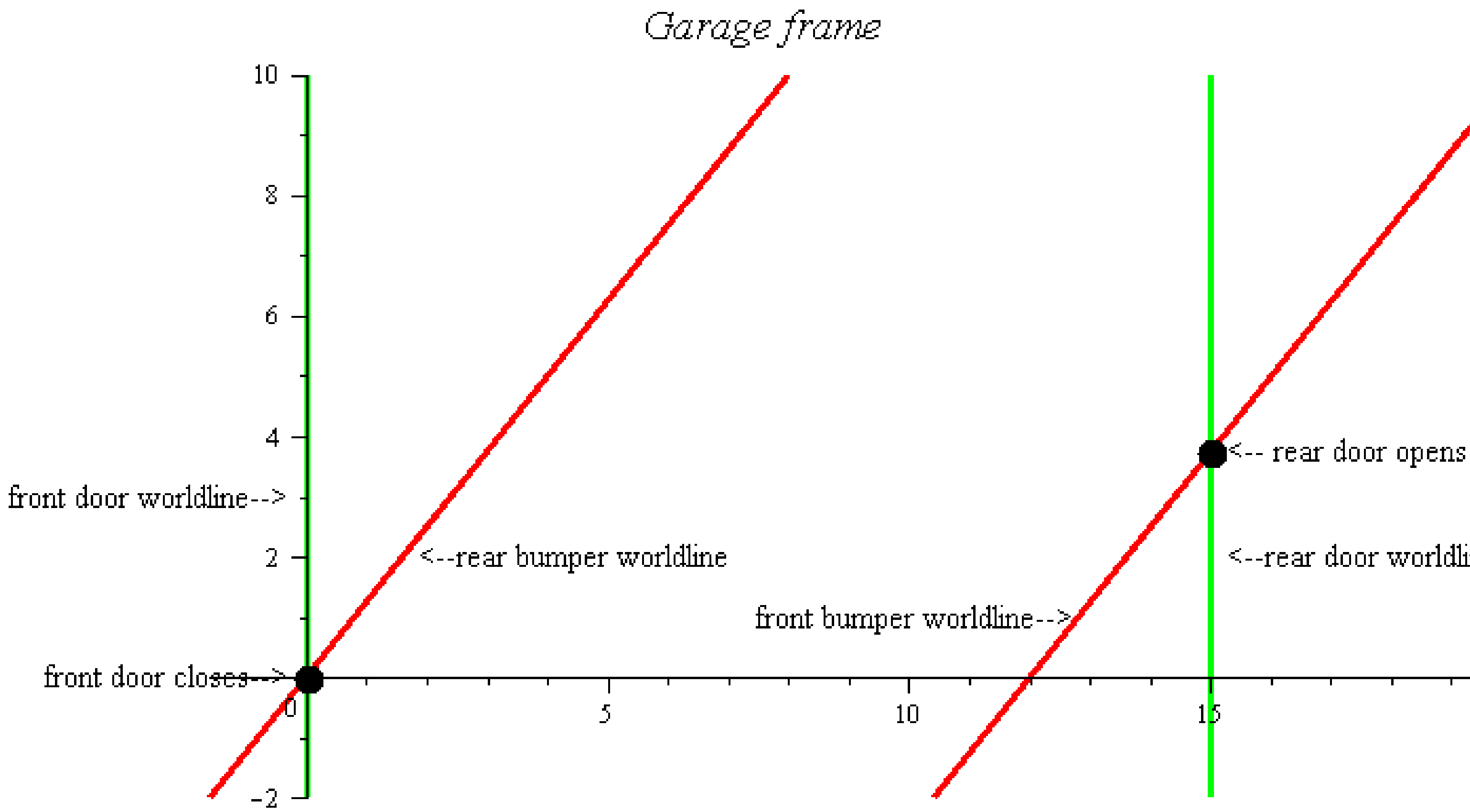
$$\gamma_v = \frac{1}{\sqrt{1 - .64}} = \frac{5}{3} \,.$$

Therefore the limousine will only be measured by the man on the ground as having a length of

$$L' = L/\gamma = \frac{3}{5}(20) = 12 \text{ meters},$$

so that it should easily fit within the garage which is 15 meters long—even in not for very much time. We have already been told to choose the origin as the event (No. 1) when the rear bumper of the car is at the front door of the garage. At that point the limousine must travel another 3 meters to get its front bumper to the back door. As it travels at a velocity of 4/5, this takes a time of $t_2 = (5/4)(3) = 15/4$ meters. Therefore Event 2 has coordinates $(3, 15/4)$. [Just for personal information, since the speed of light is 0.3 meters/nanosecond, this time can also be phrased in those units:

$$t = \frac{15}{4} \text{ m} \left[\frac{1 \text{ nanosecond}}{0.3 \text{ m}}\right] = 12.5 \text{ nanosec.}$$



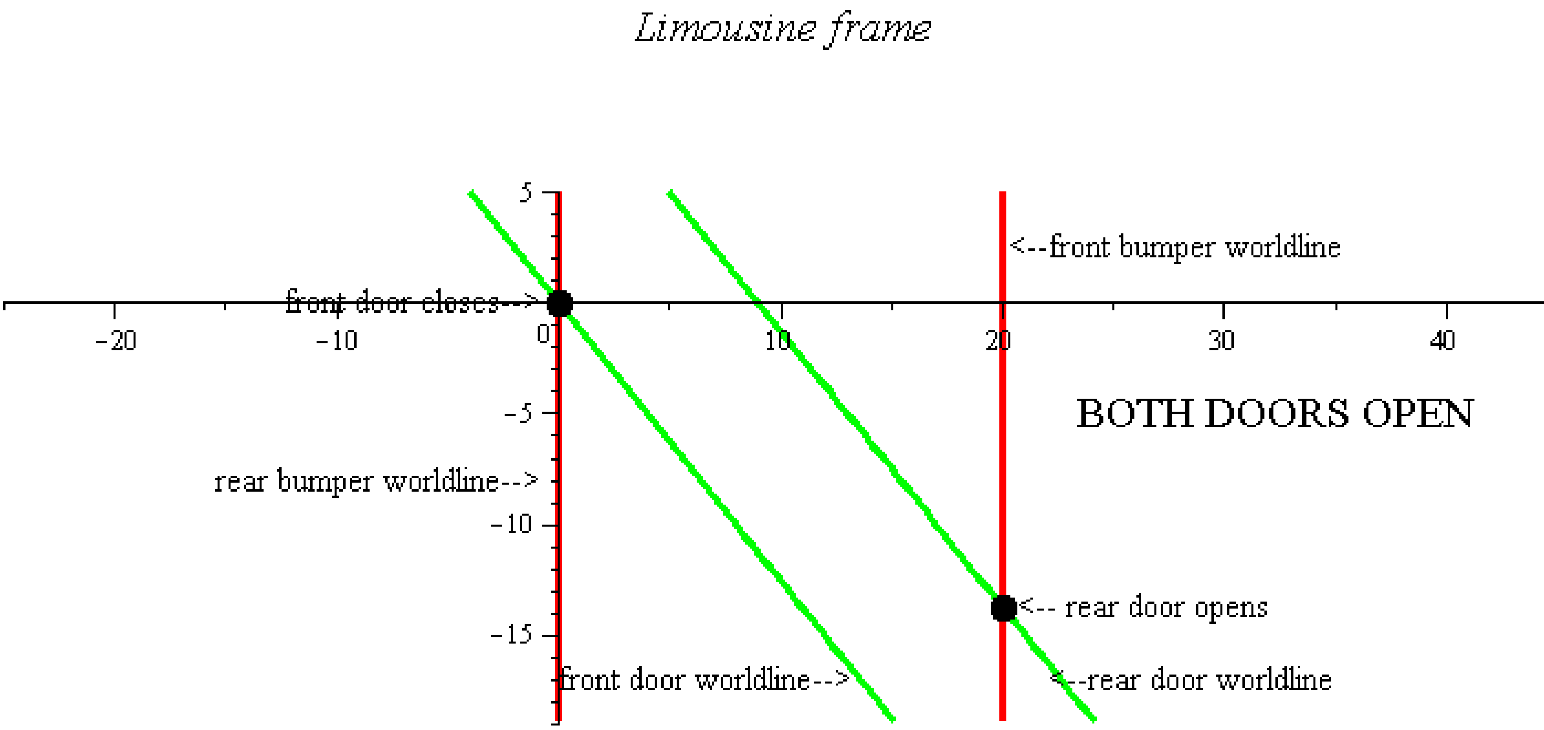
Note that the Minkowski diagram itself makes it clear that the separation between our two events is *spacelike.*

Now we want to go to the reference frame of the chauffeur in the limousine. He knows that his vehicle is 20 meters in length, and he sees a rather shorter garage headed toward him. The Lorentz contraction factor is the same as before, i.e., $\gamma = 5/3$, so the garage has a measured length of only $L' = (3/5)(15) = 9$ meters. Their two origins coincide, so that Event 1 is still at the origin. However, let's calculate his coordinates for Event 2:

$$x_2 = 15\,, \quad t_2 = 15/4\,;$$

$$x_2' = (5/3)[15 - (4/5)(15/4)] = 20 \text{ m}\,, \quad t_2' = (5/3)[(15/4) - (4/5)(15)] = -55/4 \text{ m}.$$

As this time is negative, in this frame, we see that Event 2 comes BEFORE Event 1 in this frame. This means the following: The garage approaches our stationary car—as seen in the car's reference frame—with its front door already open. The rear door then opens, at Event 2, prior to that rear end hitting the limousine. Both doors stay open for some time while the garage passes over the limousine. When the garage gets to the point that its front door has passed the rear end of the limousine that door closes, at Event 1, and no collision occurs. Below is a Minkowski diagram from the point of view of the chauffeur:



We see that the problem is resolved by the fact that the time ordering of the two events is different in the two frames, which is possible because the two events are spacelike separated.
