Physics 495

Homework No. 4 **Solutions:** due Wednesday, 30 September, 2009

1. Let $f = f(x, y, z, t)$ be a function of the four (independent) variables shown, which are being measured by observers in the reference frame $\mathcal{O}$. Set the differential of f in the standard way:

$$df \equiv \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt \ .$$

Now let $\mathcal{O}'$ be a second reference frame, who $\mathcal{O}$ measures as moving with velocity $\vec{\beta} = \beta\hat{z}$. Using the Lorentz transformation between the two frames, determine the components of the differential df that would be measured by $\mathcal{O}'$; i.e., how is $\partial f/\partial x'$, and the other partial derivatives, related to those derivatives measured by $\mathcal{O}$?

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Transformations between the two frames are initiated by the standard Lorentz transformation for the differences of coordinates of a pair of events:

$$\Delta\widetilde{r}' = \Lambda(\vec{\beta})\,\Delta\widetilde{r} \quad \text{or} \quad \Delta r'^{\alpha} = \Lambda^{\alpha}{}_{\mu}\,\Delta r^{\mu} \ ,$$

with Λ the usual 4×4 matrix relating the two, and I am using the symbol r^{μ} to denote the entire set of $\{x, y, z, t\}$ as μ varies from 1 to 4. As well, we may always choose one of the events as a constant one, especially if we choose it as the origin, so that we could also write this in the form

$$dr'^{\alpha} = \Lambda^{\alpha}{}_{\mu}\,dr^{\mu} \quad \Longrightarrow \quad \frac{\partial r'^{\alpha}}{\partial r^{\mu}} = \Lambda^{\alpha}{}_{\mu} \ .$$

Therefore, when we want to think of $f = f(r^{\mu})$ as actually a function of the primed variables, via $f[r^{\mu}(r'^{\alpha})]$, we may use the chain rule from calculus to write

$$\frac{\partial f}{\partial r^{\mu}} = \frac{\partial r'^{\alpha}}{r^{\mu}}\frac{\partial f}{\partial r'^{\alpha}} = \Lambda^{\alpha}{}_{\mu}\frac{\partial f}{\partial r'^{\alpha}} \ ,$$

indicating that these partial derivatives transform oppositely (or inversely) to the transformation of the coordinates themselves.

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2. Using the matrices $\mathfrak{K}_i$ that are the generators for pure Lorentz transformations, and also the matrices $\mathfrak{J}_j$ that are the generators for rotations, calculate the following matrices:

$$(\vec{\beta}\cdot\vec{\mathfrak{K}})^3\,, \qquad (\vec{\theta}\cdot\vec{\mathfrak{J}})^3\,, \qquad [\vec{\theta}\cdot\vec{\mathfrak{J}},\vec{\beta}\cdot\vec{\mathfrak{J}}]\,, \quad [\vec{\theta}\cdot\vec{\mathfrak{J}},\vec{\beta}\cdot\vec{\mathfrak{K}}]\,, \quad [\vec{\theta}\cdot\vec{\mathfrak{K}},\vec{\beta}\cdot\vec{\mathfrak{K}}]\,.$$

We note that this "dot product" of an ordinary 3-dimensional vector and a *matrix-valued* 3-dimensional vector should be thought of as follows:

$$\begin{aligned}\vec{\beta}\cdot\vec{\mathfrak{K}} &= \beta^x\mathfrak{K}_x+\beta^y\mathfrak{K}_y+\beta^z\mathfrak{K}_z\\ &=\begin{pmatrix}0&0&0&-\beta^x\\0&0&0&0\\0&0&0&0\\-\beta^x&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&-\beta^y\\0&0&0&0\\0&-\beta^y&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-\beta^z\\0&0&-\beta^z&0\end{pmatrix}\\ &=\begin{pmatrix}0&0&0&-\beta^x\\0&0&0&-\beta^y\\0&0&0&-\beta^z\\-\beta^x&-\beta^y&-\beta^z&0\end{pmatrix}.\end{aligned}$$

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All the calculations considered below were performed in Maple, and then transcribed here. Beginning with the form of $\vec{\beta}\cdot\vec{\mathfrak{K}}$ already given in the problem, I first calculate its square

$$(\vec{\beta}\cdot\vec{\mathfrak{K}})^2=\begin{pmatrix}\vec{\beta}\vec{\beta}^T&\vec{0}\\\vec{0}^T&\vec{\beta}^2\end{pmatrix}=\begin{pmatrix}\beta^x\beta^x&\beta^x\beta^y&\beta^x\beta^z&0\\\beta^y\beta^x&\beta^y\beta^y&\beta^y\beta^z&0\\\beta^z\beta^x&\beta^z\beta^y&\beta^z\beta^z&0\\0&0&0&\vec{\beta}^2\end{pmatrix},$$

and then its cube:

$$(\vec{\beta}\cdot\vec{\mathfrak{K}})^3=\begin{pmatrix}\mathbf{0}_3&-\vec{\beta}^2\vec{\beta}\\-\vec{\beta}^2\vec{\beta}^T&0\end{pmatrix}=-\vec{\beta}^2\begin{pmatrix}0&0&0&\beta^x\\0&0&0&\beta^y\\0&0&0&\beta^z\\\beta^x&\beta^y&\beta^z&0\end{pmatrix}=\vec{\beta}^2(\vec{\beta}\cdot\vec{\mathfrak{K}})\,.$$

This recursion relation, to within a scalar multiple, obviously allows us to easily calculate any (analytic) function of this matrix that we desire.

For the next question we need to recall the form for a generator of rotations:

$$\vec{\theta}\cdot\vec{\mathfrak{J}}=\begin{pmatrix}0&-\theta^z&+\theta^y&0\\\theta^z&0&-\theta^x&0\\-\theta^y&+\theta^x&0&0\\0&0&0&0\end{pmatrix},$$

which gives us for the square, and for the cube the following:

$$(\vec{\theta}\cdot\vec{\mathfrak{J}})^2 = \begin{pmatrix} \vec{\theta}\vec{\theta}^T - \vec{\theta}^2\mathbf{I}_3 & \vec{0} \\ \vec{0}^T & 0 \end{pmatrix} = \begin{pmatrix} \theta^x\theta^x - \vec{\theta}^2 & \theta^x\theta^y & \theta^x\theta^z & 0 \\ \theta^y\theta^x & \theta^y\theta^y - \vec{\theta}^2 & \theta^y\theta^z & 0 \\ \theta^z\theta^x & \theta^z\theta^y & \theta^z\theta^z - \vec{\theta}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

and for the cube we then obtain

$$(\vec{\theta}\cdot\vec{\mathfrak{J}})^3 = -\vec{\theta}^2(\vec{\theta}\cdot\vec{\mathfrak{J}}) ,$$

which again makes it simple to calculate infinite power series in this matrix.

Now we calculate the various requested commutators:

$$\left[\vec{\theta}\cdot\vec{\mathfrak{J}},\vec{\beta}\cdot\vec{\mathfrak{J}}\right] = \begin{pmatrix} 0 & \theta^y\beta^x - \theta^x\beta^y & \theta^z\beta^x - \theta^x\beta^z & 0 \\ \theta^x\beta^y - \theta^y\beta^x & 0 & \theta^z\beta^y - \theta^y\beta^z & 0 \\ \theta^x\beta^z - \theta^z\beta^x & \theta^y\beta^z - \theta^z\beta^y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (\vec{\theta}\times\vec{\beta})\cdot\vec{\mathfrak{J}} ;$$

$$\left[\vec{\theta}\cdot\vec{\mathfrak{J}},\vec{\beta}\cdot\vec{\mathfrak{K}}\right] = \begin{pmatrix} 0 & 0 & 0 & \theta^z\beta^y - \theta^y\beta^z \\ 0 & 0 & 0 & -\theta^z\beta^x + \theta^x\beta^z \\ 0 & 0 & 0 & \theta^y\beta^x - \theta^x\beta^y \\ \theta^z\beta^y - \theta^y\beta^z & -\theta^z\beta^x + \theta^x\beta^z & \theta^y\beta^x - \theta^x\beta^y & 0 \end{pmatrix} = (\vec{\theta}\times\vec{\beta})\cdot\vec{\mathfrak{K}} ;$$

$$\left[\vec{\theta}\cdot\vec{\mathfrak{K}},\vec{\beta}\cdot\vec{\mathfrak{K}}\right] = \begin{pmatrix} 0 & \theta^x\beta^y - \theta^y\beta^x & \theta^x\beta^z - \theta^z\beta^x & 0 \\ \theta^y\beta^x - \theta^x\beta^y & 0 & \theta^y\beta^z - \theta^z\beta^y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -(\vec{\theta}\times\vec{\beta})\cdot\vec{\mathfrak{J}} .$$

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3. Given the following vectors as measured by $\mathcal{O}$:

$$\widetilde{A} \Longrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} , \quad \widetilde{B} \Longrightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \quad \widetilde{C} \Longrightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} , \quad \widetilde{D} \Longrightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \\ -3 \end{pmatrix} ,$$

a. show that they are linearly independent;

b. find the components of the 1-form $\underset{\sim}{p}$ if

$$\underset{\sim}{p}(\widetilde{A}) = 1 , \quad \underset{\sim}{p}(\widetilde{B}) = -1 , \quad \underset{\sim}{p}(\widetilde{C}) = -1 , \quad \underset{\sim}{p}(\widetilde{D}) = 0 ;$$

c. find the value of $\underset{\sim}{p}(\widetilde{E})$ for

$$\widetilde{E} \Longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ;$$

d. determine whether the 1-forms $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$, and $\underset{\sim}{s}$ are linearly independent if $\underset{\sim}{q}(\widetilde{A}) = \underset{\sim}{q}(\widetilde{B}) = 0$, $\underset{\sim}{q}(\widetilde{C}) = 1 = -\underset{\sim}{q}(\widetilde{D})$, and $\underset{\sim}{r}(\widetilde{A}) = 2$ while $\underset{\sim}{r}(\widetilde{B}) = \underset{\sim}{r}(\widetilde{C}) = \underset{\sim}{r}(\widetilde{D}) = 0$, and $\underset{\sim}{s}(\widetilde{A}) = -1 = \underset{\sim}{s}(\widetilde{B})$, along with $\underset{\sim}{s}(\widetilde{C}) = 0 = \underset{\sim}{s}(\widetilde{D})$.

. . . .

a. We first put the four 4-vectors given, as measured by $\mathcal{O}$ into a matrix form, using the individually-given 4-vectors as its columns:

$$W = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & -3 \end{pmatrix} ,$$

and determine its determinant to be $+8$. As this is not zero, they are indeed linearly independent; i.e., if any one were a linear combination of any of the others the determinant would have vanished.

b. We may now determine the desired 1-form, $\underset{\sim}{p}$, by putting it as a row vector to be multiplied by W, on the right of $\underset{\sim}{p}$, which must then give the 4 results given as another row vector. This matrix equation is then resolved by multiplying by the inverse of W, which exists since it has a non-zero determinant:

$$(\,p_x \quad p_y \quad p_z \quad p_t\,)\,W = (\,1 \quad -1 \quad -1 \quad 0\,)$$

$$\Longrightarrow \quad (\,p_x \quad p_y \quad p_z \quad p_t\,) = (\,-3/8 \quad 15/8 \quad -23/8 \quad -1/4\,)\ .$$

c. We use the same methodology to determine $\underset{\sim}{p}(\widetilde{E})$:

$$\underset{\sim}{p}(\widetilde{E}) = (\,-3/8 \quad 15/8 \quad -23/8 \quad -1/4\,) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{5}{8}\ .$$

d. To determine whether the set of 1-forms $\{\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{r}, \underset{\sim}{s}\}$ is linearly independent we do not actually need to determine their values. Instead, since we are given the results of the action on their rights by the matrix W, it is sufficient to determine whether that set of results is linearly independent. To do this of course we again put those results into a matrix and determine its determinant, which turns out to be -2, so that, yes, they are indeed linearly independent.

[If you don't quite see this statement above right away, suppose that the matrix Q is a square matrix containing as its rows these four 1-forms, and that the matrix R is the square matrix containing as its rows the four sets of results that we were given. It then follows that $QW = R$. Since the four would surely be linearly independent if the determinant of Q were non-zero, we simply note that its determinant is equal to the product of the determinant of R times the inverse of the determinant of W.]

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4. Given the components of a $\binom{2}{0}$ tensor $M^{\alpha\beta}$ as the matrix

$$M^{\alpha\beta} \Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ +2 & 0 & 0 & +1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

a. find the following:

i.) the components of the symmetric tensor $M^{(\alpha\beta)}$ and the antisymmetric tensor $M^{[\alpha\beta]}$;

ii.) the components of $M^{\alpha}{}_{\beta}$;

iii.) the components of $M_{\alpha}{}^{\beta}$;

iv.) the components of $M_{\alpha\beta}$.

b. For the $\binom{1}{1}$ tensor whose components are $M^{\alpha}{}_{\beta}$, does it make sense to speak of its symmetric and antisymmetric parts? If so, define them. If not, say why.

c. Raise an index of the metric tensor to prove

$$\eta^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} \ .$$

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a. Again we calculate these quantities in Maple:

i.) The symmetric and anti-symmetric parts are given as

$$M^{(\alpha\beta)} = \frac{1}{2}\left(M^{\alpha\beta} + M^{\beta\alpha}\right) = \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{pmatrix} ;$$

$$M^{[\alpha\beta]} = \frac{1}{2}\left(M^{\alpha\beta} - M^{\beta\alpha}\right) = \begin{pmatrix} 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} & 0 \end{pmatrix} .$$

ii.) The components of $M^{\alpha}{}_{\beta}$ are equal to those of $M^{\alpha\gamma}\eta_{\gamma\beta}$. The process of multiplication by the metric matrix, **J**, simply changes the sign of elements in the fourth column:

$$M^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -2 \\ +2 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 \end{pmatrix} .$$

iii.) The components of $M_{\alpha}{}^{\beta}$ are the same as those of $\eta_{\alpha\gamma}M^{\gamma\beta}$. Multiplication on the left side by **J** changes the sign of elements in the fourth row:

$$M_{\alpha}{}^{\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ +2 & 0 & 0 & +1 \\ -1 & 0 & +2 & 0 \end{pmatrix} .$$

iv.) The components of $M_{\alpha\beta}$ are the same as those of $\eta_{\alpha\gamma}M^{\gamma\delta}\eta_{\delta\beta}$; therefore, this process changes the sign of elements in both the fourth row and the fourth column:

$$\Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -2 \\ +2 & 0 & 0 & -1 \\ -1 & 0 & +2 & 0 \end{pmatrix} .$$

b. The question is whether the statement that some particular tensor is symmetric, or antisymmetric, is a Lorentz invariant statement, or, if you prefer, as to whether it is a true relation between tensors. Since we know that the statement that a matrix is symmetric means that it equals its own transpose, we consider that question, first, for the original matrix, $M^{\alpha\beta}$. We first write down its Lorentz transform where, as usual, we normalize by giving the statement for the location 4-vector, where we denote its matrix presentation simply by r, and the matrix presentation of $M^{\alpha\beta}$ by $M2u$:

$$r'^{\alpha} = \Lambda^{\alpha}{}_{\mu}r^{\mu} \quad \text{or} \quad r' = \Lambda r$$

$$M'^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\, M^{\mu\nu} \quad \text{or} \quad M2u' = \Lambda M2u\Lambda^{T} .$$

Therefore we now propose the question: Suppose that $M2u = (M2u)^{T}$, then what is the status in that regard with respect to its Lorentz transform; or, differently phrased, is this a property that is preserved under Lorentz transformations. We test that by determining the transpose of the Lorentz transform:

$$(M2u')^T = \Lambda (M2u)^T \Lambda^T = \Lambda M2u \Lambda^T = M2u \ ,$$

and we see that, YES, this property is preserved. I believe it clear that the same proof would work for an antisymmetric tensor. Now, lets denote the matrix presentation of the tensor $M^\alpha{}_\beta$ by Mud and try the same thing:

$$M'^\alpha{}_\beta = \Lambda^\alpha{}_\mu M^\mu{}_\nu (\Lambda^{-1})^\nu{}_\beta \quad \text{or} \quad Mud' = \Lambda (Mud) \Lambda^{-1} \quad \Longrightarrow \quad (Mud')^T = (\Lambda^{-1})^T (Mud)^T \Lambda^T \ .$$

Now we will suppose that, at least in this frame, $Mud = (Mud)^T$, and see what this implies for the Lorentz transformed matrix:

$$(Mud')^T = (\Lambda^{-1})^T (Mud) \Lambda^T = (\Lambda^{-1})^T \Lambda^{-1} (Mud) \Lambda \Lambda^T \ .$$

We see that this works, i.e., preserves the symmetry in the one frame only when the matrix Λ is an orthogonal matrix, i.e., has its transpose as its inverse. This is true for rotations, but definitely not for Lorentz boosts! Therefore, it is not a reasonable thing to declare about this matrix, since we want statements we make to be true in an arbitrary inertial frame.

d. To determine the desired relationship, we simply raise one index of $\eta_{\alpha\beta}$:

$$\eta^\alpha{}_\beta = \eta^{\alpha\delta}\eta_{\delta\beta} \implies \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \impliedby \quad \delta^\alpha_\beta \ ,$$

as was desired.

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5. The fact that the product of two Lorentz transformations is again a Lorentz transformation, the fact that the identity transformation is a Lorentz transformation, and also the fact that the inverse of a Lorentz transformation is a Lorentz transformation tells us that these transformations **form a group, called the Lorentz group,** usually denoted by $\boldsymbol{O}(3,1)$, or sometimes by $\boldsymbol{L}(4)$.

a. Find the matrices of the identity element of the Lorentz group and of the element inverse to that whose matrix is implicit in the definition of a Lorentz transformation, for the case of $\vec{\beta}$ along the $\hat{x}$-direction.

b. Prove that the determinant of any matrix representing a Lorentz transformation is ± 1.

c. Prove that those elements whose matrices have determinant $+1$ form a subgroup, while those with -1 do not.

d. Show that the $(4,4)$-element in an arbitrary matrix representing the Lorentz group, $\boldsymbol{O}(3,1)$, i.e., $L^4{}_4$, has the property that $(L^4{}_4)^2 \geq +1$.

e. The three-dimensional orthogonal group $\boldsymbol{O}(3)$ is the analogous group for the metric of three-dimensional Euclidean space, which can be represented by the set of orthogonal matrices, i.e., those matrices which are such that their transpose is equal to their inverse. Show that the orthogonal matrices, so defined, do form a group, and then show that $\boldsymbol{O}(3)$ is isomorphic to a subgroup of $\boldsymbol{O}(3,1)$.

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a. The matrix $\Lambda(\beta\hat{x})$ is given by

$$\Lambda(\beta\hat{x}) = \begin{pmatrix} \gamma_\beta & 0 & 0 & -\gamma_\beta\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_\beta\beta & 0 & 0 & \gamma_\beta \end{pmatrix} .$$

Therefore its inverse is the same matrix, but with β replaced by $-\beta$, which is obvious from the physical meaning of the transformations:

$$\Lambda^{-1}(\beta\hat{x}) = \begin{pmatrix} \gamma_\beta & 0 & 0 & +\gamma_\beta\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma_\beta\beta & 0 & 0 & \gamma_\beta \end{pmatrix} .$$

Their product of course gives the identity in the group, which is presented simply by the 4×4 matrix identity:

$$\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

b. Since a Lorentz transformation is defined by a matrix L such that $L^T \eta L = \eta$, where η is the matrix representing the metric, we can take the determinant of the entire equation, which tells us that $-[\det(L)]^2 = -1$, which then tells us that $\det(L) = \pm 1$, since we know that the elements in the matrix are all real.

c. If we pick on L_1 and L_2 which are Lorentz transformations, then their product $L_1 L_2$ is a Lorentz transformation, since those transformations form a group. If, on the other hand, we now restrict those two initial transformations to have determinant $+1$ then their product will also have determinant of $+1$, showing that this subset contains its products. As well the determinant of the 4×4 identity is $+1$, and therefore the determinant of the inverse of L_i will also be $+1$, confirming that this subset forms a subgroup.

On the other hand if L_1 and L_2 come from the subset which have determinant -1, then their product will have determinant which is their product, which is $+1$, so that this subset does not even contain its own products and so certainly cannot form a subgroup.

d. We return to the defining transformation requirement:

$$\Lambda^T \eta \Lambda = \eta \ ,$$

and consider the $(4,4)$-component. Considering that η is a diagonal matrix, it gives

$$-1 = \eta_{44} = \Lambda^\alpha{}_4 \eta_{\alpha\beta} \Lambda^\beta{}_4 = -(\Lambda^4{}_4)^2 + \sum_{i=1}^{3} (\Lambda^i{}_4)^2 \ .$$

Moving terms around we have the desired result:

$$((\Lambda^4{}_4)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_4)^2 \quad \Longrightarrow \quad |\Lambda^4{}_4| \geq +1 \ .$$

e. The 3×3 matrices that are orthogonal, $R \in \boldsymbol{O}(3)$ satisfy $R^T R = \mathbf{I}_3$.

i.) It is clear that the identity 3×3 matrix satisfies this equation; therefore, the identity is an element of this set.

ii.) If R_1 and R_2 are elements then we have

$$(R_1 R_2)^T (R_1 R_2) = R_2^T R_1^T R_1 R_2 = R_2^T \mathbf{I}_3 R_2 = R_2^T R_2 = \mathbf{I}_3 \ ,$$

so that the product also is an orthogonal matrix.

iii.) If we take the inverse of both sides of our defining equation, noting that the inverse of the identity matrix is itself, then we have

$$\mathbf{I}_3^{-1} = \mathbf{I}_3 = R^{-1}(R^{T})^{-1} = (R^{-1})^{T} R^{-1} ,$$

confirming that the inverse of an orthogonal matrix is orthogonal, where we had to use the facts that the transpose of a matrix commutes with that matrix, and that the operations of transposing and inversion also commute.

iv.) Lastly, matrix multiplication is always associative.

Therefore we have shown that the orthogonal matrices do indeed form a group. It is straightforward to look at them, isomorphically, as if they were Lorentz transformations, by viewing a 3×3 matrix R in the form

$$L_{R} \equiv \begin{pmatrix} R & \vec{0} \\ \vec{0}^T & 1 \end{pmatrix} ,$$

where the statement that this 4×4 matrix is a Lorentz transformation simply means that $L_{R}^{T} \eta L_{R} = \eta$:

$$\begin{pmatrix} R^T & \vec{0} \\ \vec{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I}_3 & \vec{0} \\ \vec{0}^T & -1 \end{pmatrix} \begin{pmatrix} R & \vec{0} \\ \vec{0}^T & 1 \end{pmatrix} = \begin{pmatrix} R^T & \vec{0} \\ \vec{0}^T & -1 \end{pmatrix} \begin{pmatrix} R & \vec{0} \\ \vec{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_3 & \vec{0} \\ \vec{0}^T & -1 \end{pmatrix} .$$

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6. Consider the coordinates $u \equiv t - x$, $v \equiv t + x$ in Minkowski space.

a. Define $\widetilde{e}_u$ to be the vector connecting the events with coordinates $\{u = 1, v = 0, y = 0, z = 0\}$ and the origin, i.e., $\{u = 0, v = 0, y = 0, z = 0\}$, and analogously for $\widetilde{e}_v$.

a. Show that

$$\widetilde{e}_u = \tfrac{1}{2}(\widetilde{e}_t - \widetilde{e}_x) , \quad \widetilde{e}_v = \tfrac{1}{2}(\widetilde{e}_t + \widetilde{e}_x) ,$$

and draw $\widetilde{e}_u$ and $\widetilde{e}_v$ in a spacetime diagram of the t, x-plane.

b. Show that $\{\widetilde{e}_u, \widetilde{e}_v, \widetilde{e}_y, \widetilde{e}_z\}$ are a basis for vectors in Minkowski space.

c. Find the components of the metric tensor on this basis.

d. Show that $\widetilde{e}_u$ and $\widetilde{e}_v$ are null, and not orthogonal. (They are called a *null basis* for the t, x-plane.

e. Compute the four one forms du, dv, and $\mathbf{g}(\widetilde{e}_u,\ )$, $\mathbf{g}(\widetilde{e}_v,\ )$, in terms of dt and dx.

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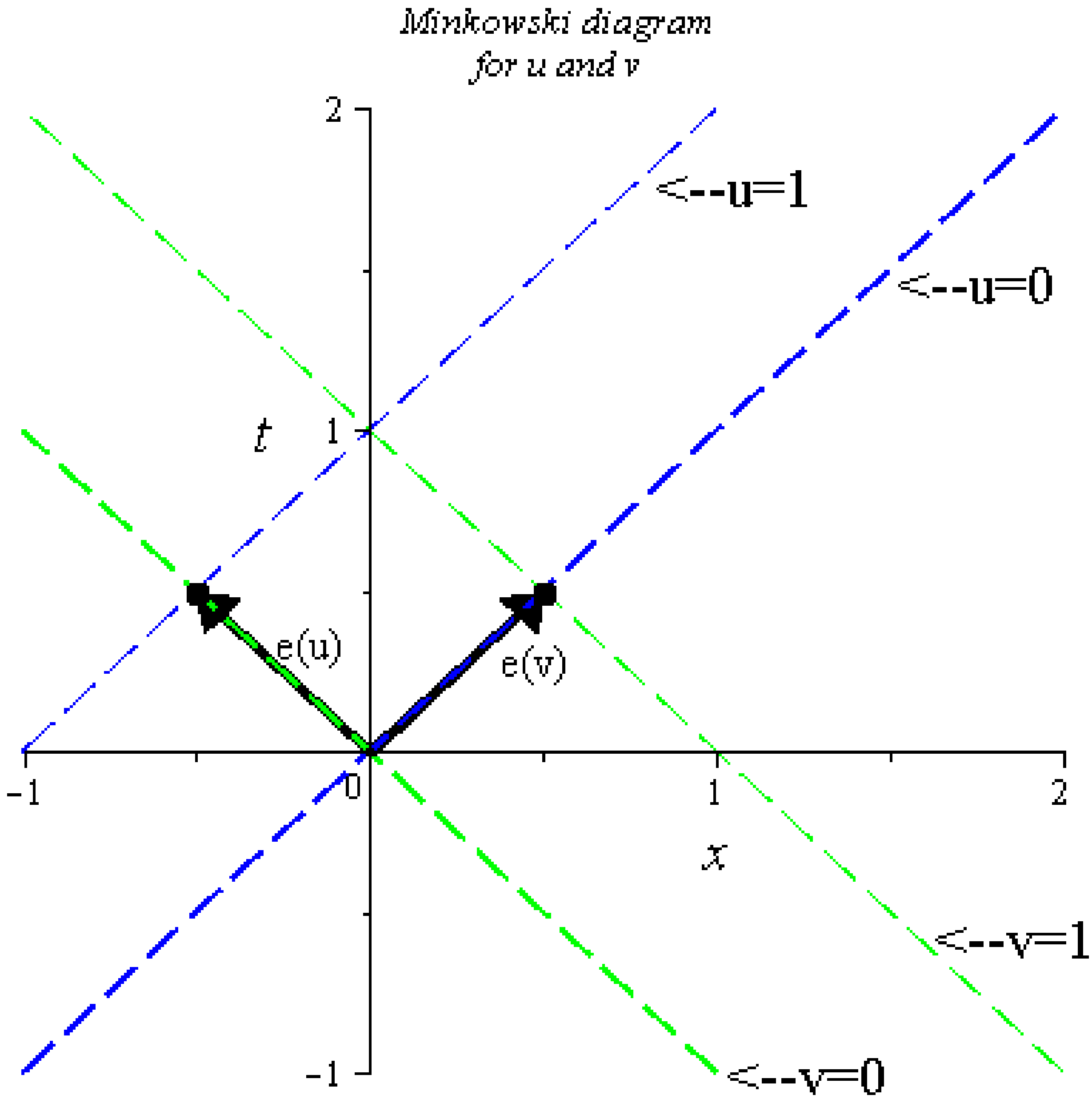
On the standard Minkowski diagram just below, we have shown lines for $u = 0$ and $u = 1$ and also $v = 0$ and $v = 1$, which are of course all at 45° relative to the horizontal.



a. One can see the x, t-coordinates of the point $u = 1$, $v = 0$ which is supposed to be the tip of the arrow, from the origin, for $\widetilde{e}_u$, namely $x = -1/2 = -t$, as could also of course be determined by solving the two equations. We have drawn that arrow there. Likewise one can see the x, t-coordinates of the point $u = 0$, $v = 1$, which is the tip of the arrow, from the origin, for $\widetilde{e}_v$, namely $x = +1/2 = t$, again as could also have been determined by solving the two equations.

Therefore, remembering that $\widetilde{e}_x$ is of unit length and runs, from left to right, along the positive x-axis while $\widetilde{e}_t$ is of unit length and runs, toward the top, along the positive t-axis, one can see that the desired equations relating the unit vectors are indeed true:

$$\widetilde{e}_u = \tfrac{1}{2}(\widetilde{e}_t - \widetilde{e}_x) , \quad \widetilde{e}_v = \tfrac{1}{2}(\widetilde{e}_t + \widetilde{e}_x) .$$

It is also worthwhile re-writing all this in terms of the form of basis vectors for tangent vectors written as partial derivative operators:

$$\partial_u = \tfrac{1}{2}(\partial_t - \partial_x) , \quad \partial_v = \tfrac{1}{2}(\partial_t + \partial_x) ,$$

where I use the abbreviated symbolism that ∂_x, say, is written instead of the more lengthy form $\partial/\partial x$. With this form in hand, for example, I can now see immediately that these operators have exactly the correct effect on the original definitions of the coordinates:

$$\partial_u(u) = \tfrac{1}{2}(\partial_t - \partial_x)(t - x) = \tfrac{1}{2}(1 - 0 - 0 + 1) = 1 ,$$

$$\partial_u(v) = \tfrac{1}{2}(\partial_t - \partial_x)(t + x) = \tfrac{1}{2}(1 + 0 - 0 - 1) = 0 ,$$

$$\partial_v(u) = \tfrac{1}{2}(\partial_t + \partial_x)(t - x) = \tfrac{1}{2}(1 - 0 + 0 - 1) = 0 ,$$

$$\partial_v(v) = \tfrac{1}{2}(\partial_t + \partial_x)(t + x) = \tfrac{1}{2}(1 + 0 + 0 + 1) = 1 .$$

b. The criterion that I have been using for a basis is that they should be linearly independent and span the space. Since we are assuming that the set $\{\widetilde{e}_x, \widetilde{e}_y, \widetilde{e}_z, \widetilde{e}_t\}$ forms a basis, we then simply note that we can replace the pair $\{\widetilde{e}_x, \widetilde{e}_t\}$ with any other pair that is linearly independent and span the same 2-space that they did, which is shown, for instance, by inverting the pair of equations just above:

$$\widetilde{e}_x = \widetilde{e}_v - \widetilde{e}_u , \quad \widetilde{e}_t = \widetilde{e}_v + \widetilde{e}_u .$$

c. The components of the metric tensor, presented as a matrix, are simply $\underset{\sim}{\eta}(\widetilde{e}_i, \widetilde{e}_j) \equiv \widetilde{e}_i \cdot \widetilde{e}_j$ in the i-th row and j-th column. Therefore we need

$$\widetilde{e}_u \cdot \widetilde{e}_u = \tfrac{1}{4}(1+0+0-1) \ = 0 = \tfrac{1}{4}(1+0+0-1) = \widetilde{e}_v \cdot \widetilde{e}_v \ ,$$
$$\widetilde{e}_u \cdot \widetilde{e}_u = \tfrac{1}{4}(-1+0+0-1) \ = -\frac{1}{2} = \tfrac{1}{4}(-1+0+0-1) = \widetilde{e}_v \cdot \widetilde{e}_u \ .$$

Therefore our metric matrix, relative to this basis, with the ordering for the basis as $\widetilde{e}_y$, $\widetilde{e}_z$, $\widetilde{e}_u$, $\widetilde{e}_v$, is given by

$$\mathbf{g}(\widetilde{e}_\mu, \widetilde{e}_\nu) = \begin{array}{c} \\ \widetilde{e}_y \\ \widetilde{e}_z \\ \widetilde{e}_u \\ \widetilde{e}_v \end{array} \begin{array}{c} \begin{array}{cccc} \widetilde{e}_y & \widetilde{e}_z & \widetilde{e}_u & \widetilde{e}_v \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \end{array} \equiv \lambda_{\mu\nu} \ ,$$

and I have used the symbol $\lambda_{\mu\nu}$ to denote the metric relative to this basis, so as to distinguish it from our "standard" form of the metric matrix, $\eta_{\mu\nu}$), which we reserve for the case of a Cartesian choice of basis vectors.

d. It is apparent from the calculations just above that both $\widetilde{e}_u$ and $\widetilde{e}_v$ are null, i.e., of zero length. Moreover, they are obviously not orthogonal since their scalar product is not zero.

e. To determine these four 1-forms, we first attack the easy (or "obvious") pair, where we simply take the defining equations for u and v and use the exterior differential on them:

$$\left.\begin{array}{l} u \equiv t - x \ , \\ v \equiv t + x \ , \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} du = dt - dx \ , \\ dv = dt + dx \ . \end{array}\right.$$

At this point we may immediately show that these have the correct form for a reciprocal basis, relative to the basis for tangent vectors above:

$$du(\partial_u) = (dt - dx)\tfrac{1}{2}(\partial_t - \partial_x) = \tfrac{1}{2}(1-0-0+1) = 1 \ ,$$
$$du(\partial_v) = (dt - dx)\tfrac{1}{2}(\partial_t + \partial_x) = \tfrac{1}{2}(1+0-0-1) = 0 \ ,$$
$$dv(\partial_u) = (dt + dx)\tfrac{1}{2}(\partial_t - \partial_x) = \tfrac{1}{2}(1-0+0-1) = 0 \ ,$$
$$dv(\partial_v) = (dt + dx)\tfrac{1}{2}(\partial_t + \partial_x) = \tfrac{1}{2}(1+0+0+1) = 1 \ .$$

However, as already suggested, the understanding about the other pair of 1-forms requested is somewhat "trickier." Therefore, please let me do the calculation in two somewhat different ways. Firstly, we try a very straightforward approach to determine $\mathbf{g}(\widetilde{e}_u,\ )$ and $\mathbf{g}(\widetilde{e}_v,\ )$. These are clearly defined as objects that require the insertion of one more (tangent) vector into them in order that they determine a number. This is the definition of a 1-form; therefore, the forms shown are simply somewhat different way of defining a particular 1-form. With that premise, then, let us consider their action on some arbitrary vector, say $\widetilde{A} = A^x\partial_x + A^y\partial_y + A^z\partial_z + A^t\partial_t$, where we insert the already-determined values of $\widetilde{e}_u$ and $\widetilde{e}_v$:

$$\mathbf{g}(\widetilde{e}_u, \widetilde{A}) = \tfrac{1}{2}[\mathbf{g}(\widetilde{e}_t, \widetilde{A}) - \mathbf{g}(\widetilde{e}_x, \widetilde{A})] = \tfrac{1}{2}(-A^t - A^x) = -\tfrac{1}{2}(A^t + A^x) \ ,$$

$$\mathbf{g}(\widetilde{e}_v, \widetilde{A}) = \tfrac{1}{2}[\mathbf{g}(\widetilde{e}_t, \widetilde{A}) + \mathbf{g}(\widetilde{e}_x, \widetilde{A})] = \tfrac{1}{2}(-A^t + A^x) = -\tfrac{1}{2}(A^t - A^x) \ .$$

Looking at these results, we now notice the following continuations of those calculations:

$$\mathbf{g}(\widetilde{e}_u, \widetilde{A}) = -\tfrac{1}{2}(A^t + A^x) = -\tfrac{1}{2}dv(\widetilde{A}) \ ,$$

$$\mathbf{g}(\widetilde{e}_v, \widetilde{A}) = -\tfrac{1}{2}(A^t - A^x) = -\tfrac{1}{2}du(\widetilde{A}) \ ,$$

from which we "easily" determine the desired mappings:

$$\mathbf{g}(\widetilde{e}_u,\ ) = -\tfrac{1}{2}\, dv = -\tfrac{1}{2}(dt + dx) \ ,$$

$$\mathbf{g}(\widetilde{e}_v,\ ) = -\tfrac{1}{2}\, du = -\tfrac{1}{2}(dt - dx) \ .$$

On the other hand, a second, quite different, approach is to recall that the component form of the action of the metric on a vector, turning it into a 1-form, is given by

$$\mathbf{g}(\widetilde{B},\ ) = g_{\mu\nu}B^\nu \underset{\approx}{\omega}^\mu \equiv B_\mu \underset{\approx}{\omega}^\mu \ ,$$

where we have used simply the symbol $\mathbf{g}$ for whatever, arbitrary, choice of metric that has been made, and used this to define explicitly the form of the components of the 1-form associated, via the action of the metric, on a tangent vector. In our particular case, we should use the form of the metric appropriate to this new basis, as computed in section (c) above, and also to use the appropriate form of a column vector for the components of our basis vectors, relative to this new choice of basis. In particular then, a basis set has an "obvious" form of matrix representation relative to itself and its ordering, i.e., $\{\widetilde{e}_y, \widetilde{e}_z, \widetilde{e}_u, \widetilde{e}_z\}$ as chosen above:

$$\widetilde{e}_y \Longrightarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \ , \quad \widetilde{e}_z \Longrightarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \ , \quad \widetilde{e}_u \Longrightarrow \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \ , \quad \widetilde{e}_v \Longrightarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \ .$$

On the other hand the corresponding matrix presentations for our (reciprocal) basis for 1-forms, i.e., for the set $\{\underset{\approx}{\omega}^\mu\}_{\mu=1}^4$ subject to $\underset{\approx}{\omega}^\mu(\widetilde{e}_\nu) = \delta^\mu_\nu$, would be

$$\underset{\approx}{\omega}^y \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \quad \underset{\approx}{\omega}^z \Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} ,$$

$$\underset{\approx}{\omega}^u \Longrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} , \quad \underset{\approx}{\omega}^v \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} .$$

With that agreement in hand, and the form of the metric matrix in this basis from section (c), we may write

$$(\widetilde{e}_u)_\mu = [\mathbf{g}(\widetilde{e}_u, \ )]_\mu = \lambda_{\mu\nu}(\widetilde{e}_u)^\nu$$

$$\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Longleftarrow \quad -\tfrac{1}{2}[(\underset{\approx}{\omega}^v)^T]_\mu ;$$

$$(\widetilde{e}_v)_\mu = [\mathbf{g}(\widetilde{e}_v, \ )]_\mu = \lambda_{\mu\nu}(\widetilde{e}_v)^\nu$$

$$\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Longleftarrow \quad -\tfrac{1}{2}[(\underset{\approx}{\omega}^u)^T]_\mu .$$

This of course results in the same results as the first derivation given above, but looks at things in a very-much-more coordinate-dependent (or matrix-dependent) approach.

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7. Consider a particle of mass m moving with a velocity $\vec{v}(t)$, that is not too much below the speed of light, and an acceleration $d\vec{v}/dt \equiv \vec{a}(t)$. The acceleration is due to the action on the particle by a force $\vec{F}(t)$. Divide the force and the acceleration into their components that are either parallel or perpendicular to the particle's velocity, at any particular instant, and determine the scalar proportionality factors between the force and the acceleration, for each of the two sorts, i.e., for the parallel components and the perpendicular components. As these two proportionality components are different, this assures us that the force 3-vector is not parallel to the acceleration 3-vector.

. . . .

We have defined the force acting on a particle as the time derivative of its momentum. We only consider the case when the mass is constant; therefore, we have

$$\vec{F} = \frac{d}{dt}\vec{p} = m\frac{d}{dt}\gamma_v\vec{v} = m\gamma_v \left[\vec{a} + \gamma_v^2(\vec{v}\cdot\vec{a})\vec{v}\right] .$$

This simplifies enormously when we consider, first, those components of $\vec{F}$ and $\vec{a}$ that are (momentarily) perpendicular to the velocity:

$$\vec{F}_\perp = \gamma_v m \vec{a}_\perp \ .$$

When we now consider those components parallel to $\vec{v}$ we have additional terms:

$$\vec{F}_\| = \gamma_v m \vec{a}_\| (1 + v^2 \gamma_v^2) = \gamma_v^3 m \vec{a}_\| \ .$$

This makes it clear that the two full 3-vectors are not parallel, since these two pairs of components have different proportionalities.