Physics 495

Homework No. 5 **Solutions:** due Wednesday, 7 October, 2009

1. Let us work in the usual 3-dimensional space, with Cartesian coordinates, $\{x, y, z\}$. Let the electric field vector be denoted as

$$\vec{E} = E^x \partial_x + E^y \partial_y + E^z \partial_z \ ,$$

where the partial derivative operators with respect to the coordinates are being taken as the basis set for (tangent) vector fields. [This particular vector is *tangent* to Maxwell's lines of force!]

a. First show that the (dual) basis set for 1-forms is $\{dx, dy, dz\}$.

b. Then determine the 1-form $\underset{\sim}{E}$, which has the effect that, for some arbitrary 3-vector field, $\widetilde{A}$, is such that

$$\vec{E} \cdot \vec{A} = \underset{\sim}{E}(\vec{A}) \ ,$$

where the metric for the "dot product" is just the usual 3×3 identity matrix.

c. Determine the explicit form of the components of the 2-form $d\underset{\sim}{E}$. Compare this to the standard form for the components of the curl of $\vec{E}$, i.e., $\nabla \times \vec{E}$.

d. Determine the (Hodge) dual of $\underset{\sim}{E}$, i.e., $*\underset{\sim}{E}$, which is a 2-form. Note that the set $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$ is a basis for the space of 2-forms.

e. Determine the exterior derivative of this Hodge dual, i.e., $d * \underset{\sim}{E}$, which is then a 3-form, and compare it to the usual form of the divergence of $\vec{E}$, i.e., $\nabla \cdot \vec{E}$. Note that the usual Cartesian basis for 3-forms is the one quantity $dx \wedge dy \wedge dz$.

f. Determine explicitly the components of the 2-form $d * d\underset{\sim}{E}$.

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a. The generic definition of the dual basis is that it should reproduce the elements of the identity matrix when the members of that dual basis operate on the vectors to which they are dual, i.e., we should have $\underset{\sim}{\omega}^i(\widetilde{e}_j) = \delta^i_j$. Therefore we only need to show that

$$dx^i(\partial_{x^j}) = \delta^i_j \ .$$

However, this is straightforward because the action is the same as the other way around, i.e., this is equivalent to

$$\frac{\partial}{\partial x^j}(x^i) = \delta^i_j \ ,$$

which is true.

b. In this case we simply write out $\underset{\sim}{E} = f_i dx^i$, where the symbols f_i are simply a labeling of the elements of the components of $\underset{\sim}{E}$ relative to the basis for 1-forms we have just discussed in part (a) above, and we use that symbol just so as not to prejudice the final result. Then the requirement is that

$$E^x A^x + E^y A^y + E^z A^z = \vec{E} \cdot \vec{A} \equiv \underset{\sim}{E}(\vec{A}) = f_x A^x + f_y A^y + f_z A^z \ ,$$

from which, since $\vec{A}$ is arbitrary, we deduce that the components of $\underset{\sim}{E}$, relative to the basis dx^i are the same as the components of $\vec{E}$ relative to the basis ∂_{x^i}; i.e., we have

$$f_i dx^i \equiv \underset{\sim}{E} \equiv E_x dx + E_y dy + E_z dz = E^x dx + E^y dy + E^z dz \ .$$

This is hardly unexpected since the more generic way to find these components might simply be

$$E_i \equiv g_{ij} E^j \ ,$$

and in this case g_{ij} is just 0 or 1 depending on whether $i \neq j$ or $i = j$, respectively.

c. To determine $d\underset{\sim}{E}$ we have

$$\begin{aligned} d\underset{\sim}{E} &= dE_k \wedge dx^k = (\partial_m E_k) dx^m \wedge dx^k \\ &= (\partial_x E_y) dx \wedge dy + (\partial_x E_z) dx \wedge dz + (\partial_y E_x) dy \wedge dx \\ &\quad + (\partial_y E_z) dy \wedge dz + (\partial_z E_x) dz \wedge dx + (\partial_z E_y) dz \wedge dy \\ &= (\partial_x E_y - \partial_y E_x) dx \wedge dy + (\partial_z E_x - \partial_x E_z) dz \wedge dx + (\partial_y E_z - \partial_z E_y) dy \wedge dz \ . \end{aligned}$$

However, we know that we may determine the curl of $\vec{E}$ as follows:

$$\begin{aligned} \vec{E} &= E^x \hat{x} + E^y \hat{y} + E^z \hat{z} \\ \nabla \times \vec{E} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E^x & E^y & E^z \end{vmatrix} = (\partial_x E_y - \partial_y E_x)\hat{z} + (\partial_z E_x - \partial_x E_z)\hat{y} + (\partial_y E_z - \partial_z E_y)\hat{z} \ . \end{aligned}$$

We see that the two sets of components are identical, so that the two quantities are "identical" if we agree to make the following identifications:

$$dx \wedge dy \leftrightarrow \hat{z}\ , \quad dy \wedge dz \leftrightarrow \hat{x}\ , \quad dz \wedge dx \leftrightarrow \hat{y}\ ,$$

which is in fact the identification that would be made under the Hodge duality mapping of 3-dimensional 2-forms into 1-forms, and vice-versa. [This also assumes the identification of the two different ways of thinking about bases of tangent vectors, i.e.,

$$\hat{x} \leftrightarrow \partial_x\ , \quad \hat{y} \leftrightarrow \partial_y\ , \quad \hat{x} \leftrightarrow \partial_z\ .$$

d. Again, using the ideas of the Hodge dual, we may write out the 2-form, in the way just described in the previous section:

$$*\underset{\sim}{E} = E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy\ .$$

e. The exterior derivative of this form is determined in the same way as above, but, now, for a 2-form:

$$\begin{aligned} d * \underset{\sim}{E} &= d(E_x)\, dy \wedge dz + d(E_y)\, dz \wedge dx + d(E_z)\, dx \wedge dy \\ &= (\partial_x E_x + \partial_y E_y + \partial_z E_z) dx \wedge dy \wedge dz\ , \end{aligned}$$

where the scalar function multiplying our volume 3-form is just the usual definition of the divergence of $\vec{E}$, namely $\nabla \cdot \vec{E}$, provided as before that we recall that $E_x = E^x$, etc.

f. Above we determined $d\underset{\sim}{E}$. To compute its Hodge dual we simply make the mappings that we have already discussed; therefore, we have

$$*d\underset{\sim}{E} = (\partial_x E_y - \partial_y E_x) dz + (\partial_z E_x - \partial_x E_z) dy + (\partial_y E_z - \partial_z E_y) dx\ .$$

We may then determine its exterior derivative:

$$\begin{aligned} d * d\underset{\sim}{E} &= d(\partial_x E_y - \partial_y E_x) \wedge dz + (\partial_z E_x - \partial_x E_z) \wedge dy + (\partial_y E_z - \partial_z E_y) \wedge dx \\ &= \partial_x(\partial_x E_y - \partial_y E_x) dx \wedge dz + \partial_y(\partial_x E_y - \partial_y E_x) dy \wedge dz + \partial_x(\partial_z E_x - \partial_x E_z) dx \wedge dy \\ &\qquad + \partial_z(\partial_z E_x - \partial_x E_z) dz \wedge dy + \partial_y(\partial_y E_z - \partial_z E_y) dy \wedge dx + \partial_z(\partial_y E_z - \partial_z E_y) dz \wedge dx \\ &= [\partial_y(\partial_x E_x + \partial_z E_z) - (\partial_x^2 + \partial_z^2) E_y] dz \wedge dx + [\partial_x(\partial_y E_y + \partial_z E_z) - (\partial_y^2 + \partial_z^2) E_x] dy \wedge dz \\ &\qquad + [\partial_z(\partial_x E_x + \partial_y E_y) - (\partial_x^2 + \partial_y^2) E_z] dx \wedge dy\ . \end{aligned}$$

This really looks rather "nasty"; however, if we add and subtract a single term in each of the three components above, and also take the Hodge dual of the final result we have

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\begin{aligned}
*d*d\underset{\sim}{E} = {} & [\partial_y(\partial_x E_x + \partial_z E_z + \partial_y E_y) - (\partial_x^2 + \partial_z^2 + \partial_y^2)E_y]dy + [\partial_x(\partial_y E_y + \partial_z E_z + \partial_x E_x) \\
& - (\partial_y^2 + \partial_z^2 + \partial_x^2)E_x]dx + [\partial_z(\partial_x E_x + \partial_y E_y + \partial_z E_z) - (\partial_x^2 + \partial_y^2 + \partial_z^2)E_z]dz \\
& \leftrightarrow \quad \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times (\nabla \times \vec{E}) \ ,
\end{aligned}
$$

which is a fairly reasonable result, and, at least after the fact, hopefully not unexpected. It is worth noting that while all of this is not clearly worth the effort in Cartesian coordinates and three dimensions, it turns out to be a much simpler calculational scheme for things like this either in some more complicated form of coordinates or in more than three dimensions.

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2. Let us consider an electric field in 3-dimensions, but in spherical coordinates. Recall that

$$x = r \sin\theta \cos\varphi \ , \quad y = r \sin\theta \sin\varphi \ , \quad z = r \cos\theta \ .$$

Recall that the metric in Cartesian coordinates is just the identity matrix, so that, if we want to present it as a $\binom{0}{2}$ tensor we could do so as

$$\mathbf{g} = dx^2 + dy^2 + dz^2 \ .$$

a. Re-determine the metric, from this point of view in terms of $\{dr, d\theta, d\varphi\}$, and use this to show that the following is a basis set for 1-forms, $\{\underset{\sim}{\omega}^r \equiv dr, \underset{\sim}{\omega}^\theta \equiv r\, d\theta, \underset{\sim}{\omega}^\varphi \equiv r \sin\theta\, d\varphi\} \equiv \{\underset{\sim}{\omega}^a\}_{a=1}^3$, and that each of these is of unit length, and is orthogonal to both of the other elements of the set. Now calculate the exterior derivatives of each of these $\underset{\sim}{\omega}^a$. Define so-called connection 1-forms $\underset{\sim}{\Gamma}^a{}_b$ such that

$$d\underset{\sim}{\omega}^a = \underset{\sim}{\omega}^b \wedge \underset{\sim}{\Gamma}^a{}_b \ ,$$

and determine these quantities. The fact that they are non-zero shows that this basis set is not simply the exterior derivatives of some other set of three coordinates, since, in that case, these connection 1-forms would simply vanish.

b. Determine the dual basis, for tangent vectors, relative to this orthonormal set of basis 1-forms, given above. Label them $\{\widetilde{e}_r, \widetilde{e}_\theta, \widetilde{e}_\varphi\}$, and calculate their (three) commutators, i.e., the quantities ${C^i}_{jk}$ determined via

$$[\widetilde{e}_j, \widetilde{e}_k] \equiv {C^i}_{jk}\widetilde{e}_i \ .$$

The fact that these commutators are non-zero is a verification of the fact that there are not some other variables such that these basis vectors are derivatives with respect to them. How are these 3×3 commutator quantities related to the connection 1-forms?

c. Determine the components of the metric matrix, g^{ij}, in the basis $\{dr, d\theta, d\varphi\} \equiv \{dr^i\}_{i=1}^3$ and, again, for g^{ab}, in the orthonormal spherical basis. Determine the transformation matrix, ${A^i}_a$, which is such that $dr^i = {A^i}_a \underset{\sim}{\omega}^a$, and then use this to show that it is true that

$$g^{ij} = {A^i}_a\, {A^j}_b\, g^{ab} \ .$$

d. Determine the components of the tensors η^{ijk} and η_{ijk}, relative to this spherical orthonormal basis set, and then use them to determine $*\underset{\sim}{E}$ and $*d\underset{\sim}{E}$, where the desired basis of 2-forms is $\{r^2 \sin\theta \, d\theta \wedge d\varphi, r \sin\theta \, d\varphi \wedge dr, r dr \wedge d\theta\}$.

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a. Beginning from the definitions of x, y, and z as functions of r, θ, and φ, we may differentiate them, square them, and sum them:

$$\begin{aligned}
dx &= \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi \ , \\
dy &= \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi \ , \\
dz &= \cos\theta\, dr - r\sin\theta\, d\theta \ ;
\end{aligned}$$

$$\begin{aligned}
\mathbf{g} &= dx^2 + dy^2 + dz^2 \\
&= [\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta]dr^2 + r^2[\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta]d\theta^2 \\
&\qquad + r^2\sin^2\theta[\cos^2\varphi + \sin^2\varphi]d\varphi^2 + 2r\sin\theta\cos\theta[\cos^2\varphi + \sin^2\varphi - 1]dr\, d\theta \\
&\qquad\quad + r\sin^2\theta\sin\varphi\cos\varphi(-1+1)dr\, d\varphi + r^2\sin\theta\cos\theta\sin\varphi\cos\varphi(-1+1)d\theta\, d\varphi \\
&= dr^2 + r^2\, d\theta^2 + r^2\sin^2\theta\, d\varphi^2 = (\underset{\sim}{\omega}^r)^2 + (\underset{\sim}{\omega}^\theta)^2 + (\underset{\sim}{\omega}^\varphi)^2 \ , \\
&\equiv g_{\hat{i}\hat{j}}\underset{\sim}{\omega}^{\hat{i}}\underset{\sim}{\omega}^{\hat{j}} \ ,
\end{aligned}$$

where we use the “hat”’s over the indices to remind ourselves that the quantities $g_{\hat{\imath}\hat{\jmath}}$ are simply the components of the identity matrix, which of course is the same as the statement that the particularly-defined $\{\underset{\sim}{\omega}^{\hat{\imath}}\}$ are in fact of unit length and orthogonal, i.e., what is usually called orthonormal. Next we calculate their exterior derivatives:

$$\begin{aligned}
d\underset{\sim}{\omega}^r = d(dr) &= 0 \equiv \underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}^r{}_a \ , \\
d\underset{\sim}{\omega}^\theta = d(r\,d\theta) &= dr \wedge d\theta = \frac{1}{r}\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\theta \equiv \underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}^\theta{}_a \ , \\
d\underset{\sim}{\omega}^\varphi = d(r\sin\theta\,d\varphi) &= \sin\theta\,dr\wedge d\varphi + r\cos\theta\,d\theta\wedge d\varphi \\
&= \frac{1}{r}\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\varphi + \frac{\cos\theta}{r\sin\theta}\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^\varphi \equiv \underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}^\varphi{}_a \ .
\end{aligned}$$

The given equations, to determine the connection 1-forms, are somewhat ambiguous, or, if you prefer, need more input. Nonetheless, let’s go look at them, and see what you should have been able to do:

$$\underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}^r{}_a = \underset{\sim}{0} \quad \Longrightarrow \quad \underset{\sim}{\Gamma}^r{}_a \propto \underset{\sim}{\omega}^a \ .$$

One could have simply said that the “solution” is that all three of these one-forms is zero; that is clearly the simplest solution. However, since the wedge product of any 1-form with itself is zero, it follows that the solution that says that there are unknown scalar (functions) a, b, and c, such that

$$\underset{\sim}{\Gamma}^r{}_r = a\underset{\sim}{\omega}^r \ , \quad \underset{\sim}{\Gamma}^r{}_\theta = b\underset{\sim}{\omega}^\theta \ , \quad \underset{\sim}{\Gamma}^r{}_\varphi = c\underset{\sim}{\omega}^\varphi$$

is a much more general solution, that gives us more freedom. As I said earlier we do not, yet, have enough data to make a more exact determination. Nonetheless, let us continue with the remaining two triplets of desired 1-forms:

$$\underset{\sim}{\Gamma}^\theta{}_a = \frac{1}{r}\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\theta \quad \Longrightarrow \quad \begin{cases} \underset{\sim}{\Gamma}^\theta{}_r = \dfrac{1}{r}\underset{\sim}{\omega}^\theta + f\underset{\sim}{\omega}^r \ , \\ \underset{\sim}{\Gamma}^\theta{}_\theta = g\underset{\sim}{\omega}^\theta \ , \\ \underset{\sim}{\Gamma}^\theta{}_\varphi = h\underset{\sim}{\omega}^\varphi \ , \end{cases}$$

where f, g, and h, are arbitrary scalar functions. Then we also have

$$\underset{\sim}{\Gamma}^\varphi{}_a = \frac{1}{r}\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\varphi + \frac{\cos\theta}{r\sin\theta}\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^\varphi \quad \Longrightarrow \quad \begin{cases} \underset{\sim}{\Gamma}^\varphi{}_r = \dfrac{1}{r}\underset{\sim}{\omega}^\varphi + j\underset{\sim}{\omega}^r \ , \\ \underset{\sim}{\Gamma}^\varphi{}_\theta = \dfrac{\cot\theta}{r}\underset{\sim}{\omega}^\theta + k\underset{\sim}{\omega}^\theta \ , \\ \underset{\sim}{\Gamma}^\varphi{}_\varphi = \ell\underset{\sim}{\omega}^\varphi \ , \end{cases}$$

where again j, k and ℓ are arbitrary scalar functions.

b. The dual basis is straightforward:

$$\widetilde{e}_r = \partial_r \ , \qquad \widetilde{e}_\theta = \tfrac{1}{r}\partial_\theta \ , \qquad \widetilde{e}_\varphi = \tfrac{1}{r\sin\theta}\partial_\varphi \ ,$$

since these are the coefficients necessary to insure the usual reciprocal relationships, namely

$$\underset{\sim}{\omega}^{\hat{i}}(\widetilde{e}_{\hat{j}}) = \delta^i_j \ .$$

We may then go ahead and calculate the required commutators:

$$[\widetilde{e}_r, \widetilde{e}_\theta] = \partial_r(\tfrac{1}{r}\partial_\theta) - \tfrac{1}{r}\partial_\theta(\partial_r) = -\tfrac{1}{r^2}\partial_\theta = -\tfrac{1}{r}\widetilde{e}_\theta \equiv C^i{}_{r\theta}\widetilde{e}_i \ ,$$

$$[\widetilde{e}_r, \widetilde{e}_\varphi] = \partial_r(\tfrac{1}{r\sin\theta}\partial_\varphi) - \tfrac{1}{r\sin\theta}\partial_\varphi(\partial_r) = -\tfrac{1}{r^2\sin\theta}\partial_\varphi = -\tfrac{1}{r}\widetilde{e}_\varphi = C^i{}_{r\varphi}\widetilde{e}_i \ ,$$

$$[\widetilde{e}_\theta, \widetilde{e}_\varphi] = \tfrac{1}{r}\partial_\theta(\tfrac{1}{r\sin\theta}\partial_\varphi) - \tfrac{1}{r\sin\theta}\partial_\varphi(\tfrac{1}{r}\partial_\theta) = -\frac{\cos\theta}{r^2\sin^2\theta}\partial_\varphi = -\frac{\cot\theta}{r}\widetilde{e}_\varphi = C^i{}_{\theta\varphi}\widetilde{e}_i \ .$$

There is no apparent ambiguity here, and we can easily go and pick off the commutator coefficients that we need, noting, along the way, that it is clear that since $[\widetilde{e}_i, \widetilde{e}_j] = -[\widetilde{e}_j, \widetilde{e}_i]$ then $C^i{}_{jk} = -C^i{}_{kj}$:

$$C^r{}_{r\theta} = 0 \ , \quad C^\theta{}_{r\theta} = -\frac{1}{r} \ , \quad C^\varphi{}_{r\theta} = 0 \ ,$$

$$C^r{}_{r\varphi} = 0 = C^\theta_{r\varphi} \ , \quad C^\varphi_{r\varphi} = -\frac{1}{r} \ ,$$

$$C^r{}_{\theta\varphi} = 0 = C^\theta_{\theta\varphi} \ , \quad C^\varphi{}_{\theta\varphi} = -\frac{\cot\theta}{r} \ .$$

It should be clear that there is some relation between the coefficients of the commutator 1-forms and these commutator coefficients, since they have the same sort of values. Just how they align is not yet determined, but might be possible to ascertain from the requirements of a reciprocal basis.

c. We can easily write down, from part (a) above, the matrix presentations of the metric in the two different bases, where we label the borders of the matrices with the basis set that are being used:

$$\mathbf{g}_{ij} = \begin{array}{c} \\ dr \\ d\theta \\ d\varphi \end{array} \!\!\begin{array}{c} \begin{array}{ccc} dr & d\theta & d\varphi \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \end{array} , \qquad \mathbf{g}_{\hat{a}\hat{b}} = \begin{array}{c} \\ \underset{\sim}{\omega}^r \\ \underset{\sim}{\omega}^\theta \\ \underset{\sim}{\omega}^\varphi \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \underset{\sim}{\omega}^r & \underset{\sim}{\omega}^\theta & \underset{\sim}{\omega}^\varphi \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} ,$$

where now I have gone ahead and put extra little "hats" on the indices when we are using orthonormal basis vectors.

To obtain the metric matrices presentations with upper indices we simply invert the matrices, and note that the defining basis sets are now the reciprocal basis sets, i.e., those for tangent vectors:

$$\mathbf{g}^{ij} = \begin{array}{c} \\ \partial_r \\ \partial_\theta \\ \partial_\varphi \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \partial_r & \partial_\theta & \partial_\varphi \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix} \end{array} , \qquad \mathbf{g}^{\hat a\hat b} = \begin{array}{c} \\ \underset{\sim}{e}_r \\ \underset{\sim}{e}_\theta \\ \underset{\sim}{e}_\varphi \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \underset{\sim}{e}_r & \underset{\sim}{e}_\theta & \underset{\sim}{e}_\varphi \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} ,$$

As the one matrix is just the identity and the other one is diagonal, then the obvious "guess" for the transformation matrix A is just

$$G^{-1} = A\,\hat{G}^{-1}A^T \quad \Longrightarrow \quad A \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/(r\sin\theta) \end{pmatrix} = A^i{}_{\hat a} \quad \text{and} \quad dr^i = A^i{}_{\hat a}\underset{\approx}{\omega}^a \ .$$

We notice here that the determinant of the matrix A is $1/(r^2\sin\theta)$, which will be needed below.

d. To determine η^{ijk} we first need to have the quantity labeled as m in the notes on the Hodge dual, which is the square root of the detrminant of the matrix G, which is the same as the inverse of the determinant of A, i.e., $m = r^2\sin\theta$. Then our equations tell us that

$$\eta^{ijk} = \frac{1}{r^2\sin\theta}\,\epsilon[ijk] \ , \qquad \eta_{ijk} = r^2\sin\theta\,\epsilon[ijk] \ ,$$

where $\epsilon[ijk]$ is just the Levi-Civita symbol which is skew-symmetric when any two adjacent indices are interchanged, and so is zero unless all indices are different, and then is exactly such that $\epsilon[123] = +1$. We will need this to determine the requested duals. Let us first perform the calculations in the coordinate basis for 1-forms, i.e., with the basis $\{dr^i\}_{i=1}^3 = \{dr, d\theta, d\varphi\}$. Then we may write out the original 1-form in terms of that basis in the obvious way:

$$\underset{\sim}{E} = E_r\,dr + E_\theta\,d\theta + E_\varphi\,d\varphi \ .$$

The dual is then given by

$$*\underset{\sim}{E} = -\frac{1}{2}E^i\eta_{ijk}\,dr^j\wedge dr^k = -\frac{1}{2}g^{im}E_m\eta_{ijk}dr^j\wedge dr^k\ ,$$

where we must use both the components of the (inverse) metric as well as those of the Levi-Civita tensor:

$$\begin{aligned}*\underset{\sim}{E} =\ & -\frac{1}{2}[E_r\eta_{rjk} + \tfrac{1}{r^2}E_\theta\eta_{\theta jk} + \tfrac{1}{r^2\sin^2\theta}E_\varphi\eta_{\varphi jk}]\,dr^j\wedge dr^k\\ &= -r^2\sin\theta[E_r\,d\theta\wedge d\varphi + \tfrac{1}{r^2}E_\theta\,d\varphi\wedge dr + \tfrac{1}{r^2\sin^2\theta}E_\varphi\,dr\wedge d\theta]\\ &\quad = -\left[r^2\sin\theta E_r\,d\theta\wedge d\varphi + \sin\theta\,E_\theta\,d\varphi\wedge dr + \tfrac{1}{\sin\theta}E_\varphi\,dr\wedge d\theta\right]\end{aligned}$$

.

We are also supposed to calculate $*d\underset{\sim}{E}$, so we first calculate the exterior derivative itself, using our current coordinates:

$$\begin{aligned}d\underset{\sim}{E} &= d(E_r)\wedge dr + d(E_\theta)\wedge d\theta + d(E_\varphi)\wedge d\varphi\\ &= (\partial_r E_\theta - \partial_\theta E_r)dr\wedge d\theta + (\partial_\theta E_\varphi - \partial_\varphi E_\theta)d\theta\wedge d\varphi + (\partial_\varphi E_r - \partial_r E_\varphi)d\varphi\wedge dr\ .\end{aligned}$$

An alternative approach to the one that we used with the previous dual, just above, is to calculate the Hodge duals of the basis set, using the definitions we have:

$$*(dr\wedge d\theta) = -\sin\theta\,d\varphi\ ,\quad *(d\theta\wedge d\varphi) = -\frac{1}{r^2\sin\theta}dr\ ,\quad *(d\varphi\wedge dr) = -dfrac1\sin\theta d\theta\ .$$

Just as a place to record these sorts of details, one could invert these, using the fact that the dual of the dual is just the identity transformation, noting that this set would have generated the expression above for $*\underset{\sim}{E}$:

$$*dr = -r^2\sin\theta\,d\theta\wedge d\varphi\ ,\quad *d\theta = -\sin\theta\,d\varphi\wedge dr\ ,\quad *d\varphi = -\frac{1}{\sin\theta}\,dr\wedge d\theta\ .$$

At any event we may now use these relationships to write the desired quantity:

$$*d\underset{\sim}{E} == -(\partial_r E_\theta - \partial_\theta E_r)\sin\theta\,d\varphi - \frac{1}{r^2\sin\theta}(\partial_\theta E_\varphi - \partial_\varphi E_\theta)\,dr - \frac{1}{\sin\theta}(\partial_\varphi E_r - \partial_r E_\varphi)\,d\theta\ .$$

NOTE TO THE GRADER: the overall minus signs above are due to Finley's particular choice of sign in his definition of the Hodge dual; there are other

definitions that do not include that sign, and they should also be allowed!

Alternatively we could have done all this in the orthonormal basis. Some parts are simpler there, and some are not. We begin with the following:

$$\underset{\approx}{E} = E_{\hat{r}}\underset{\approx}{\omega}^{\hat{r}} + E_{\hat{\theta}}\underset{\approx}{\omega}^{\hat{\theta}} + E_{\hat{\varphi}}\underset{\approx}{\omega}^{\hat{\varphi}} \ , \qquad \begin{cases} E_{\hat{r}} = E_r \ , \\ \\ E_{\hat{\theta}} = \frac{1}{r}E_\theta \ , \\ \\ E_{\hat{\varphi}} = \frac{1}{r\sin\theta}E_\varphi \ . \end{cases}$$

Since the metric for this orthonormal basis is just the 3 × 3 identity, its determinant is $+1$ and the η^{abc} tensor simply has just ± 1 and 0 as its values; therefore, the dual is quite straightforward to calculate:

$$*\underset{\approx}{E} = E_{\hat{r}}\,\underset{\approx}{\omega}^{\hat{\theta}} \wedge \underset{\approx}{\omega}^{\hat{\varphi}} + E_{\hat{\theta}}\,\underset{\approx}{\omega}^{\hat{\varphi}} \wedge \underset{\approx}{\omega}^{\hat{r}} + E_{\hat{\varphi}}\,\underset{\approx}{\omega}^{\hat{r}} \wedge \underset{\approx}{\omega}^{\hat{\theta}} \ .$$

However, when calculating the exterior derivative it is much to perform the calculation in the coordinate basis and then transform, since the exterior derivative of a coordinate basis form vanishes; therefore, we write

$$\begin{aligned} d\underset{\approx}{E} &= (\partial_r E_\theta - \partial_\theta E_r)dr \wedge d\theta + (\partial_\theta E_\varphi - \partial_\varphi E_\theta)d\theta \wedge d\varphi + (\partial_\varphi E_r - \partial_r E_\varphi)d\varphi \wedge dr \\ &= \frac{1}{r}(\partial_r E_\theta - \partial_\theta E_r)\,\underset{\approx}{\omega}^{\hat{r}} \wedge \underset{\approx}{\omega}^{\hat{\theta}} + \frac{1}{r^2\sin\theta}(\partial_\theta E_\varphi - \partial_\varphi E_\theta)\,\underset{\approx}{\omega}^{\hat{\theta}} \wedge \underset{\approx}{\omega}^{\hat{\varphi}} + \frac{1}{r\sin\theta}(\partial_\varphi E_r - \partial_r E_\varphi)\,\underset{\approx}{\omega}^{\hat{\varphi}} \wedge \underset{\approx}{\omega}^{\hat{r}} \end{aligned}$$

However, this should still be re-written in terms of the coordinates relative to the orthonormal basis, where we also take the Hodge dual at the same time, since in this basis that is very easy:

$$\begin{aligned} *d\underset{\approx}{E} =& \frac{1}{r}\left(\frac{\partial}{\partial r}(rE_{\hat{\theta}}) - \frac{\partial}{\partial\theta}E_{\hat{r}}\right)\underset{\approx}{\omega}^{\hat{\varphi}} + \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}(r\sin\theta E_{\hat{\varphi}}) - \frac{\partial}{\partial\varphi}(rE_{\hat{\theta}})\right)\underset{\approx}{\omega}^{\hat{r}} \\ &+ \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}E_{\hat{r}} - \frac{\partial}{\partial r}(r\sin\theta E_{\hat{\varphi}})\right)\underset{\approx}{\omega}^{\hat{\theta}} \ . \end{aligned}$$

One might hope, at least, that you recognize this formula as the more usual one for the curl of a vector field as determined in spherical coordinates.

I note that you might ought to have been asking whether E^θ or E_θ, or, for that matter, $E_{\hat{\theta}}$ was the one that you deal with in beginning physics. The answer is that it is the last one, the ones with the "hatted" indices; the reason for this is that in that approach to physics one deals always with basis vectors of unit length, and those are exactly those "hatted" basis vectors that we are referring to as orthonormal.

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3. Since any linear combination of the three 4×4 matrices $\mathfrak{J}_i$ and the three 4×4 matrices $\mathfrak{K}_j$ is an element of the Lie algebra for the Lorentz group, i.e., is such that its exponential is an element of the Lorentz group, please determine the explicit matrix form of

$$L \equiv e^{a(\mathfrak{J}_z+\mathfrak{K}_z)}\ .$$

Do you have any comments concerning its physical meaning?

. . . .

As we have done in class, we first write down the particular matrix, calling it Q, that we have been given, and begin to calculate its powers:

$$Q \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad\Longrightarrow\quad Q^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ ,$$

$$Q^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad\Longrightarrow\quad Q^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ .$$

We see that the fourth power of Q is just the identity, so that Q^5 is obviously the same as Q. We may then begin to sum the desired series for the exponential:

$$e^{aQ} = (1 + a^4/4! + a^8/8! + \ \ldots\ )\mathbf{I}_4 + (a + a^5/5! + a^9/9! + \ \ldots\ )Q$$
$$+ (a^2/2! + a^6/6! + a^{10}/10! + \ \ldots\ )Q^2 + (a^3/3! + a^7/7! + \ \ldots\ )Q^3\ .$$

We want to explicitly sum the series; therefore, at this point we notice that the 4×4 matrix is easily split up into a so-called *block-diagonal form*, it being two 2×2 matrices along the diagonal. Looking just at the upper 2×2 matrix in that diagonal, and referring to it as Q_1, we see that $Q_1^2 = -\mathbf{I}_2$ and $Q_1^3 = -Q_1$; therefore, for this upper 2×2 matrix our sums may be re-distributed:

$$e^{aQ_1} = (1 - a^2/2! + a^4/4! - a^6/6! + \ \ldots\ )\mathbf{I}_2 + (a/1! - a^3/3! + a^5/5! - a^7/7! + \ \ldots\ )Q_1$$
$$= \cos a\,\mathbf{I}_2 + \sin a\,Q_1 = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}\ .$$

Referring to the second element along the block diagonal as Q_2, we see that $Q_2^2 = +\mathbf{I}_2$ and $Q_2^3 = +Q_2$; therefore, for this upper 2×2 matrix our sums may be re-distributed:

$$e^{aQ_2} = (1 + a^2/2! + a^4/4! + a^6/6! + \ \ldots\ )\mathbf{I}_2 + (a/1! + a^3/3! + a^5/5! + a^7/7! + \ \ldots\ )Q_2$$
$$= \cosh a\,\mathbf{I}_2 + \sinh a\,Q_2 = \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix}\ .$$

The result is then that our final 4 × 4 matrix is

$$e^{aQ} = \begin{pmatrix} \cos a & -\sin a & 0 & 0 \\ \sin a & \cos a & 0 & 0 \\ 0 & 0 & \cosh a & -\sinh a \\ 0 & 0 & -\sinh a & \cosh a \end{pmatrix} .$$

Such a motion is referred to as a *screw motion*, which rotates about—in this case the $\hat{z}$-axis—at exactly the same rate as it boosts along that axis.

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4. If at a certain event an electromagnetic field satisfies the relations $\vec{E} \cdot \vec{B} = 0$ while the magnitude of neither of the vectors is zero, prove that there exists a reference frame in which either $\vec{E} = 0$ or $\vec{B} = 0$—although not both. Then show that there are infinitely many such reference frames, all in standard configuration with each other.

……

We first note that we are certainly not interested in the case where both fields are zero. If this particular invariant is zero, then either one or the other of the two fields must vanish, or they can simply be perpendicular. As the other invariant is $\vec{B}^2 - \vec{E}^2$, then the sign of this invariant would tell us which one can be made to vanish if we move to the correct frame.

There would appear to be two possible cases; let us first consider the case where $E < B$, so that the second invariant is positive. Since it will always be positive, we need to find a second frame where $\vec{E} = \vec{0}$. We know that the component of $\vec{E}$ that would be parallel to the relative velocity between the two frames, $\vec{\beta}$ will not change; therefore, it must already have been zero earlier. This is the same as saying that $\vec{\beta}$ must be perpendicular to $\vec{E}$. Therefore now let's look at the transformation equation for $\vec{E}_\perp$ which in this case is all there is to transform:

$$\vec{E}' = \vec{E}'_\perp = \gamma_\beta [\vec{E} + \vec{\beta} \times \vec{B}] ,$$

where so far all we know is that $\vec{\beta} \perp \vec{E}$ and that $\vec{B} \perp \vec{E}$. However, if the quantity above is to be zero, then the second term inside the bracket must be proportional to $\vec{E}$. A plausible "guess" is therefore that $\vec{B} \propto \vec{E} \times \vec{B}$. Inserting this we have

$$[\vec{E} \times \vec{B}] \times \vec{B} = (\vec{B} \cdot \vec{E})\vec{B} - B^2\vec{E} = -B^2\vec{E} .$$

This looks quite good and suggests that we choose

$$\vec{\beta} = \frac{\vec{E} \times \vec{B}}{B^2} \quad \Longrightarrow \quad \vec{E}' = \vec{0} \; .$$

We then note that since $\vec{E}$ and $\vec{B}$ are perpendicular the magnitude of $\vec{\beta}$ is just

$$|\vec{\beta}| = \frac{EB}{B^2} = \frac{E}{B} < 1 \; ,$$

so that it is acceptable as a velocity of some frame. We then note that in this frame we have only a perpendicular component of $\vec{B}$:

$$\vec{B}' = \vec{B}'_{\perp} = \gamma_{\beta}[\vec{B} - (\vec{E} \times \vec{B})\vec{E}/B^2] = \gamma_{\beta}[1 - (E^2/B^2)]\vec{B} \; .$$

We must therefore calculate γ:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(E/B)^2}} \; ,$$

which gives us the final value for the transformed electric and magnetic fields:

$$\vec{E}' = 0 \; , \qquad \vec{B}' = \sqrt{1-(E/B)^2}\,\vec{B} \; , \qquad\qquad \vec{\beta} = \frac{\vec{E} \times \vec{B}}{B^2} \; .$$

If, instead, we had considered the case where $B < E$, then $B/E < 1$, and we need to find a frame where $\vec{B}' = \vec{0}$. Not having done the calculation it should be clear that the result is

$$\vec{E}' = \sqrt{1-(B/E)^2}\vec{E} \; , \qquad \vec{B}' = \vec{0} \; , \qquad\qquad \vec{\beta} = \frac{\vec{E} \times \vec{B}}{E^2} \; .$$

As to his reference to the fact that there should be infinitely many such reference frames, they are actually already taken care of in the form that I have written, without having chosen any basis vectors. On the other hand, another way to look at them is the fact that we can always perform an arbitrary rotation about the axis $\hat{\beta}$ without changing any of the details above.