Physics 495

Homework No. 6 **Solutions:** due Wednesday, 14 October, 2009

1. A charge q is released from rest at the origin, in the presence of a uniform electric field $\vec{E} = E_0\hat{z}$ and a uniform magnetic field, $\vec{B} = B_0\hat{x}$, where we assume that $E_0 < B_0$. Determine the trajectory of the particle by first transforming to a reference frame in which $\vec{E} = 0$, finding the path in that frame, and then transforming back to the original frame.

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We begin by recalling, from the previous homework set, that when $\vec{E}\cdot\vec{B} = 0$ and $B^2 > E^2$ then there is indeed a reference frame, $\mathcal{O}'$, that sees $\vec{E}' = \vec{0}$. As measured by the original frame, for the fields given in this problem, that frame is traveling with velocity

$$\vec{\beta} = \frac{E_0}{B_0}\hat{y} \quad\Longrightarrow\quad \gamma = \frac{1}{\sqrt{1-\left(\frac{E_0}{B_0}\right)^2}} \equiv \frac{1}{\sqrt{1-\beta^2}}\ , \quad \beta \equiv \frac{E_0}{B_0} < 1\ .$$

We therefore move everything to that reference frame, where the fields are given by

$$\vec{E}' = \vec{0}\ , \qquad \vec{B}' = B_0\sqrt{1-\beta^2}\,\hat{x} = (B_0/\gamma)\hat{x}\ ,\ .$$

Since, in the original frame, $\mathcal{O}$, our charged particle was at rest until $t = 0$, in this frame it has an initial velocity $\vec{v}' = -\vec{\beta} = -\beta\hat{y}$; therefore it does feel a force from the magnetic field that exists in this frame, which will of course change that velocity. The initial force will be in the $\hat{x}\times\hat{y} = \hat{z}$ direction, so that the velocity should have components in both the $\hat{y}$- and $\hat{z}$-directions:

$$\begin{aligned} m\frac{d}{dt'}\gamma_v\vec{v} = \frac{d}{dt'}\vec{p}' = \vec{F}' = q\vec{v}'\times\vec{B}' = qB_0\gamma^{-1}[v'_y\hat{y}\times\hat{x} + v'_z\hat{z}\times\hat{x}] = qB'[-v'_y\hat{z} + v'_z\hat{y}] \\ \Longrightarrow\quad \frac{d}{dt'}\gamma_{v'}\,v'_y = \frac{qB'}{m}v'_z\ , \quad \frac{d}{dt'}\gamma_{v'}\,v'_z = -\frac{qB'}{m}v'_y\ , \end{aligned}$$

where of course the magnetic field in this frame, $B' = B_0\sqrt{1-\beta^2}$, is constant. However, as the only field generating a force is that magnetic field no work can be done (at least in this frame), so that the energy cannot change, telling us that $\gamma_{v'}$ is constant. This then gives us the following equation:

$$\frac{d}{dt'}(v'_y + iv'_z) = -i\frac{qB'}{\gamma_{v'}m}(v'_y + iv'_z) \quad \Longrightarrow \quad v'_y + iv'_z = -\beta\, e^{-i\omega t'} \ , \qquad \omega \equiv \frac{qB'}{\gamma_{v'}m} = \frac{qB_0}{\gamma_{v'}\gamma_\beta m} \ ,$$
$$\text{or} \ \ v'_y = -\beta\,\cos\omega t' \ , \quad v'_z = -\beta\,\sin\omega t' \ ,$$

where we have included the initial condition into the determination of the solution, so that at $t' = 0$ the velocity is just $-\vec{\beta}$. Checking up on the calculations we see that in fact $(\vec{v}')^2 = \beta^2$ is indeed constant, and in fact we have $\gamma_{v'} = \gamma_\beta$. Next, of course, we can move this vector back to the original frame, $\mathcal{O}$, which is somewhat more convenient if we have the 4-momentum instead of just the velocity:

$$\vec{p}' = \gamma_{v'} m\vec{v}' = -\frac{m\beta}{\sqrt{1-\beta^2}}\ [\cos\omega t'\,\hat{y} + \sin\omega t'\,\hat{z}] \ ,$$
$$E' = \gamma_{v'}m = \gamma_\beta\, m \ .$$

We know that the perpendicular component of the momentum doesn't change, while the parallel component is mixed with the energy:

$$p_z = p'_z = -\frac{m\beta}{\sqrt{1-\beta^2}}\sin\omega t' = -m\gamma_\beta\beta\sin\omega t' \ ,$$
$$p_y = \gamma_\beta[p'_y + \beta E'] = \gamma_\beta[-m\beta\gamma_\beta\cos\omega t' + \beta\gamma_\beta m] = m\gamma_\beta^2\beta[1-\cos\omega t'] \ ,$$
$$\Longrightarrow \quad \vec{p} = m\gamma_\beta\beta\,[\gamma_\beta(1-\cos\omega t')\hat{y} - \sin\omega t'\hat{z}] \ ,$$
$$E = \gamma_\beta[E' + \beta p'_y] = \gamma_\beta[\gamma_\beta m - m\beta^2\gamma_\beta\cos\omega t'] = m\gamma_\beta^2\beta[1-\beta^2\cos\omega t'] \ .$$

The velocity is of course the ratio of these two:

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{\vec{p}}{E} = \frac{1-\cos\omega t'}{1-\beta^2\cos\omega t'}\hat{y} - \frac{\sin\omega t'}{\gamma_\beta(1-\beta^2\cos\omega t')}\hat{z} \ .$$

These equations appear very difficult to integrate, especially since they involve dt on one side and t' on the other. Let us go back and integrate in the frame without the electric field:

$$\frac{dy'}{dt'} = v'_y = -\beta\cos\omega t' \ , \quad \frac{dz'}{dt'} = v'_z = -\beta\sin\omega t' \ ,$$
$$\Longrightarrow \quad y' = y'(t') = -\frac{\beta}{\omega}\sin\omega t' \ , \quad z' = z'(t') = -\frac{\beta}{\omega}(1-\cos\omega t') \ ,$$

where I have chosen constants of integration so that the charged particle begins at the origin, and we see that the particle is executing circular motion in the y', z'-plane, with frequency $\omega = (q/m)B_0(1-\beta^2)$.

We may then move these solutions, via the Lorentz transformation back to the original frame:

$$
\begin{aligned}
t' &= \gamma(t - \beta y) \ , \\
z = z' &= -\frac{\beta}{\omega}(1 - \cos\omega t') = -\frac{\beta}{\omega}\{1 - \cos[\gamma\omega(t - \beta y)]\} \ , \\
y = \gamma[y' + \beta t'] &= \gamma\frac{\beta}{\omega}[\omega t' - \sin\omega t'] = \gamma\frac{\beta}{\omega}\{\gamma\omega(t - \beta y) - \sin[\omega\gamma(t - \beta y)]\} \ .
\end{aligned}
$$

I can at least take the terms linear in y in the last expression and combine them:

$$
y = \beta t - \frac{\beta}{\omega\gamma}\sin[\gamma\omega(t - \beta y)] \ , \qquad z = -\frac{\beta}{\omega}\{1 - \cos[\gamma\omega(t - \beta y)]\} \ .
$$

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2. Consider the following 4×4 matrix:

$$L = \begin{pmatrix} -1 & 0 & a & -a \\ 0 & 1 & 0 & 0 \\ -a & 0 & -1 + a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1 + a^2/2 \end{pmatrix} .$$

a. Please show that L is in fact a special, orthochronous Lorentz transformation, for all real values of the constant parameter a. However, explain why we are sure that it is neither a pure rotation nor a pure Lorentz boost.

b. Even though L is in the connected part of the Lorentz group that contains the identity, it is nevertheless not sufficiently near the identity that it can be written as a single exponential. However, it can be written as the product of a rotation $R(\theta; \hat{y})$ multiplied on the right by a pure boost. What is the direction of the velocity associated with this boost?

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We begin by showing that L is a Lorentz transformation, i.e., that $L^T H L = H$, or, with indices, that $L^\mu{}_\alpha L^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$, which is a calculation that I perform inside Maple and present the result here:

$$\begin{pmatrix} -1 & 0 & -a & a \\ 0 & 1 & 0 & 0 \\ a & 0 & -1 + a^2/2 & -a^2/2 \\ -a & 0 & -a^2/2 & 1 + a^2/2 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & a & -a \\ 0 & 1 & 0 & 0 \\ -a & 0 & -1 + a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1 + a^2/2 \end{pmatrix}$$
$$= \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Next one calculates its determinant to be $+1$, and, lastly $L^4{}_4 > 1$ so that it is orthochronous, i.e., preserves the direction of time.

Lastly, we may say that it is

i.) not a rotation since a rotation should not mix the 4th components of vectors with the other 3, i.e., it should have zero entries in the 4 row and column, except for the 4,4-entry which should be 1, which is not true for this matrix, and

ii.) not a boost since all boosts are symmetrical matrices, i.e., equal to their own transposes, which again this matrix is not.

b. Next we should be able to determine the desired boost by writing $B = R(-\theta;\hat{y})L$:

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & a & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1+a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix}$$
$$= \begin{pmatrix} -\cos\theta + a\sin\theta & 0 & a\cos\theta + (1-a^2/2)\sin\theta & -a\cos\theta + (a^2/2)\sin\theta \\ 0 & 1 & 0 & 0 \\ -\sin\theta - a\cos\theta & 0 & a\sin\theta - (1-a^2/2)\cos\theta & -a\sin\theta - (a^2/2)\cos\theta \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix} .$$

Knowing that the 4th row and 4th column of a boost should be equal we quickly write down two equations to determine the sine and cosine of θ:

$$\begin{aligned} -a\,\cos\theta + \frac{a^2}{2}\,\sin\theta &= a \\ -a\,\sin\theta - \frac{a^2}{2}\,\cos\theta &= -\,\frac{a^2}{2} \\ \Longrightarrow \qquad \cos\theta = \frac{a^2-4}{a^2+4}\,&,\ \sin\theta = \frac{4a}{a^2+4}\ . \end{aligned}$$

It is straightforward to show that, indeed, these are proper values for the cosine and sine, i.e., that $\cos^2\theta + \sin^2\theta = 1$. Note that for $a = 0$ the angle θ must equal π, while as a increases toward infinity, that angle decreases toward 0.

We must now insert these values into the remainder of the matrix, to acquire our desired boost. This gives

$$B(\vec{v}) = \begin{pmatrix} 1+\frac{2a^2}{a^2+4} & 0 & -\frac{a^3}{a^2+4} & a \\ 0 & 1 & 0 & 0 \\ -\frac{a^3}{a^2+4} & 0 & 1+\frac{a^4/2}{a^2+4} & -a^2/2 \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix} .$$

Knowing that the 4th row and/or column is just the components of $\widetilde{u}$, we may immediately say that

$$\gamma v^x = a\ , \quad \gamma v^y = 0\ , \quad \gamma v^z = -a^2/2\ , \quad \gamma = 1 + a^2/2\ ,$$
$$\Longrightarrow \qquad \vec{v} = \frac{1}{a^2+2}\left\{2a\,\hat{x} - a^2\,\hat{z}\right\}\ .$$

It is straightforward to insert these values into the standard form for boosts and to observe that the entire matrix is consistent with this choice for $\vec{v}$.

The conclusion is that

$$L = R(\theta;\hat{y})B(\vec{v})\ ; \qquad \cos\theta = \frac{a^2-4}{a^2+4}\ , \quad \vec{v} = \frac{2a\,\hat{x} - a^2\,\hat{z}}{a^2+2}\ .$$

Again we see that when $a = 0$ this is just the identity, i.e., the associated velocity is zero, while as a increases the velocity increases, making an angle in the x, z-plane with tangent of $-a/2$, and therefore beginning slowly in the $\hat{x}$-direction and rotating, as a increases, to point more and more in the negative $\hat{z}$-direction.

NOTE to GRADER: as usual, all this interpretation for varying values of a is NOT required.

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3. Please calculate the matrix presentation, in terms of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, for the energy-momentum tensor or the electromagnetic field:

$$4\pi M^{\mu\nu} \equiv \mathcal{F}^{\mu}{}_{\lambda}\mathcal{F}^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} \; .$$

Try to present it in the following form:

$$\begin{pmatrix} 3\times 3 \text{ dyadic matrix} & \text{3-vector} \\ \text{3-vector}^T & \text{3-scalar} \end{pmatrix} ,$$

and then a physical interpretation/name for these different portions.

(Note: a *dyadic matrix* is one created from the matrix product of a column vector multiplied by a row vector on the right, i.e., of the form $\vec{A}\vec{B}^T$.)

. . . .

We first agree that we will use regular Cartesian (or Minkowski) coordinates, i.e., $\{x, y, z, t\}$. Then we begin with what should be the simplest one of these to calculate, namely the $4,4$-component:

$$4\pi M^{44} = \mathcal{F}^{4}{}_{\lambda}\mathcal{F}^{4\lambda} - (-1)\tfrac{1}{4}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} \; .$$

This reminds us that we need to first have some reasonably-formulated definitions of the spatial-temporal separation of the components of the Faraday tensor:

$$\mathcal{F}^{\alpha\alpha} = 0 \; , \quad \mathcal{F}^{i4} = -E^i = -\mathcal{F}^{4i} \; , \qquad \mathcal{F}^{ij} = \eta^{ijk}B_k \; ,$$

where in the second statement above—the one involving the magnetic field—the tensor η^{ijk} is really just the Levi-Civita $\epsilon[ijk]$ since we are in Cartesian coordinates. [However, do note that it would be more complicated if we were in fact in, say, spherical coordinates.] The next thing that we are reminded to do is to note that, every time, we will need that invariant; therefore, let's calculate it first:

$$\begin{aligned}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} = \mathcal{F}^{i\beta}\mathcal{F}_{i\beta} + \mathcal{F}^{4\beta}\mathcal{F}_{4\beta} &= \mathcal{F}^{ij}\mathcal{F}_{ij} + \mathcal{F}^{i4}\mathcal{F}_{i4} + \mathcal{F}^{4i}\mathcal{F}_{4i} \\ &= \eta^{ijk}B_k\eta_{ijm}B^m + (-E^i)(+E_i) + (+E^i)(-E_i) \\ &= 2\delta^k_m B_k B^m - 2E_iE^i = 2[B_mB^m - E_iE^i] = 2[\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E}] \ ,\end{aligned}$$

which we remember as one of the Lorentz invariants of the electromagnetic field. With all that in hand we may now return to the calculation of M^{44}:

$$4\pi M^{44} = E_mE^m + \tfrac{1}{4}[2(-\vec{E}^2 + \vec{B}^2] = \tfrac{1}{2}[\vec{E}^2 + \vec{B}^2] \equiv e \ ,$$

where we have used the symbol e to denote this quantity since it is the energy density, i.e., energy per unit volume of the field. I might also **note that** the overall sign of this tensor has been chosen so that this quantity, the energy density, should be positive!

Next we may proceed to the 3-vector portion:

$$4\pi M^{i4} = \mathcal{F}^i{}_j\mathcal{F}^{4j} = \eta^{imk}B_k(+E)_m = (\vec{E}\times\vec{B})^i \ ,$$

which is the usual form for the Poynting vector, i.e., the flux of intensity. Lastly we need the 3×3 matrix portion:

$$4\pi M^{ij} = \mathcal{F}^i{}_\lambda\mathcal{F}^{j\lambda} - \tfrac{1}{2}g^{ij}(\vec{B}^2 - \vec{E}^2) \ .$$

Next we simply concentrate on the complicated portion, i.e., the first term:

$$\begin{aligned}\mathcal{F}^i{}_\lambda\mathcal{F}^{j\lambda} &= \mathcal{F}^{im}g_{mn}\mathcal{F}^{jn} + \mathcal{F}^i{}_4\mathcal{F}^{j4} = \eta^{imk}B_kg_{mn}\eta^{jnb}B_b + (-E^i)E^j = \eta^{imk}B_k\eta_{rmb}B^bg^{rj} - E^iE^j \\ &= (\delta^k_b\delta^i_r - \delta^k_r\delta^i_b)B_kB^bg^{rj} - E^iE^j = B_bB^bg^{ij} - B_rB^ig^{rj} - E^iE^j = -E^iE^j - B^iB^j + g^{ij}\vec{B}^2 \ .\end{aligned}$$

Therefore we may now add back on the second term of the calculation, which we have been ignoring more recently, and we have

$$\begin{aligned}4\pi M^{ij} = &-E^iE^j - B^iB^j + g^{ij}\vec{B}^2 - \tfrac{1}{2}g^{ij}(\vec{B}^2 - \vec{E}^2) \\ &= -E^iE^j - B^iB^j + \tfrac{1}{2}g^{ij}[\vec{B}^2 + \vec{E}^2] = \{-\vec{E}\vec{E}^T - \vec{B}\vec{B}^T + \tfrac{1}{2}\mathbf{I}_3[\vec{B}^2 + \vec{E}^2]\}^{ij} \ ,\end{aligned}$$

where of course we have used the fact that g^{ij} are the components of the identity matrix in 3 dimensions.

To put this into an explicit matrix form we first define the energy density $e \equiv \frac{1}{2}(E^2 + B^2)$ and write out

$$((M^{\mu\nu})) \equiv \mathbf{M} \Longrightarrow \begin{pmatrix} e\mathbf{I}_3 - \vec{E}\vec{E}^T - \vec{B}\vec{B}^T & (\vec{E} \times \vec{B}) \\ (\vec{E} \times \vec{B})^T & e \end{pmatrix} ,$$

although it is not required that you really put it completely into this form of a presentation.

Do note, although not required for the problem, that the trace of this matrix is

$$4\pi M^{\mu\nu}\eta_{\mu\nu} = 4\pi[M^{ij}g_{ij} + M^{44}\eta_{44}] = 4\pi[-E^2 - B^2 + \tfrac{3}{2}(E^2 + B^2) - \tfrac{1}{2}(E^2 + B^2)] = 0 \ .$$
