Physics 495

Homework No. 7 **Solutions:** due Monday, 9 November, 2009

1. Please determine the 2×2 matrix representations $D(1/2,0)$ and $D(0,1/2)$ for the 4-dimensional version of the six generators, $\mathfrak{J}^{\alpha\beta}$, preferably in terms of the three 2×2 Pauli matrices, σ_x, σ_y, and σ_z.

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We begin by recalling that the relation between the 4-dimensional version of the generators and the 3-dimensional version is given by the following presentation, where we are using the (upper) indices on $\mathfrak{J}^{\alpha\beta}$ as matrix indices for this presentation:

$$\mathfrak{J}^{\alpha\beta} \Longrightarrow \begin{pmatrix} 0 & \mathfrak{J}_3 & -\mathfrak{J}_2 & -\mathfrak{K}^1 \\ -\mathfrak{J}_3 & 0 & \mathfrak{J}_1 & -\mathfrak{K}^2 \\ \mathfrak{J}_2 & -\mathfrak{J}_1 & 0 & -\mathfrak{K}^3 \\ +\mathfrak{K}^1 & +\mathfrak{K}^2 & +\mathfrak{K}^3 & 0 \end{pmatrix} .$$

Next we recall that the representations are labeled by $D(f,g)$, where f and g are the half-integers associated with representations of $\mathfrak{F}^i$ and $\mathfrak{G}^j$, while these are related to the standard ones by

$$\mathfrak{J}^i = \mathfrak{F}^i + \mathfrak{G}^i , \qquad \mathfrak{K}^i = i(\mathfrak{G}^i - \mathfrak{F}^i) .$$

Lastly we need the notion that the $D(1/2)$ representation for the rotation group corresponds to a representation of its generators by $\mathfrak{T}^i \Longrightarrow -\frac{i}{2}\sigma^i$, where of course $\mathfrak{T}^i$ is just a name for the generators of whichever version of the rotation group we want. Therefore, we will have the following:

$$D^{(0,1/2)} : \begin{cases} \mathfrak{J}^i \Longrightarrow & D^{(0)}(\mathfrak{F}^i) \oplus D^{(1/2)}(\mathfrak{G}^i) = -\frac{i}{2}\sigma^i , \\ \mathfrak{K}^i \Longrightarrow & i[D^{(1/2)}(\mathfrak{G}^i) \oplus (-1)D^{(0)}(\mathfrak{F}^i)] = \frac{1}{2}\sigma^i . \end{cases}$$

$$D^{(1/2,0)} : \begin{cases} \mathfrak{J}^i \Longrightarrow & D^{(1/2)}(\mathfrak{F}^i) \oplus D^{(0)}(\mathfrak{G}^i) = -\frac{i}{2}\sigma^i , \\ \mathfrak{K}^i \Longrightarrow & i[D^{(0)}(\mathfrak{G}^i) \oplus (-1)D^{(1/2)}(\mathfrak{F}^i)] = -\frac{1}{2}\sigma^i . \end{cases}$$

Having those we may simply insert their values into the overall matrix presentation for the 6 independent generators, in that 4-dimensional format. We begin with the $D(0,1/2)$ representation:

$$D^{(0,1/2)}(\mathfrak{J}^{\alpha\beta}) \Longrightarrow -\frac{i}{2}\begin{pmatrix} 0 & \sigma^3 & -\sigma^2 & -i\sigma^1 \\ -\sigma^3 & 0 & \sigma^1 & -i\sigma^2 \\ \sigma^2 & -\sigma^1 & 0 & -i\sigma^3 \\ +i\sigma^1 & +i\sigma^2 & +i\sigma^3 & 0 \end{pmatrix} ,$$

and then also present the other one, i.e., the $D(1/2,0)$ representation:

$$D^{(1/2,0)}(\mathfrak{J}^{\alpha\beta}) \Longrightarrow \quad -\frac{i}{2}\begin{pmatrix} 0 & \sigma^3 & -\sigma^2 & +i\sigma^1 \\ -\sigma^3 & 0 & \sigma^1 & +i\sigma^2 \\ \sigma^2 & -\sigma^1 & 0 & +i\sigma^3 \\ -i\sigma^1 & -i\sigma^2 & -i\sigma^3 & 0 \end{pmatrix} ,$$

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2. Please determine the invariants of the Lie algebra for the Lorentz group, beginning directly with the 4-dimensional version of the six generators, $\mathfrak{J}^{\alpha\beta}$. Note that the analogy with the electromagnetic Faraday tensor should be helpful. Then apply that knowledge to determine their numerical values, as multiples of the identity matrix, for the two 2×2 representations discussed above in problem (1).

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Since the matrix presentation of the components of $\mathfrak{J}^{\alpha\beta}$ looks exactly the same as the matrix presentation of the components of the Faraday 2-form, $\mathcal{F}^{\alpha\beta}$, then the invariants must surely be the same, namely:

$$\mathfrak{J}^{\alpha\beta}\mathfrak{J}_{\alpha\beta} = 2(\vec{\mathfrak{J}}^{\,2} - \vec{\mathfrak{K}}^{\,2}) \, , \quad \mathfrak{J}^{\alpha\beta} * \mathfrak{J}_{\alpha\beta} = 4\vec{\mathfrak{J}} \cdot \vec{\mathfrak{K}} \, ,$$

where the $*$ indicates the Hodge dual—recall that it is obtained by the changes in symbols where $\vec{B} \to \vec{E}$ and $\vec{E} \to -\vec{B}$, and since there are possibilities of various overall minus signs I have simply ignored them, since the negatives of the invariants are also invariants.

When I now apply the representations above to these invariants I obtain the following 2×2 matrices, which I recall that the matrix square of each of the three Pauli sigma matrices is just the 2×2 identity matrix:

$$2(\vec{\mathfrak{J}}^{\,2} - \vec{\mathfrak{K}}^{\,2}) : \quad \begin{cases} \underset{D(0,1/2)}{\Longrightarrow} \quad -\vec{\sigma}^{\,2} = -3\mathbf{I}_2 \, , \\ \underset{D(1/2,0)}{\Longrightarrow} \quad -\vec{\sigma}^{\,2} = -3\mathbf{I}_2 \, , \end{cases}$$

$$4\vec{\mathfrak{J}} \cdot \vec{\mathfrak{K}} : \quad \begin{cases} \underset{D(0,1/2)}{\Longrightarrow} \quad -i\vec{\sigma}^{\,2} = -3i\mathbf{I}_2 \, , \\ \underset{D(1/2,0)}{\Longrightarrow} \quad +i\vec{\sigma}^{\,2} = 3i\mathbf{I}_2 \, , \end{cases}$$

Do NOTE that an entirely different approach to this question is to recall that $\vec{\mathfrak{F}}^2$ and $\vec{\mathfrak{G}}^2$ are invariants for the Lorentz group, since they correspond to different ways of looking at the subalgebras isomorphic to rotations. Therefore, we may calculate

$$\vec{\mathfrak{F}}^2 = \tfrac{1}{4}[\vec{\mathfrak{J}} + i\vec{\mathfrak{K}}]^2 = \tfrac{1}{4}[\vec{\mathfrak{J}}^2 - \vec{\mathfrak{K}}^2 + 2i\vec{\mathfrak{J}}\cdot\vec{\mathfrak{K}}]\ ,$$
$$\vec{\mathfrak{G}}^2 = \tfrac{1}{4}[\vec{\mathfrak{J}} - i\vec{\mathfrak{K}}]^2 = \tfrac{1}{4}[\vec{\mathfrak{J}}^2 - \vec{\mathfrak{K}}^2 - 2i\vec{\mathfrak{J}}\cdot\vec{\mathfrak{K}}]\ ,$$

which then show us two different linear combinations of the same two invariants. The particular values for these invariants have of course more obvious values, since they originated from the squares of $\vec{\mathfrak{F}}$ and $\vec{\mathfrak{G}}$. We calculate them below:

$$D(0,1/2):\quad \begin{cases} \vec{\mathfrak{F}}^2 \Longrightarrow \tfrac{1}{4}[-\tfrac{3}{2} - 2i\tfrac{3i}{4}]\mathbf{I}_2 = \mathbf{0}_2\ , \\ \vec{\mathfrak{G}}^2 \Longrightarrow \tfrac{1}{4}[-\tfrac{3}{2} + 2i\tfrac{3i}{4}]\mathbf{I}_2 = -\tfrac{3}{4}\mathbf{I}_2\ , \end{cases}$$
$$D(1/2,0):\quad \begin{cases} \vec{\mathfrak{F}}^2 \Longrightarrow \tfrac{1}{4}[-\tfrac{3}{2} - 2i\tfrac{-3i}{4}]\mathbf{I}_2 = -\tfrac{3}{4}\mathbf{I}_2\ , \\ \vec{\mathfrak{G}}^2 \Longrightarrow \tfrac{1}{4}[-\tfrac{3}{2} + 2i\tfrac{-3i}{4}]\mathbf{I}_2 = \mathbf{0}_2\ , \end{cases}$$

where we note that for a value of $j = 1/2$, we would have $\vec{\mathfrak{J}}^2 \Longrightarrow (-i)^2(1/2)(1+1/2) = -\tfrac{3}{4}$.

Note to Grader: certainly not necessary to give both of these approaches, and it is conceivable even a third, correct one exists.
