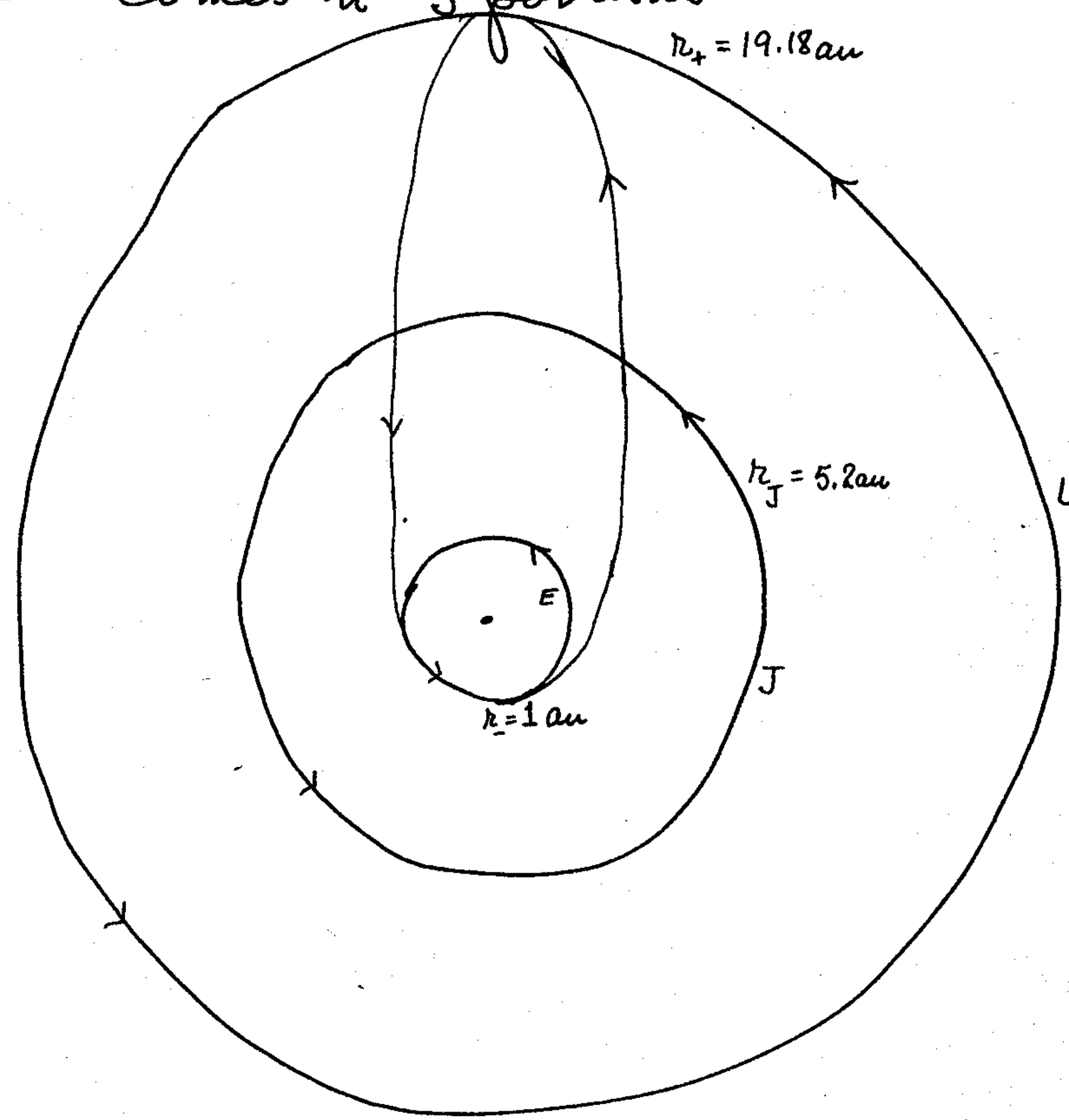
Trip to Uranus via Jupiter

— comes in 3 portions

a)



originally the spacecraft is put into an orbit as shown:

$$a = \frac{r_+ + r_-}{2} = 10.1\,au \qquad e = -\frac{k}{2a} = -\frac{4\pi^2}{2a} = -1.954\,(au/yr)^2$$

$$\epsilon = \frac{r_+ - r_-}{2a} = .901 \qquad l = \sqrt{k\frac{r_+ r_-}{a}} = 8.66\,au^2/yr$$

$$\tau = a^{3/2} = 32.1\,yr.$$

$v_- = \ell / r_- = 8.66\,\text{au/yr}$

Note: $1\,\text{au/yr} \doteq 4.75\,\text{km/sec}$.

but $v_{Earth} = 2\pi\,\text{au/yr}$

So $v_{rel} = 2.38\,\text{au/yr}$

As it passes $r = r_J = 5.2\,\text{au}$,

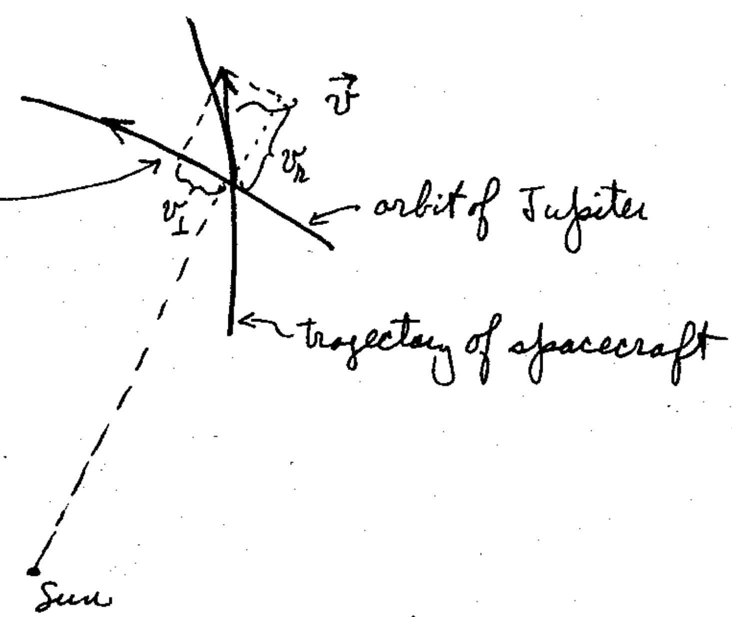
$v_\perp = \ell / r_J = 1.67\,\text{au/yr}$ and $e = \frac{1}{2}\vec{v}^2 - \frac{k}{r_J}$

$\Rightarrow |\vec{v}| = 3.36\,\text{au/yr}$

$\Rightarrow v_r = \sqrt{\vec{v}^2 - v_\perp^2} = 2.91\,\text{au/yr}$

Closeup picture

$v_J = \sqrt{\frac{k}{r_J}} = 2.76\,\text{au/yr}$

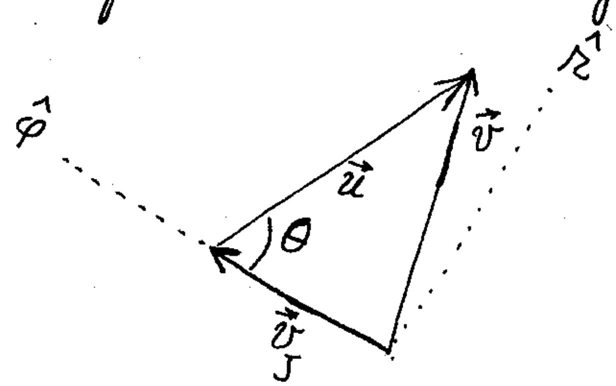


All of this happens, since $r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\varphi} = \frac{1.901}{1+.901\cos\varphi}$,

at $\varphi_J = \cos^{-1}\left\{ \frac{a(1-\epsilon^2)}{r_J} - 1 \right\} = 134.84°$

Assuming Jupiter is nearby it is reasonable to re-look at things from coordinates based on Jupiter. Let $\vec{u} \equiv \vec{v} - \vec{v}_J$ be velocity of spacecraft as measured on Jupiter

Closeup picture again

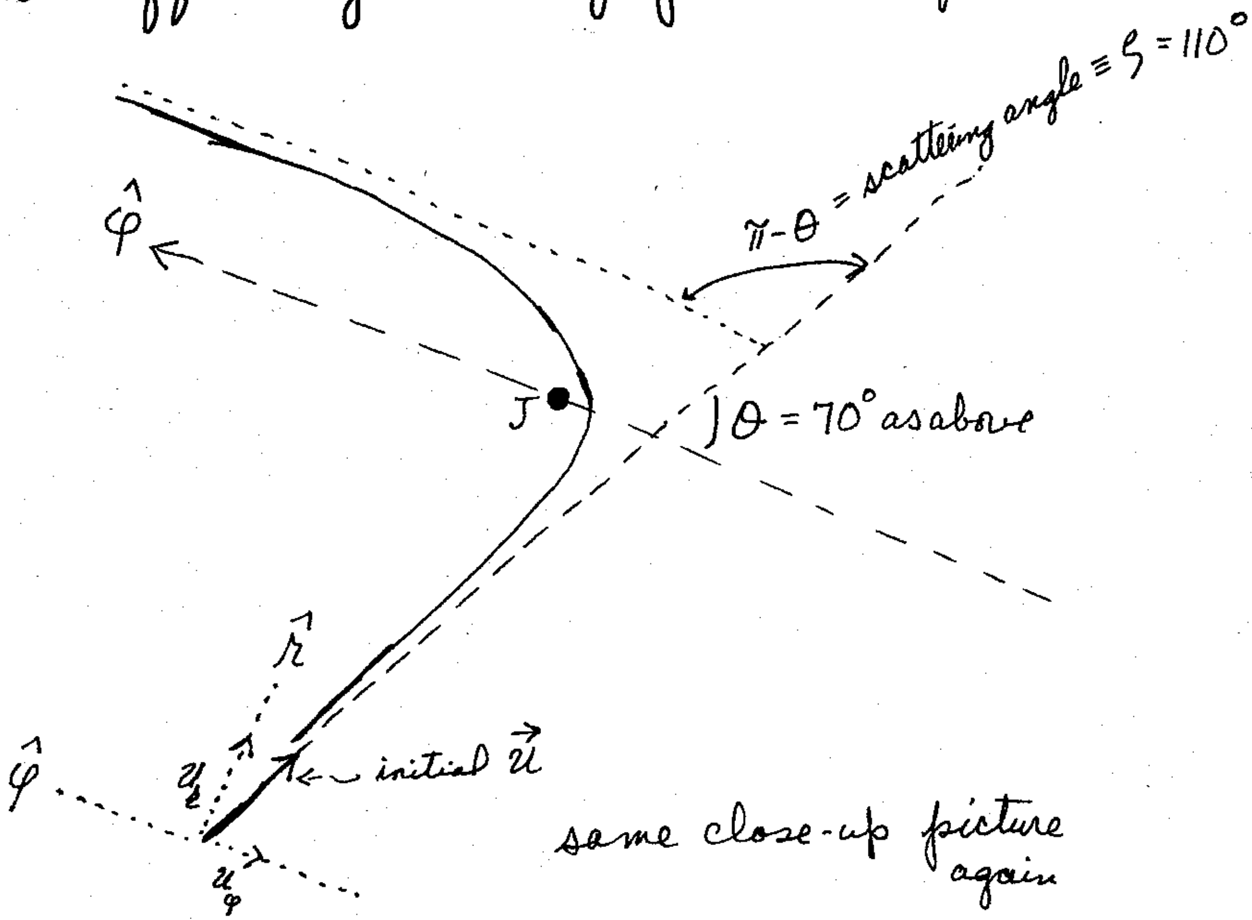


$$u_r = v_r = 2.91\,\text{au/yr}$$

$$u_\perp = v_\perp - v_J = -1.09\,\text{au/yr}$$

$$\Rightarrow u = \sqrt{u_r^2 + u_\perp^2} = 3.11\,\text{au/yr} \qquad \tan\theta = \frac{|u_r|}{|u_\perp|} \Rightarrow \theta = 70^\circ$$

When sufficiently close to Jupiter we ignore the sun



same close-up picture again

We come now to phase (2)!
What is the orbit near Jupiter and how does it affect the spacecraft

— to produce trajectory on next page

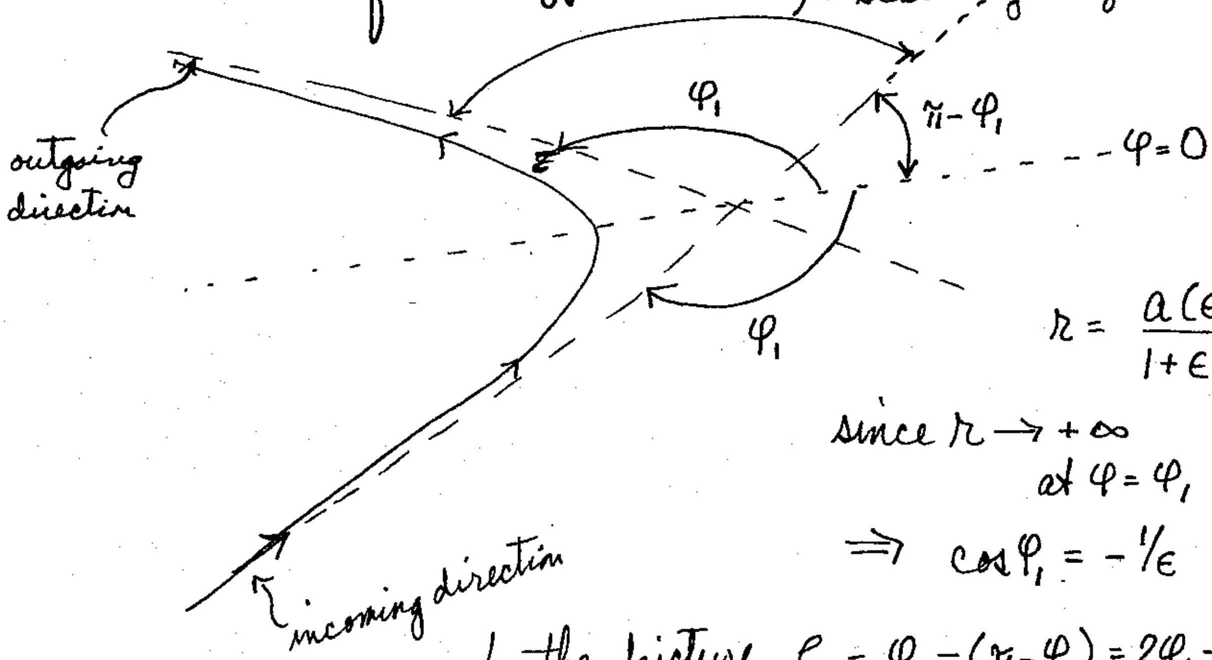
Relatively far from Jupiter

$$e = \frac{1}{2}u^2 - \cancel{\frac{k_J}{r}} \quad \approx 0 \text{ because far}$$

$$= 4.84\,(\text{au}/\text{yr})^2 > 0$$

$\Rightarrow$ hyperbolic orbit

In general, for a hyperbolic orbit



$$r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon\cos\varphi}$$

since $r \to +\infty$ at $\varphi = \varphi_1$

$$\Rightarrow \cos\varphi_1 = -1/\epsilon$$

by the picture $\varsigma = \varphi_1 - (\pi - \varphi_1) = 2\varphi_1 - \pi$

or $\varphi_1 = \frac{\varsigma + \pi}{2}$

$$\Rightarrow -\frac{1}{\epsilon} = \cos\varphi_1 = \cos\frac{\varsigma + \pi}{2} = -\sin\varsigma/2$$

General relation for hyperbolic orbit

But, previous picture shows that $\zeta = 110°$ for our problem because we choose the initial timing so that Jupiter will be at the correct place to swing the spacecraft into the direction of $\hat{\varphi}$ — the direction of Jupiter's own motion — so as to get maximum increase of speed. What is the right place? Well

$$\zeta = 110° \Rightarrow \epsilon = \frac{1}{\sin\frac{110°}{2}} = 1.22$$

and $a = \frac{k_J}{2e}$.

To calculate k_J, use $\frac{k_J}{R_J} = \frac{1}{2} v_{ESC}^2 \overset{\text{look it up}}{=} \frac{1}{2}(12.84\,au/yr)^2$

$$\Rightarrow a = 8.52 R_J$$

and $r_- = a(\epsilon - 1) = 8.52 R_J (1.22 - 1) = 1.85 R_J > R_J$.

So we have given a complete description of this portion and the final $\vec{u}$ has the same magnitude as the initial one, since, faraway, $e = \frac{1}{2}u^2 - \frac{k_J}{r} \overset{\sim 0}{}$, but now in the $+\hat{\varphi}$-direction.

The third portion cometh!
We go back to a sun-based reference frame

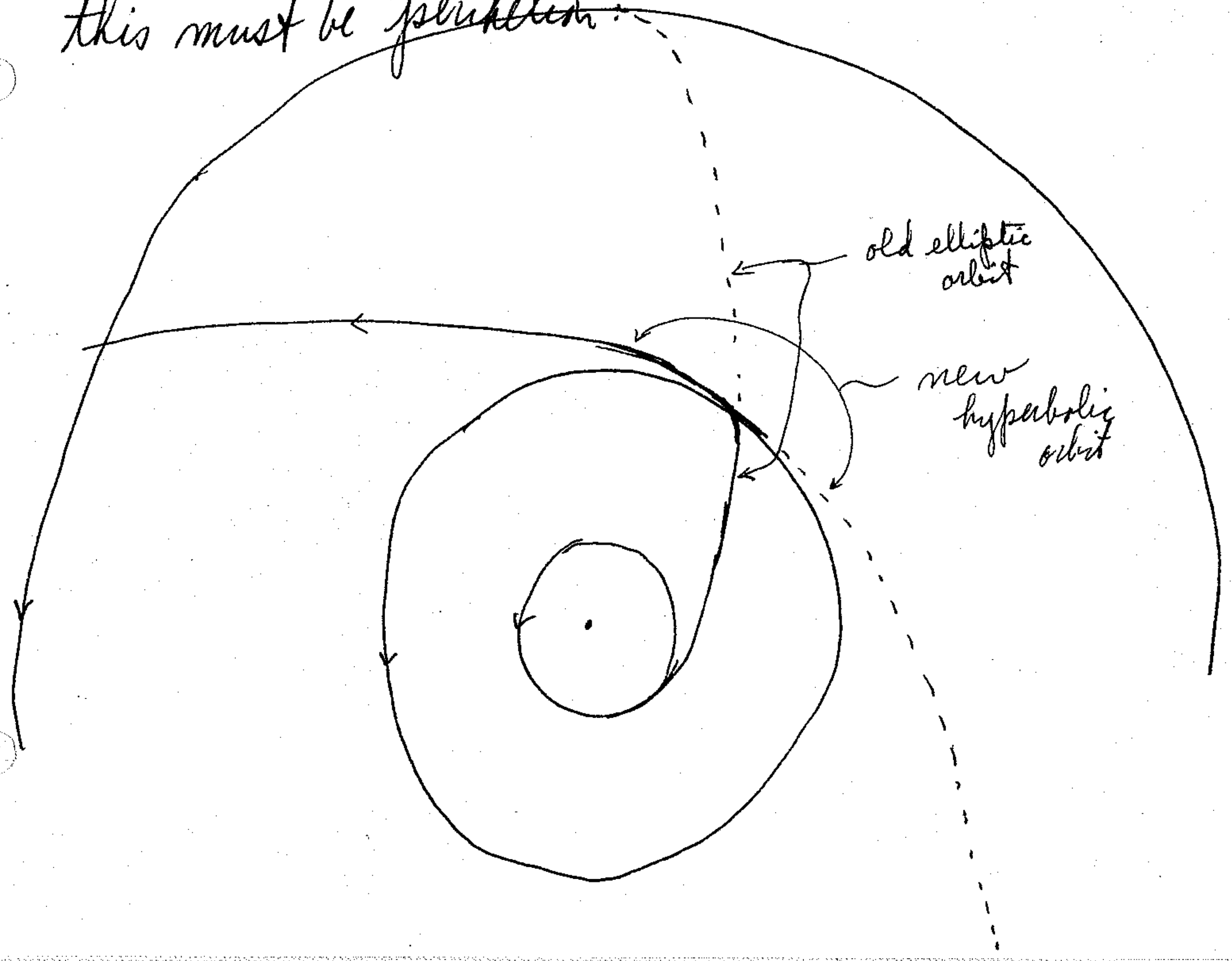
$$\vec{v} = \vec{u} + \vec{v}_J = (3.11 + 2.76)\hat{\varphi} = 5.87\hat{\varphi}$$

i.e., $\dot{r} = 0$ "v_r"

So now, from the sun,

$$e = \frac{1}{2}\vec{v}^{\,2} - \frac{k}{r_J} = \frac{1}{2}(5.87)^2 - \frac{4\pi^2}{5.2} = +9.636 (\mathrm{au}/\mathrm{yr})^2$$

So the orbit is hyperbolic, and since $\dot{r} = 0$ at $r = r_J$) this must be perihelion:



Knowing e, we have $a = \frac{k}{2e} = \frac{2\pi^2}{9.636} = 2.0485 \text{au}$

and knowing $r_- = 5.2 \text{au} = a(\epsilon - 1)$

$\Rightarrow \epsilon = 3.54$

and $r = \frac{23.6}{1 + 3.54\cos(\varphi - \varphi_0)}$ where here $\varphi_0 = \text{old } \varphi_J = 134.84°$.

and this orbit will pass Uranus and, being hyperbolic go on and escape the Solar System.

In order to find the time for travel,

use the eccentric anomaly. — See that handout.

First — on the first portion — from Earth to Jupiter

$r = a(1 - \epsilon \cos u)$ So $r_J = 5.2 \text{au} = (10.1)(1 - .901 \cos u_J)$

$\Rightarrow \cos u_J = .538$ and $u_J = 57.5°$

Then $2\pi \frac{t_J}{\tau} = u - \epsilon \sin u = .2438$

and with τ (from p. 1) $= 32.1 \text{yr}$

$\Rightarrow t_J = 1.245 \text{yr}.$

Note that without the kick given by Jupiter, it would have taken $\frac{1}{2}\tau = 16.05$ yrs to get to Uranus, most of that time after having passed Jupiter.

Then — on last portion — from Jupiter to Uranus on hyperbolic orbit.

$$19.18_{au} = r_U = a(\epsilon \cosh v_U - 1) = 2.0485\{3.54 \cosh v_U - 1\}$$

$\Rightarrow \cosh v_U = 2.9274$

$$\Rightarrow v_U = \log\{\cosh v_U + \sqrt{\cosh^2 v_U - 1}\} = 1.7367$$

Then $n \equiv \sqrt{\dfrac{k}{a^3}} = 2.143/\text{yr}$

& $n(t_U - t_0) = \epsilon \sinh v_U - v_U$

$$= 3.54(2.7512) - 1.7367 = 8.00266$$

where $t_0 = t_J$ from previous page

So $t_U - t_J = 3.734$ years

$\Rightarrow \quad t_U = 3.734 + 1.245 = \underline{4.979 \text{ years}}$

Considerably less than the original 16.05 years, but it doesn't ever come back.