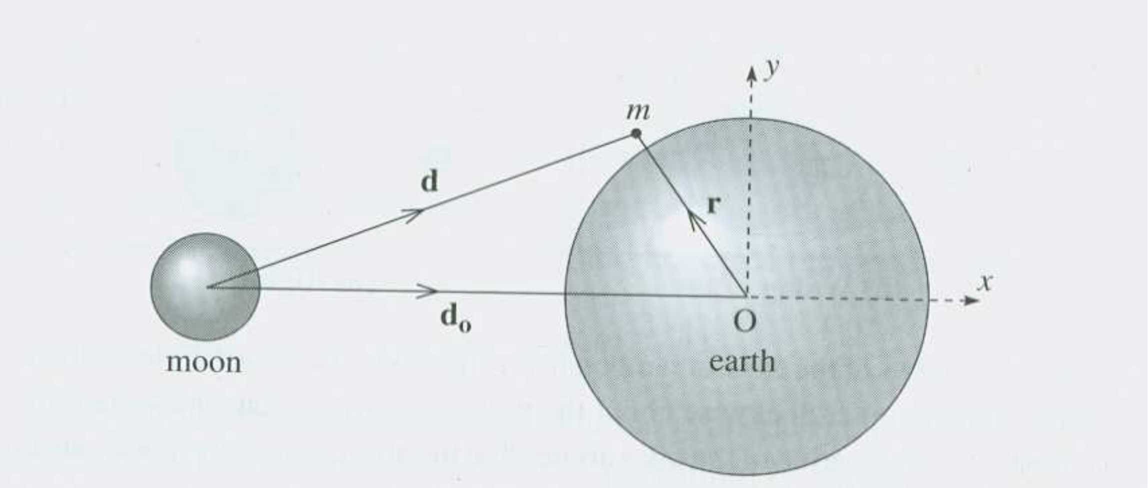
Physics 303

Tidal Forces on the Earth

The figure below shows the moon, the earth, and a small object of mass m on the surface of the earth, as well as vectors between each two of those objects, and a choice of Cartesian coordinate axes, with $\hat{z}$ coming out of the paper (and therefore not actually shown).



The forces on the small mass, as measured by an observer on the earth, i.e., in an accelerated coordinate system, are given in the text, which I reproduce here, but divided by its mass, so that the equation describes its acceleration measured in this non-inertial frame:

$$\ddot{\vec{r}} = -g\hat{r} - GM_m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right) + \tfrac{1}{m}\vec{F}_{\rm nongrav}\ , \tag{1}$$

where I assumed that the force of gravity on the mass caused by the earth—with magnitude g—is in the same direction as the vector locating the object itself. An important note is that the vector $\vec{d_0}$ is a constant vector; in the figure, one sees that we will choose that direction for $\hat{x}$. As well, note that the vector between the moon and the small mass, $\vec{d}$, is such that

$$\vec{d} = \vec{d_0} + \vec{r}\ . \tag{0}$$

The description I will give here will go into slightly more depth than the text, but will endeavor to follow his approach, and notation, as much as possible. As the text proceeds, he ignores the non-gravitational forces—which I think altogether reasonable—and considers only what he refers to as the tidal forces, caused by the moon. However, as the force of gravity caused by the earth is also a gravitational force, I would like to include them all together, into a single *multi-gravitational force,* caused by both the moon and the earth, so that I will be interested in the following acceleration of our mass, which is caused by both the nearby, large sources of gravity, and still remembering that it is being measured in our (somewhat) non-inertial coordinate system, based on the earth:

$$\frac{1}{m}\vec{F}_{\rm grav} = -GM_e\frac{\hat{r}}{r^2} - GM_m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right) \ , \tag{2}$$

where I have replaced g by its form for an arbitrary distance from the center of the earth. [This last action causes some of the details to seem somewhat different from that presented by the text; however, there are slightly more details here, so that I hope it will remain clear, and it applies to slightly more situations.]

With this force denoted explicitly, it is easy to see that it is a conservative force; therefore, I want to write down the potential function that creates it, **noting that I am using the word potential to describe the potential energy per unit mass**:

$$U_{\rm grav} = -\frac{GM_e}{r} - \frac{GM_m}{d} - \frac{GM_m x}{d_0^2} \ . \tag{3}$$

To ensure that this is correct, we need to take the negative gradient—with respect to the location, $\vec{r}$, of the mass—of this potential. What we want, of course, is that

$$\frac{1}{m}\vec{F}_{\rm grav} = -\nabla_{\vec{r}}U_{\rm grav} \ ; \qquad \nabla_{\vec{r}} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \ . \tag{4}$$

Since $\vec{d}_0$ is a constant, it follows—from the chain rule, if you like—that the gradient with respect to $\vec{d}$ or with respect to $\vec{r}$ are exactly the same, differing only by zero, the gradient of a constant; therefore, we may perform this calculation as follows, and drop the subscript on the symbol for the gradient, since they are all the same:

$$-\nabla_{\vec{r}}U_{\rm grav} = +GM_e\nabla\frac{1}{r} + \frac{GM_m}{d_0^2}\nabla x + GM_m\nabla\frac{1}{d} = -GM_e\frac{\hat{r}}{r^2} + GM_m\frac{\hat{x}}{d_0^2} - GM_m\frac{\hat{d}}{d^2} \ , \tag{5}$$

which is the same as the force above, given that $\hat{d}_0$ is the same as $\hat{x}$.

I now want to simplify this complicated-appearing form by noting that all distances on earth are considerably smaller than the distance between the earth and the moon; i.e., $r << d_0$, independent of just where $\vec{r}$ points on the earth. Therefore, I note the following use of the binomial approximation:

$$
\begin{aligned}
(1+x)^\alpha &= 1 + \alpha\, x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \;\ldots\;\; ; \\
d^2 &= |\vec{d}_0 + \vec{r}|^2 = d_0^2 + 2\vec{d}_0\cdot\vec{r} + r^2 = d_0^2\left[1 + 2\frac{x}{d_0} + \left(\frac{r}{d_0}\right)^2\right] \;, \qquad (6) \\
\Longrightarrow \quad \frac{1}{d} &= \frac{1}{d_0}\left[1 + 2\frac{x}{d_0} + \left(\frac{r}{d_0}\right)^2\right]^{-1/2} = \frac{1}{d_0}\left[1 - \frac{x}{d_0} - \frac{1}{2}\left(\frac{r}{d_0}\right)^2 + \frac{3}{2}\left(\frac{x}{d_0}\right)^2 + \;\ldots\right]
\end{aligned}
$$

Taking ψ as the name of the angle between $\hat{d}_0$ and $\hat{r}$, we can write $x = r\,\cos\psi$, and approximate the infinite series above by only the terms that I have shown explicitly:

$$
\frac{1}{d} \approx \frac{1}{d_0} - \frac{r}{d_0^2}\cos\psi + \frac{r^2}{d_0^3}\left[\tfrac{1}{2}(3\cos^2\psi - 1)\right] \;. \qquad (7)
$$

We now insert this approximation into our potential, the approximation being really quite good for points anywhere on the earth:

$$
\begin{aligned}
U_{\text{grav}} = &\; -\frac{GM_e}{r} - \frac{GM_m}{d} - \frac{GM_m x}{d_0^2} \\
&\approx -\frac{GM_e}{r} - \frac{GM_e}{d_0}\left[1 - \frac{r}{d_0}\cos\psi + \left(\frac{r}{d_0}\right)^2 P_2(\cos\psi)\right] - \frac{GM_m r}{d_0^2}\cos\psi \qquad (8) \\
&= -\frac{GM_e}{r} - \frac{GM_m r^2}{d_0^3}P_2(\cos\psi)\;; \qquad\qquad P_2(z) \equiv \tfrac{1}{2}(3z^2 - 1)\;.
\end{aligned}
$$

One of the purposes for determining this potential, for points on the surface of the earth, is now to apply an approximation to them, saying that all of the points on the surface are equally likely to move under the influence of the force which is determined by the gradient of this potential; this of course would be quite literally true if, for instance, the entire surface of the planet were covered with water! Even though this is not quite the case, it does give us a reasonable approximation for understanding the behavior of the motion of the shape of the earth caused by tides. I therefore, argue that the surface must be an equipotential for this function;

i.e., all the points on the surface move until they are at the same value for the potential function, U. This constant value, say U_S, could be taken as determining the points on the earth. Since the potential function involves both r and ψ, this says that the actual shape of the earth will be such that they change together so as to keep U constant. We want to know what that shape is. A simple way to do that is to ask how much it deviates from a sphere. We can therefore ascribe some fictitious value to the radius of that sphere, namely R_E. If the earth were spherical then this potential would simply have the value $-GM_e/R_E$, which would be constant. Therefore we may express U in that form, and determine the shape, by solving that equation for r as a function of ψ as follows:

$$\begin{aligned} -\frac{GM_e}{R_E} \equiv U_S &= -\frac{GM_e}{r} - \frac{GM_m r^2}{d_0^3} P_2(\cos\psi) \\ \Longrightarrow \quad \frac{1}{R_E} - \frac{1}{r} &= \frac{M_m}{M_e}\frac{r^2}{d_0^3} P_2(\cos\psi) \\ \Longrightarrow \quad r - R_E \equiv \Delta r &= \frac{M_m}{M_e}\frac{R_E r^3}{d_0^3} P_2(\cos\psi)\ . \end{aligned} \tag{9}$$

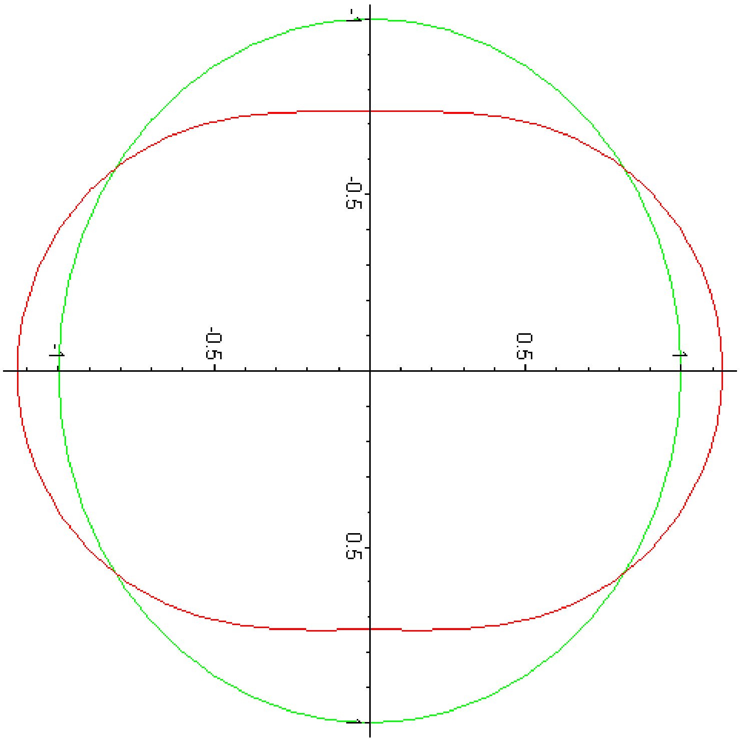
This is a complicated-appearing, cubic equation for r; however, as we know that Δr is really quite small, we believe it is sufficient to solve it approximately; therefore, we can replace the r^3 on the right-hand side by

$$r^3 = (R_E + \Delta r)^3 = R_E^3 \left[1 + 3\frac{\Delta r}{R_E} + 3\left(\frac{\Delta r}{R_E}\right)^2 + \left(\frac{\Delta r}{R_E}\right)^3\right] \approx R_E^3\ ,$$

where we have ignored all the higher-order terms since they were very small. Inserting this into that right-hand side, we obtain

$$r - R_E \equiv \Delta r = \frac{M_m R_E^4}{M_e d_0^3} P_2(\cos\psi) = \frac{M_m R_E^4}{2\, M_e d_0^3}(3\cos^2\psi - 1)\ . \tag{10}$$

This is then an equation for the shape of a fairly-flexible covering over the earth, as a result of the moon being overhead, showing the deviation from being a sphere with ψ the angle between the overhead direction and the place in question, with location $\vec{r}$, on the earth, ignoring higher-order corrections.



The figure shows a spherical shape, and then the shape just calculated above, with a choice of the factor $M_m R_E^4 / M_e d_0^3$ as 0.265 to the basic radius of 1; i.e., the tidal distortion has been enormously exaggerated, so that the basic idea would be clear. The actual factor, when calculated, comes out to be only 0.358 meter. On the other hand, exactly the same relation would have been obtained if we had used the sun instead of the moon, with, of course d_0 being the distance from the earth to the sun; in that case, that factor is, instead, 0.165 meter, which is on the order of half the effect of the moon. Therefore, when they happen to be working together, i.e., both in the same direction from the earth, or exactly opposite direction, then these effects add; i.e., at new moon and full moon.

The last, important comment is that the moon rotates around the earth, with a period of approximately 27 1/3 days. Therefore, all that has been said above has been appropriate at a particular moment of time, with the bulge of the surface of the earth toward, and also away from, the moon. This means that the bulge goes around the earth as the moon rotates around the earth, or, if you prefer, the earth rotates under the moon. The earth rotates once every 24 hours, but during that time the moon, going in the same direction, progresses through 1/(27 1/3) of its orbit. This means that the bulge comes back to its initial position every 24[1+1/(27 1/3)] hours, which equals 24 hours and 52.7 minutes. High tide therefore comes every half of this time period, i.e., it repeats every 12 hours and 26.4 minutes. In principle the height of the tide would be the same every one of these repetitions. The reasons it does not repeat exactly are many, beginning with the fact that there are continents, bays, etc. However, a very important effect is also that the inclination angle of the Earth's axis to that of the moon varies, between $17°$ to $29°$.

Appendix:

The expansion above for the inverse square root of the sum of two vectors is actually a very important one, as you might imagine, I hope, for many physics problems; therefore, it is common enough to have been studied many times, and generates objects called *Legendre polynomials*, examples of the very simplest ones we saw above. The more general statement is the following. Let $\vec{r}_1$ and $\vec{r}_2$ be two vectors, named so that $r_1 > r_2$, i.e., r_1 is the longer of these two vectors, and let ψ name the angle between the two. Then we may expand the inverse of the distance between the two vectors as follows:

$$\begin{aligned}\frac{1}{|\vec{r}_1 - \vec{r}_2|} &= \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\psi + r_2^2}} = \frac{1}{r_1}\left[1 - 2\frac{r_2}{r_1}\cos\psi + \left(\frac{r_2}{r_1}\right)^2\right]^{-1/2} \\ &\equiv \frac{1}{r_1}\sum_{\ell=0}^{\infty}\left(\frac{r_2}{r_1}\right)^{\ell} P_\ell(\cos\psi)\ , \qquad r_1 > r_2\ .\end{aligned} \tag{A1}$$

Here $P_\ell(\cos\psi)$ is simply the sum of all the coefficients of $(r_2/r_1)^\ell$ in the binomial expansion of the inverse square root in question. As we have seen above, in our expansion of the lowest order terms, we know that

$$P_0(w) = 1\ , \qquad P_1(w) = w\ , \qquad P_2(w) = \frac{1}{2}(3w^2 - 1)\ , \qquad \ldots \qquad ,$$

where I have used the symbol w for whatever the argument of the polynomial might be; in the main discussion above, we had $w = \cos\psi$; more generally, if we used spherical coordinates, $\hat{x}\cdot\vec{r} = \sin\theta\,\cos\varphi$, and that would have been what is meant here by just the symbol w.

More generally, it is fairly easy to see that all of these quantities, for any integer ℓ, these coefficients are polynomials in their argument, and in fact, ℓ-th-order polynomials. Next semester we will spend some more using these polynomials to determine the behavior of the gravitational potential of an object larger than a point and not (exactly) in the shape of a sphere!