Physics 303

The Eccentric Anomaly

as a useful device for determining the passage of time on a planetary orbit

We begin with the equation for the total energy per unit mass, and also the angular momentum per unit mass:

$$e \equiv \frac{E}{\mu} = \frac{1}{2}\vec{v}^2 - \frac{GM}{r} \quad = \frac{1}{2}\dot{r}^2 + \frac{\xi^2}{2r^2} - \frac{GM}{r} \ , \qquad \xi \equiv \frac{\ell}{\mu} = r^2\dot{\phi} \ , \tag{0.1}$$

which can be used to solve for the radial velocity in the following form:

$$r\,\dot{r} = \pm\sqrt{2er^2 + 2GMr - \xi^2} \ . \tag{0.2}$$

The orbit in question could be linear, circular, elliptic, parabolic, or hyperbolic, depending on its angular momentum and energy. The linear case corresponds to the very special case when the angular momentum is zero; all the other cases must have non-zero angular momentum. We will consider these in turn, beginning with the elliptical one, which has the circular orbit as simply a special case.

I. The elliptical case:

We describe it by introducing its semi-major axis, a, and its eccentricity, ϵ, recalling that the circular case is simply the one where $\epsilon = 0$. In that case, we may express many of the desired parameters in terms of these quantities:

$$e = -\frac{GM}{2a} \ , \qquad r_{\rm min} = a(1-\epsilon) \ , \quad r_{\rm max} = a(1+\epsilon) \ , \quad \xi^2 = GMa(1-\epsilon^2) \equiv GMc \ . \tag{1.1}$$

We also may re-think the equation above for the radial velocity, in the sense that it must vanish at the minimum and maximum radial values, so that it must be so that

$$\begin{aligned} r\,\dot{r} = \ \pm\sqrt{-2e(r-r_{\rm min})(r_{\rm max}-r)} &= \pm\sqrt{\frac{GM}{a}[r-a(1-\epsilon)][a(1+\epsilon)-r]} \\ &= \pm\sqrt{\frac{GM}{a}[(a\epsilon)^2-(r-a)^2]} \ . \end{aligned} \tag{1.2}$$

The last useful thing one can do with this is to consider the factor in front of the polynomial (under the square root). Using Kepler's third law, for the period, we can write that overall constant factor in a very nice way:

$$\frac{(2\pi)^2}{GM}\,a^3 = \left(\frac{a}{GM}\right)(2\pi a)^2 = \tau^2 \quad \Longrightarrow \quad \sqrt{\frac{GM}{a}} = \frac{2\pi a}{\tau}\ , \tag{1.3}$$

This last expression is the average velocity around a circle which would have the same radius as the ellipse has semimajor axis, namely a. Therefore the "best" form of this expression for the radial speed is

$$\frac{\tau}{2\pi}\frac{r}{a}\dot{r} = \pm\sqrt{[(a\epsilon)^2 - (r-a)^2]}\ . \tag{1.4}$$

Our next task is to create a parametrization which allows this equation to be easily integrated; more precisely, we intend to create a parameter η, referred to as *the eccentric anomaly*, which allows us to determine fairly simple forms for both $r(\eta)$ and $t(\eta)$. The final answer is in fact that we want to define η so that

elliptic (or circular) orbits: $\qquad r = a(1-\epsilon\cos\eta)\ , \qquad \dfrac{2\pi}{\tau}t = \eta - \epsilon\sin\eta\ ,$ (1.5)

where we have arranged the beginning of η so that $\eta = 0$ corresponds to the periastron point, i.e., for $r(\eta)$ we have $r(0) = r_{\rm min} = a(1-\epsilon)$; as well, we also allow our clocks to begin running then i.e., for $t(\eta)$ we have $t(0) = 0$. However, we must now go and explain two relevant facts:

1.) given the equation above for $r(\eta)$, which can be taken as defining η, we must show that the equation for $t(\eta)$ follows, and

2.) we should give a nice physical meaning for this parameter as well.

To proceed with step 1 above, we should insert this form for $r(\eta)$ into our speed equation above. We first rewrite $\dot{r}$ as dr/dt, move the dt to the other side of the equation, and then note the following substitutions:

$$r - a = -a\epsilon\cos\eta\ , \quad dr = a\epsilon\sin\eta\, d\eta\ . \tag{1.6}$$

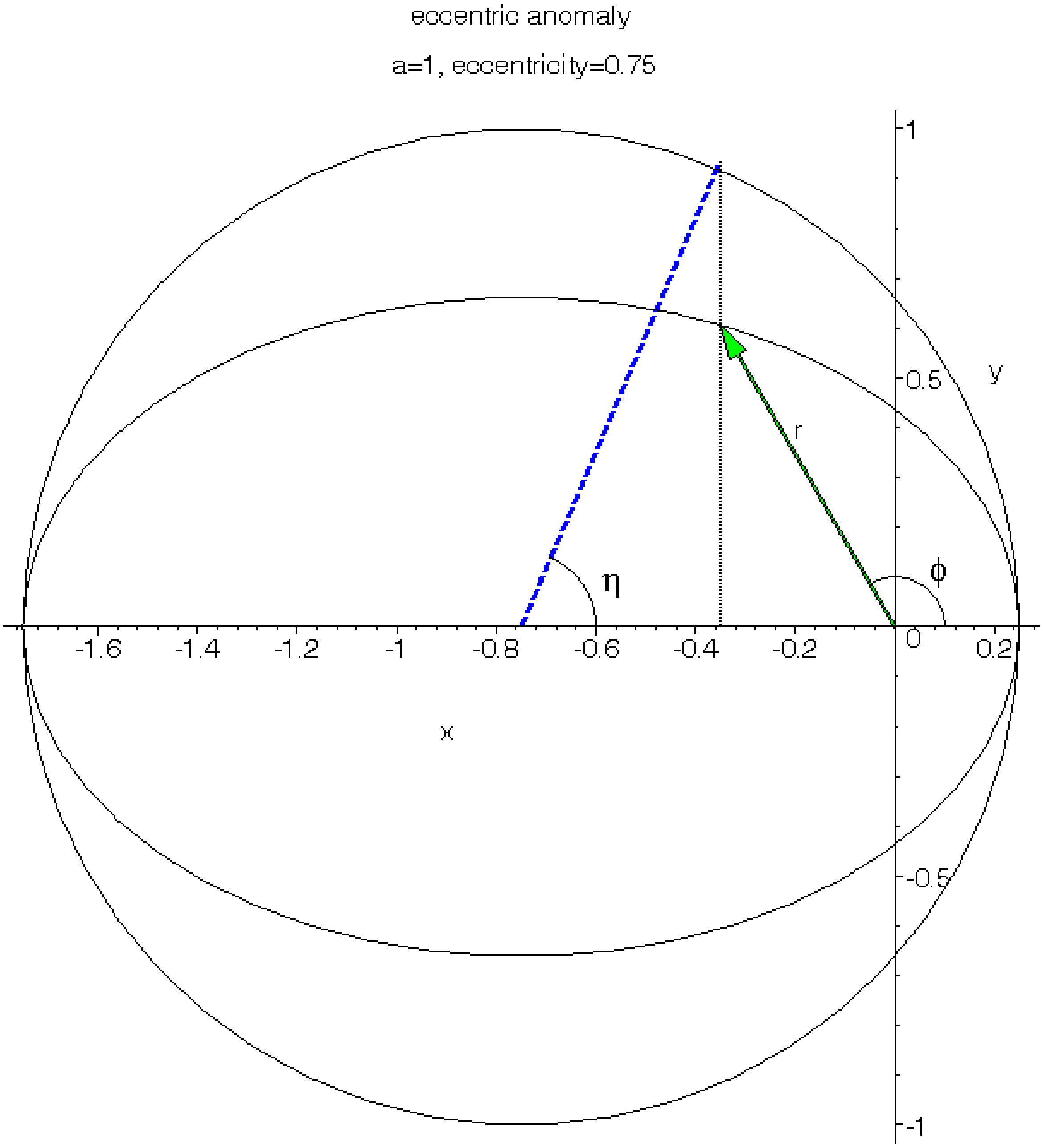
Inserting these quantities into the primary equation above, Eq. (1.4), we have the following, where we drop the $\pm$ sign for the moment:

$$\begin{aligned} &\frac{\tau}{2\pi}(1-\epsilon\cos\eta)\,a\epsilon\sin\eta\,d\eta = \sqrt{(a\epsilon)^2(1-\cos\eta)}\,dt = a\epsilon\sin\eta\,dt \\ &\Longrightarrow \quad \int_0^\eta d\eta\,(1-\epsilon\cos\eta) = \frac{2\pi}{\tau}\int_0^t dt \quad \Longrightarrow \quad \mu \equiv \frac{2\pi}{\tau}t = \eta - \epsilon\sin\eta\ , \end{aligned} \tag{1.7}$$

which was the formula desired, where this new quantity, μ, a measure of the time from $\eta = 0$ to the current value of η relative to the total period, is referred to as *the mean anomaly.*

The next task, called step 2 above, is to explain the "physical" meaning of the parameter η, which will in fact be useful in many instances of integrations of this type. We begin by drawing around our ellipse the so-called "exscribed" circle mentioned above, namely a circle with the same center as the ellipse and with a radius equal to the semimajor axis of the ellipse. This circle is tangent to the ellipse at two (important) places, the minimum and maximum values of r. In the figure below we see a particular instance of the vector from the center of the force, i.e., the focus of the ellipse, to the point where the orbiting object happens to be at this moment, labeled r, and a second line from the center of the ellipse to the point directly above the orbiting object, on the "fictitious circular orbit" above it. These two lines constitute hypotenuses for two right triangles that can be seen, created as well by the line from that fictitious location through the actual location down to the horizontal axis. The distance along that horizontal axis between the center of the ellipse and the focus at the origin of the coordinates has the value $a\epsilon$, while the base of the right-hand right triangle has length $r\cos(\pi-\phi)$. Therefore the base of the left-hand right triangle is the difference of these two lengths, and gives us an expression for the cosine of the angle η, the eccentric anomaly:

$$a\cos\eta = a\epsilon - r\cos(\pi-\phi) = a\epsilon + r\cos\phi\ . \tag{1.8}$$



However, we can now take the explicit expression for the orbit, and solve it for the quantity $r\cos\phi$ which we need:

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\phi} \quad \Longrightarrow \quad r\cos\phi = [a(1-\epsilon^2) - r]/\epsilon\ , \tag{1.9}$$

which we may then insert into our equation for the cosine of the eccentric anomaly:

$$a\cos\eta = a\epsilon + \frac{a}{\epsilon} - a\epsilon - \frac{r}{\epsilon} \quad \Longrightarrow \quad r = a(1-\epsilon\cos\eta)\ , \tag{1.10}$$

which was the equation that was desired to be shown.

This section concludes by re-emphasizing the "physical meaning" of η, namely that it is the angle above the horizontal that a line from the center of the ellipse toward the fictitious location of the particle on the circle that includes the ellipse. The fictitious particle there is moving on the smallest circle that contains the elliptical orbit, and is moving with constant velocity along that circle, given by $v = 2\pi a/\tau$.

The next step is to relate the physical angle, ϕ, to our fictitious angle, η; i.e., we want to determine $\phi = \phi(\eta)$, to put along with our already-shown values of $r = r(\eta)$ and $t = t(\eta)$. We begin by returning to Eq. (1.8) and resolving it for $r\cos\phi$:

$$r\cos\phi = a(\cos\eta - \epsilon) \quad \Longrightarrow \quad \cos\phi = \frac{\cos\eta - \epsilon}{1 - \epsilon\cos\eta} , \tag{1.11}$$

where the second equality comes from dividing by r and then inserting its value in terms of η from Eq. (1.10). In principle this is the desired result, which came very easily; however, a somewhat more useful form is obtained if we evaluate both $1 \pm \cos\phi$ from this expression and use them to determine the following:

$$\tan\phi/2 = \sqrt{\frac{1-\cos\phi}{1+\cos\phi}} = \sqrt{\left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{1-\cos\eta}{1+\cos\eta}\right)} = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\,\tan\eta/2 . \tag{1.12}$$

Do notice of course that in the circular case, when $\epsilon = 0$, all this was quite redundant, since the particle was moving along a circular orbit anyway.

II. Unbound orbits:

We now move onward to the case of unbound orbits. Since the energy is larger than the mass of the particle, this results in hyperbolic orbits (or the very special case of parabolic orbits when the energy is just zero and the eccentricity is exactly 1). Such an orbit may be described in terms of a parameter a and an eccentricity, ϵ, this last now being greater than or equal to 1. The equations defining the angular momentum and energy, per unit mass, as given in Eqs. (1.1) for the bound case, are essentially the same, with only a sign change:

$$e = +\frac{GM}{2a} , \qquad r_{\min} = a(\epsilon - 1) , \quad \xi^2 = GMa(\epsilon^2 - 1) , \tag{2.1}$$

while the equation for the speed has minor changes because of the size of the eccentricity, and of course including the fact that there is no maximum value for r, nor is there a period; it becomes

$$r\dot{r} = \pm\sqrt{GM/a}\,\sqrt{(r+a)^2 - (a\epsilon)^2}\ . \tag{2.2}$$

Without going into nearly as much detail as in the previous discussion, we are encouraged by that discussion, and the different signs under the square root, to create a *hyperbolic eccentric anomaly*, v, such that

$$r = a(\epsilon\cosh v - 1)\ . \tag{2.3}$$

Although there is indeed no period in this case, it is nonetheless customary to define a quantity

$$n \equiv \sqrt{\frac{GM}{a^3}}\ , \tag{2.4}$$

which is the same as the $2\pi/\tau$ that occurred in the elliptic case. Insertion of this value for n and the equation for r given above, in Eq. (2.3), into our square root allows us to separate the entire thing into an equation to find the time as a function of v:

$$na\,dt = a(\epsilon\cosh v - 1)\,dv \quad\Longrightarrow\quad nt = \epsilon\sinh v - v\ , \tag{2.5}$$

where we have chosen to measure time from the periastron point, where v also begins; i.e., we have

$$r_{\rm min} = r(\eta)\Big|_{\eta=0}\ ,\quad t(\eta)\Big|_{\eta=0} = 0\ . \tag{2.6}$$

This gives us the desired functional behavior, relative to our parameter v, for $r(v)$ and $t(v)$. The only remaining thing to do in this case is to determine the angular form, which is a derivation basically identical to the one given above for the elliptic case, and which results in the following:

$$\tan\phi/2 = \sqrt{\frac{\epsilon+1}{\epsilon-1}}\ \tanh v/2\ . \tag{2.7}$$

III. The linear case, when the orbiting object has exactly zero angular momentum.

In this case, for our attractive force, the motion could be either inward or outward, depending on the initial velocity. However, I will simply specialize to the reasonably important case when the particle begins from rest at some large distance, R, and then falls linearly into the source of the force, which means of course that the energy is negative. In that case, energy conservation gives us a related form for the speed:

$$r^2\dot{r}^2 = 2(GM/R)(R-r)r \ , \qquad e = \frac{E}{\mu} = -\frac{GM}{R} \ . \tag{3.1}$$

We then follow along with the reasoning as above, and define the anomaly η, via

$$r \equiv \tfrac{1}{2}R(1+\cos\eta) \quad \Longrightarrow \quad \begin{cases} r = R \ \text{ when } \eta = 0 \ , \\ \\ r = 0 \ \text{ when } \eta = \pi \ . \end{cases} \tag{3.2}$$

Using this equation, Eq. (3.2), to determine the needed $R-r$ for our speed equation, and differentiating it to obtain $\dot{r} = -\frac{2}{R}\sin\eta$, we have the following:

$$[\tfrac{1}{2}R(1+\cos\eta)]^2[-\tfrac{1}{2}R\sin\eta\,\dot{\eta}]^2 = GM[1-\cos\eta] \, . \tag{3.3}$$

Replacing $\sin^2\eta$ by $1-\cos^2\eta = (1-\cos\eta)(1+\cos\eta)$ allows us to take the square root of both sides. We can then move the dt to the other side of the equation, resulting in the following:

$$dt = \sqrt{\frac{R^3}{8GM}}\,(1+\cos\eta)\,d\eta \quad \Longrightarrow \quad t = \sqrt{\frac{R^3}{8GM}}\,(\eta + \sin\eta) \ . \tag{3.4}$$

This is clearly normalized so that $t = 0$ when the particle is at its largest distance from the star, and η has value π when it reaches the center of our force, i.e., $r = 0$, which gives us a very simple expression for the total infall time:

$$\Delta t = t(\pi) = \sqrt{\frac{R^3}{2GM}}\,\frac{\pi}{2} \ . \tag{3.5}$$