Physics 303

Bonus Homework No. 2 due Tuesday, 25 November, 2008

50 Homework points

Steady-state, periodic forcing functions are interesting and useful for damped oscillators; however, a more general sort of forcing function would surely be more useful, and more realistic. Therefore, this problem is set up to allow you to figure out how to deal with a quite general sort of time-dependent forcing function for our standard damped, simple harmonic system. The final result of the problem will be to determine, and use, a *Green's function* for such an oscillator.

We suppose given the usual form for an inhomogeneous differential equation for an oscillator:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \tfrac{1}{m}F(t) \equiv f(t) \ ,$$

where, however, we make no requirements on the function $f(t)$ except that it should be piecewise continuous, although surely its derivative need not be, and there is certainly no expectation that it should be periodic. The general solution to this problem is given by the Green's function $G(t,t')$ in the form

$$x(t) = x_{\text{hom}}(t) + \int_{-\infty}^{t} dt'\, G(t,t')f(t) \ ,$$

where of course $x_{\text{hom}}(t)$ is the homogeneous solution to the equation, i.e., when the forcing function is just zero. Therefore, we would have the problem solved if only we knew how to determine $G(t,t')$, which is the real purpose of this problem, at least for the underdamped case, i.e., when $\beta < \omega_0$.

1. To begin, consider the case of a *step function* as the time-dependent forcing option; i.e., consider the case where

$$f_s(t) = \begin{cases} a \ , \ t > t_0 \ , \\ \\ 0 \ , \ t < t_0 \ . \end{cases}$$

Imagine that the system is totally at rest, i.e., $x(t) = 0$ and $\dot{x}(t) = 0$ for all times prior to t_0, and then this constant force is applied. Please show that the motion can be described by the following, with the usual definition of ω_1:

$$x_s(t) = \begin{cases} \dfrac{a}{\omega_0^2}\left[1 - e^{-\beta t}\cos\omega_1 t - \dfrac{\beta}{\omega_1}e^{-\beta t}\sin\omega_1 t\right] & , t \geq t_0 \ , \\ 0 & , t < t_0 \ . \end{cases}$$

2. Using this result, now determine the response the same system would have if, instead, it were forced by an impulse function:

$$f_i(t) = \begin{cases} 0 \, , \; t < t_0 \, , \\ a \, , \; t_0 < t < t_1 \, , \\ 0 \, , \; t_1 < t \, , \end{cases}$$

where again the boundary conditions are such that the system is at rest at the origin prior to t_0. Notice, in your solution, that the behavior of the system for times much larger than t_1 is simply to return, exponentially, to its original rest state at equilibrium. While this solution is quite complicated in appearance, you should now take its limit as $\tau \equiv t_1 - t_0 \longrightarrow 0$ and, simultaneously a increases in such a way that the product $a\tau$ approaches a constant value, say b. This then would approximate well the behavior of an oscillator that is struck very forcefully but for a very short period of time. In this limit show that the solution takes on the form,

$$x_i(t) = \begin{cases} \dfrac{b}{\omega_1} e^{-\beta(t-t_0)} \sin[\omega_1(t-t_0)] \, , & t \geq t_0 \, , \\ 0 & , \; t \leq t_0 \, . \end{cases}$$

3. Now, let us suppose that we use the standard approach from integral calculus, where we want to consider the integral of a function as a determination of the area between its graph and the horizontal axis. There one approximates the function by a very large number of very narrow rectangles. More precisely, we suppose that we define an impulsive approximation to a very small portion of our arbitrary function, $f(t)$, by first dividing up the horizontal axis, i.e., the t-axis into some countable number of equal width intervals, of width Δt, that run from t_n to t_{n+1}. We may then look at impulsive functions of the form above

$$I_n(t) \equiv \begin{cases} 0 & , \; t < t_n \, , \\ f(t_n) & , \; t_n < t < t_{n+1} \, , \\ 0 & , \; t_{n+1} < t \, . \end{cases}$$

$$\Longrightarrow \quad f(t) = \sum_{n=-\infty}^{+\infty} I_n(t) \, .$$

Use this to show that the Green's function, as defined above, may be given by

$$G(t,t') = \begin{cases} \dfrac{1}{\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')] & , \ t \geq t' \ , \\ 0 & , \ t < t' \ . \end{cases}$$

4. Using the Green's function approach, now determine the behavior of our oscillator when the forcing function is given by the following damped harmonic driving term:

$$f(t) = \begin{cases} 0 & , \ t < 0 \ , \\ f_0 e^{-\gamma t} \sin \omega t & , \ t \geq 0 \ , \end{cases}$$

and provide graphs of the motion, which started from rest at equilibrium at $t = 0$. Create these graphs for three different pairs of values of β and γ, using ω as a scale factor for time:

$$\#1 : \quad \beta = 0.1\omega_0 \ , \quad \gamma = 0.3\omega_0 \ ,$$

$$\#2 : \quad \beta = 0.2\omega_0 \ , \quad \gamma = 0.2\omega_0 \ ,$$

$$\#3 : \quad \beta = 0.3\omega_0 \ , \quad \gamma = 0.1\omega_0 \ .$$