Physics 303

Final Exam **Solutions** 16 December, 2007

Part A: There are 5 problems in this part: please omit two of them. [16 pts each]

1. Explain the following words or concepts, giving examples:

a. Resonance of a linear oscillator.

b. Precession of the (bounded) orbit of a mass moving under some central force.

c. Terminal velocity.

d. Conservation law for angular momentum.

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a. We consider a simple harmonic oscillator that is being driven by (in principle one or more) harmonic driving forces, with frequency ω. The oscillator may or may not respond with large amplitude to being driven in this way. When it does, it is said to be *in resonance* with the driving term; this happens, more or less, when ω is some integer multiple of ω_0.

b. The bounded orbit of a (point) mass moving under some central force can be considered as a trajectory given by $r = r(\varphi)$. Since the motion is bounded it will vary between some r_{min} and some r_{max}. This will happen with some period τ_r. In the meantime, since the coordinate φ only varies between 0 and 2π, it has period 2π. If τ_r equals 2π, then the orbit will close on itself. More commonly, τ_r will differ from 2π by some non-zero amount, say δ. Then the orbit is said *to precess* since r will return to, say, its maximum value only after the angular variable has gone "all the way around" plus the amount δ—which of course could be positive or negative.

c. The generic sort of equation for the motion of a particle of mass m in the field of a velocity-dependent force, $f(v)$, and, usually, some additional, non-velocity-dependent force, f_0, would be given by Newton's second law:

$$m\dot{v} = f_0 + f(v) \ .$$

If $\dot{v}$ were to become zero, the velocity would remain constant, i.e., it would have reached a terminal velocity. Therefore we may state the necessary condition as simply that there is a non-zero solution to the equation

$$f(v_{\text{term}}) = -f_0 \ .$$

d. The time rate of change of the angular momentum of an object or a system of them is determined by the sum of all the torques on that system, caused by the forces acting on them. If that sum of all torques is zero, then the angular momentum remains the same.

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2. A ball is thrown directly upward on the earth, at a point with co-latitude angle θ, with an initial velocity of $v_0\hat{z}$. When it arrives at the high point of its trajectory, what will be its vector velocity, to first-order in the earth's angular velocity Ω?

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With the usual definitions that, locally on the earth, $\hat{z}$ points upward opposite to the direction of effective gravity, $\hat{x}$ points eastward and $\hat{y}$ points northward, the equations of motion for an object moving under the effect of gravity in our rotating reference frame are

$$\begin{aligned}\ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)\ ,\\ \ddot{y} &= -\,2\Omega\dot{x}\cos\theta\ ,\\ \ddot{z} &= -\,g + 2\Omega\dot{x}\sin\theta\ ,\end{aligned}$$

where θ is the co-latitude of our location on the earth and Ω its angular frequency.
As we have been doing, we solve these iteratively, using the following initial conditions:

$$v_{x0} = 0 = v_{y0}\ ,\quad v_{z0} = v_0 > 0\ ;\qquad x_0 = 0 = y_0 = z_0\ .$$

Therefore the zeroth-order solution to the equations of motion, i.e., ignoring Ω, is just given by

$$x(t) = 0 = y(t)\ ,\quad z(t) = v_0 t - \tfrac{1}{2}gt^2\ .$$

Inserting this back into the equations of motion, maintaining first-order terms in Ω, we then obtain the first-order solutions to the equations of motion:

$$x(t) = -\Omega\sin\theta\,(v_0 - \tfrac{1}{3}gt)t^2\ ,\quad y(t) = 0\ ,\quad z(t) = v_0 t - \tfrac{1}{2}gt^2\ .$$

The z-equation tells us that the time to reach the top of the trajectory, i.e., for $v_z(t) = 0$ is given by $t_1 = v_0/g$, and the maximum height is $z_{\max} = v_0^2/(2g) \equiv h$.
We are asked for the velocity at the top. The z-component is obviously zero. The x-component is given by the derivative of the x-component, evaluated at $t_1 = v_0/g$:

$$\vec{v}(t_1) = \dot{x}(t_1)\hat{x} = -2\Omega\,\sin\theta\,(v_0 - \tfrac{1}{2}gt_1)t_1\hat{x} = -\Omega\sin\theta\frac{v_0^2}{g}\hat{x}\ .$$

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3. A projectile is sent up from the ground at some initial angle and speed. Sometime during its trajectory it splits into two distinct parts. One of the parts lands 50 meters further along the trajectory than it would have landed had the projectile not exploded. Knowing that the two parts struck the ground again at the same time, how far from that initial aiming point did the other part of it land?

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The center of mass of the two pieces of the projectile will land at the same place that it would have had it not exploded into parts. Therefore, even when there are two parts the center of mass must move along the same trajectory as before, up until they hit the ground. Since they hit the ground at the same time, they were moving together, in that sense, until ground fall. Therefore, since the one part, having, say, mass m_1, lands 50 meters further along, i.e., to the right of the center of mass, then the other part, with mass m_2, must have landed $-50(m_1/m_2)$ meters from the center of mass, where the minus sign means further back.

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4. A certain mass, m, is subject to gravity and other forces in such a way that its equation of motion may be given as

$$2m\ddot{r} = \frac{\ell^2}{mr^3} - mg \ ,$$

where g and ℓ are constants.

a. Determine an equilibrium position, r_0, for the coordinate r for this mass, as a function of ℓ, m and g.

b. If the mass is put at a point near that equilibrium position, so that $r = r_0 + \epsilon$, it will oscillate about the equilibrium position for small values of ϵ. What will be the frequency of that oscillation?

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a. Since the velocity, $\dot{r}$, does not appear in the equation for the acceleration, all that is needed for an equilibrium position is to set the initial velocity to zero and to determine $r = r_0$ such that the acceleration vanishes; therefore, we easily have

$$r_0^3 = \frac{\ell^2}{m^2 g} \ .$$

b. Setting $r = r_0 + \epsilon$, in the given equation we have

$$\ddot{\epsilon} = \ddot{r} = \frac{\ell^2}{2m^2(r_0+\epsilon)^3} - \tfrac{1}{2}g = \frac{\ell^2}{2m^2 r_0^3}[1+\epsilon/r_0]^{-3} - \tfrac{1}{2}g \ .$$

In the next step we approximate this expression for small values of ϵ/r_0, and also, in the other part insert the value of r_0^3 as given above:

$$\ddot{\epsilon} = \tfrac{1}{2}g[1 - 3\frac{\epsilon}{r_0} + \ \ldots\,] - \tfrac{1}{2}g = -\tfrac{3}{2}\frac{g}{r_0}\epsilon + O(\epsilon^2)\ .$$

We see that this does indeed have the form of an undamped, SHO, with frequency such that

$$\omega^2 = \tfrac{3}{2}\frac{g}{r_0} = \tfrac{3}{2}\left(\frac{\ell}{mr_0^2}\right)^2 = \tfrac{3}{2}\left(\frac{mg^2}{\ell}\right)^{2/3}\ .$$

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5. In terms of the relative distance, $\vec{r}$, between two masses, which have a gravitational force acting on each one, caused by the other, a particular orbit can be described by

$$r = \frac{6}{5 + \cos\phi}\ ,$$

where $r = |\vec{r}|$ and it is measured in astronomical units.

a. Determine the semimajor axis, a, and the eccentricity, ϵ, of the orbit.

b. Make a reasonably-clear sketch of the orbit, taking the object to be an asteroid, showing its orbit, and also the orbits of the earth and Mars ($a_M = 1.5$ A.U.), both assumed circular.

c. What is the value of the angle ϕ when this orbit is at the same distance from the sun as is the earth?

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a. The simplest way to obtain a is to average the maximum and minimum values of r:

$$r_{\text{min}} = \frac{6}{5+1} = 1\ , \quad r_{\text{max}} = \frac{6}{5-1} = \frac{3}{2} \quad \Longrightarrow \quad a = \tfrac{1}{2}(r_{\text{min}} + r_{\text{max}}) = \frac{5}{4}\ .$$

The eccentricity is the coefficient of $\cos\phi$ when the other term in the denominator is just 1, i.e, the denominator has the form $5[1 + \frac{1}{5}\cos\phi]$, which tells us the eccentricity is just

$$\epsilon = \frac{1}{5}\ .$$

b.

c. As the earth is at $r = 1$ A.U., and this is the minimum value of r for this orbit, it follows that the angle is just $\phi_E = 0$.

Part B: There are 2 problems in this part. You should do both of them [26 pts each]

6. A cruise ship in the outskirts of Antarctica is at *South latitude* 60°, and cruising toward the south at 5 meters/second.

a. The ship's sextant wants to determine the angular velocity of the rotation of the earth. He knows that it must have a magnitude of $\Omega = 1$ rotation per day. However, he also wants the direction, measured of course in his own local coordinates. Taking as a choice of unit vectors, $\hat{W}$, westward, $\hat{S}$, southward, and $\hat{U}$, upward, what are components that he measures?

b. What are the magnitudes and directions of the centrifugal and Coriolis accelerations of the ship that he also measures?

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a. We begin with the standard formula from the text, relating Ω, $\hat{N}$, and $\hat{E}$, namely

$$\vec{\Omega} = \Omega[\hat{N}\sin\theta + \hat{U}\cos\theta] = \Omega[-\hat{S}\sin\theta + \hat{U}\cos\theta] \ .$$

We then note that South latitude 60° means its 30° before the south pole, or 150° from the north pole, i.e., $\theta = 150^\circ$. Next it is useful although not essential to note that $\sin(150^\circ) = \sin(30^\circ) = 1/2$, while $\cos(150^\circ) = -\cos(30^\circ) = -\frac{1}{2}\sqrt{3}$. Inserting all this we have

$$\hat{\Omega} = -[\hat{S}\sin(30^\circ) + \hat{U}\cos(30^\circ)] = -\tfrac{1}{2}[\hat{S} + \sqrt{3}\hat{U}] \ .$$

b. To determine the centrifugal acceleration we know it has the direction $\hat{s}$, which is perpendicular to $\hat{\Omega}$, i.e., in the current, local coordinates,

$$\vec{a}_{\text{Cf}} = \Omega^2 R_{_E}\sin\theta\hat{s} = \Omega^2 R_{_E}\sin(30^\circ)[-\hat{S}\cos(30^\circ) + \hat{U}\sin(30^\circ)] = \tfrac{1}{4}\Omega^2 R_{_E}[-\sqrt{3}\hat{S} + \hat{U}] \ .$$

Likewise the Coriolis acceleration on the ship, that is heading Southward is given by

$$\vec{a}_{\text{Cor}} = 2\vec{v}\times\vec{\Omega} = 2(5\ \text{m/s})\Omega\hat{S}\times[-\hat{S}\sin\theta + \hat{U}\cos\theta] = 10\Omega\cos(150^\circ)\hat{S}\times\hat{U} = 5\sqrt{3}\Omega\hat{E} \ .$$

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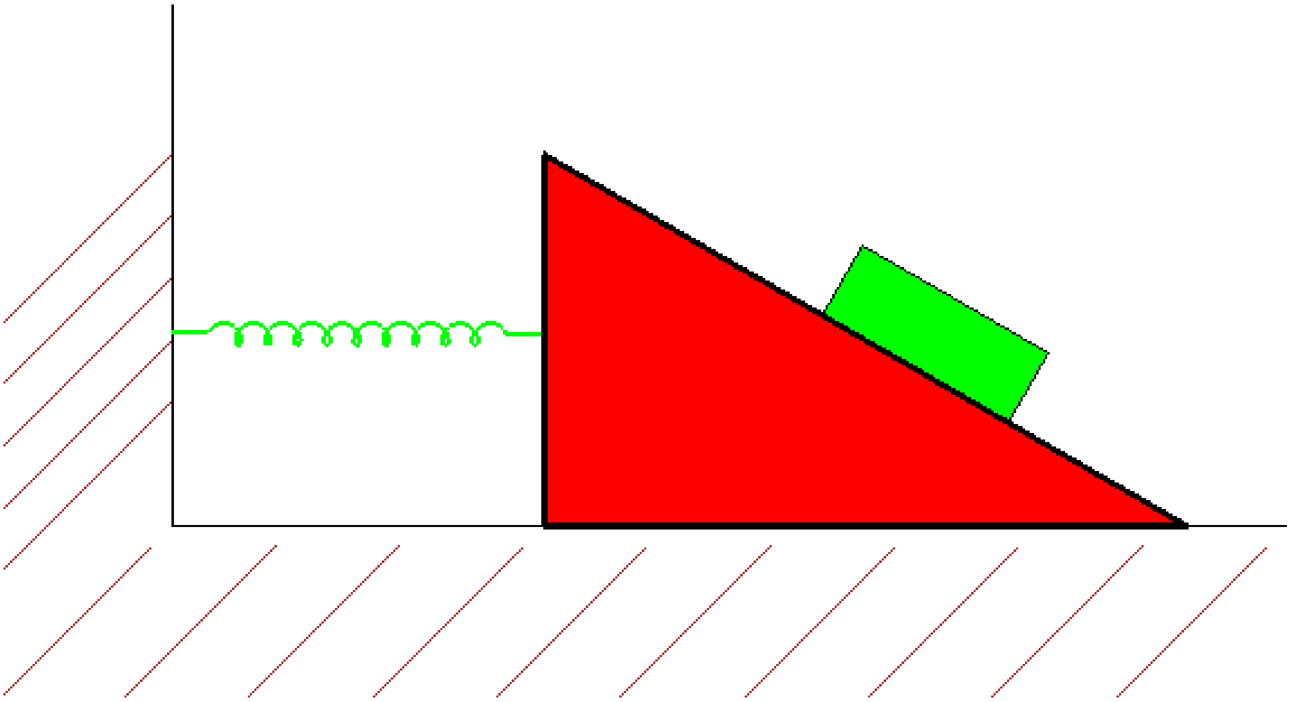
7. In the figure, a large inclined plane, of mass M and angle θ, is sliding on a horizontal surface, toward the left, with coefficient of friction, ν. The plane is attached to an adjacent wall by a massless spring, of spring constant k and equilibrium length ℓ. On the plane is a small mass, m, sliding down that plane, with coefficient of friction, μ.



a. Please first make a free-body diagram, separately, for the inclined plane and the small box.

b. Next, please ignore all the frictional forces, and write out the Lagrangian for that system—2 degrees of freedom—and determine the equations of motion of the plane and the box. You do NOT need to solve them.

c. Determine the two canonical momenta for the system, resolve the velocities in terms of the momenta, and then write out the Hamiltonian function, $\mathcal{H}(x, w, p_x, p_w)$ for this system.

d. Lastly, use that Hamiltonian function to determine the time rates of change of the two momenta.

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a. For the free-body diagram,

   I.) we first look at the top block, currently sliding downward on the bottom inclined plane. The forces acting on it are

      i. the downward force due to gravity, of amount mg,

      ii. the normal force caused by the inclined plane, perpendicular to that surface, and labeled n,

      iii. the frictional force between the two blocks, which, since the block is sliding downward, acts up the plane, with a magnitude μn.

II.) for the inclined plane, the forces acting on it are

   i. the downward force due to gravity, of amount Mg,
   ii. the reaction force caused by the top block, acting perpendicular to its upper surface, downward, and of magnitude n,
   iii. the frictional force between the two blocks, this one caused by the top block, and acting downward along the plane of the inclined, of magnitude μn,
   iv. the restoring force from the attached spring, acting horizontally and toward the right or left, depending on whether the spring is currently compressed or expanded, and
   v. the normal force caused by the floor on which it sits, and slides, which is straight upward, and named N.
   vi. the frictional force on it caused by the floor, which acts on it horizontally and toward the right.

b. We choose two coordinates, x, and w.

I.) The coordinate x gives the location of the far-left end of the inclined plane, measured from a point a distance ℓ to the right of the wall, which is the place where the spring is un-stretched, i.e., at its equilibrium position.

II.) The coordinate w gives the location of the upper side of the small mass sliding downward, measured from the top of the inclined plane.

III.) We also choose coordinate axes $\hat{x}$ horizontal, and positive toward the right, and $\hat{y}$ vertical, and positive upward.

The location of the left-hand side of the plane is just $x\hat{x}$. The location of the top side of the small box, measured from the top (and also left) side of the plane is $w(\cos\theta\,\hat{x} - \sin\theta\,\hat{y})$. Therefore the location of the small box, measured from the equilibrium position of the spring is

$$\vec{r}_{\text{box}} = (x + w\cos\theta)\hat{x} - w\sin\theta\hat{y} \ .$$

Differentiating this location we have the following:

$$\vec{v}_{\text{box}} = (\dot{x} + \dot{w}\cos\theta)\hat{x} - \dot{w}\sin\theta\,\hat{y}$$
$$\Longrightarrow \quad \vec{v}^{\,2}_{\text{box}} = \dot{x}^2 + \dot{w}^2 + 2\dot{x}\dot{w}\cos\theta \ .$$

The velocity of the inclined plane is just $\dot{x}\hat{x}$, while there are two potential energies involved:

$$U_{\text{box}} = -mgw\,\sin\theta \ , \qquad U_{\text{spring}} = \tfrac{1}{2}kx^2 \ .$$

Therefore, the Lagrangian can be written out:

$$\mathcal{L}(x,w,\dot{x},\dot{w}) = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m[\dot{x}^2 + \dot{w}^2 + 2\dot{x}\dot{w}\cos\theta] + mgw\,\sin\theta - \tfrac{1}{2}kx^2\ .$$

From here it is straightforward to determine the two associated (or canonical) momenta:

$$p_x \equiv \frac{\partial\mathcal{L}}{\partial\dot{x}} = (M+m)\dot{x} + m\dot{w}\cos\theta\ , \quad p_w \equiv \frac{\partial\mathcal{L}}{\partial\dot{w}} = m\dot{w} + m\dot{x}\cos\theta\ .$$

Next we can write down the equations of motion:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= -kx = \dot{p}_x = (M+m)\ddot{x} + m\ddot{w}\cos\theta\ ,\\ \frac{\partial\mathcal{L}}{\partial w} &= mg\sin\theta = \dot{p}_w = m\ddot{w} + m\ddot{x}\cos\theta\ .\end{aligned}$$

Remembering the definition of the Hamiltonian, we first solve these two equations for the two velocities:

$$\dot{x} = \frac{p_x - p_w\cos\theta}{M + m\sin^2\theta}\ , \qquad \dot{w} = \frac{-p_x\cos\theta + (M+m)p_w/m}{M + m\sin^2\theta}\ ,$$

and then insert these values into that definition,

$$\begin{aligned}\mathcal{H} = p_x\dot{x} + p_w\dot{w} - \mathcal{L} = &\ \frac{p_x[p_x - p_w\cos\theta] + p_w[-p_x\cos\theta + (M+m)p_w/m]}{M + m\sin^2\theta}\\ &- \mathcal{L}[x, w, \dot{x}(x,w,p_x,p_w), \dot{w}(x,w,p_x,p_w)]\ .\end{aligned}$$

When the algebra is done, one has the desired result

$$\mathcal{H}(x,w,p_x,p_w) = \frac{p_x^2 + (1 + M/m)p_w^2 - 2p_xp_w\cos\theta}{2(M + m\sin^2\theta)} + \tfrac{1}{2}kx^2 - mgw\sin\theta\ .$$

d. To determine the time derivatives of the momenta we use the Hamiltonian equations of motion:

$$\begin{aligned}\dot{p}_x &= -\frac{\partial\mathcal{H}}{\partial x} = -kx\ ,\\ \dot{p}_w &= -\frac{\partial\mathcal{H}}{\partial w} = +mg\sin\theta\ .\end{aligned}$$