Physics 304

The Gravitational (or Electric) Field
with an introduction to Legendre polynomials

I. The gravitational force, and its potential, for continuous mass distributions

The gravitational force between two point masses, m_1 and m_2, located at vector locations, $\vec{r}_1$ and $\vec{r}_2$, from some arbitrary origin, is known to be given by Newton's basic law of gravitation:

$$\begin{aligned} m_2\ddot{\vec{r}}_2 = \vec{F}_{\text{from1on2}} &= -G\frac{m_1 m_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1) \\ &= +G\frac{m_1 m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2) = -\ddot{\vec{F}}_{\text{from2on1}} = -m_1\ddot{\vec{r}}_1 \ . \end{aligned} \tag{1.1}$$

[If this were the electric field instead, then it is basically the same; the only difference is in the symbols since we would replace m_1 by q_1, m_2 by q_2, and $-G$ by $k = 1/(4\pi\epsilon_0)$. Therefore we will talk only about the gravitational case, but everything is exactly the same, with those substitutions of symbols, in the electric (Coulomb) case.]

If, then we have a distribution of matter, of total volume V_1, causing a gravitational force, such that m_1 is the total mass of some finite distribution of matter, then we would divide that distribution up into infinitesimal pieces, each of mass dm_1, located at the vector location $\vec{r}_1$, and apply the equation above to each of them in turn, adding them all up via an integral. In this general case it is convenient to describe the distribution by its mass density $\rho(\vec{r}_1)$, i.e., to describe the very small piece of mass via its density and its volume:

$$dm_1(\vec{r}_1) = \rho(\vec{r}_1)\, dV_1 \ . \tag{1.2}$$

This allows us to just sum up all the pieces that are causing the net gravitational force, via integration because the pieces are so very small: so that we may write

$$m_2\ddot{\vec{r}}_2 = \vec{F}_{\text{from1on2}} = -Gm_2\int_{V_1}\frac{dm_1(\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1) = -Gm_2\int_{V_1} dV_1\,\frac{\rho(\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1)\ , \tag{1.3}$$

where this is a **3-dimensional** integral over the entire volume, where the source of gravitational force is located, all acting on the point mass m_2. It is usual to note that this force is simply proportional to that point mass, m_2; therefore, one defines the *gravitational field,* $\vec{g}$, as the force on m_2 divided by m_2, thinking of that field as caused by all the matter in V_1, and existing always at the location $\vec{r}_2$, where, now, the mass m_2 happens to be.

As the integral in question is really rather "nasty," especially because of the existence within it of the vector, so that one has to do three different integrals, one for each component of the force vector, it is traditional to define an associated potential. [We recall that there is a potential associated with conservative forces, and this force is indeed conservative.] We therefore define the gravitational potential

$$\mathcal{G}(\vec{r}) \equiv \int_{\vec{r}_s}^{\vec{r}} d\vec{r} \cdot \vec{g}(\vec{r}) \iff \quad \vec{g}(\vec{r}) = -\nabla\, \mathcal{G}(\vec{r}) \ , \quad \text{then } \vec{F}_{\text{on2, causedby1}} = m_2\, \vec{g}(\vec{r}_2) \ , \tag{1.4}$$

where the integral (on the left) is the usual sort of line integral that one has when determining potentials from forces, with $\vec{r}_s$ as some particular, **fixed** point from which the path begins to take us to the desired arbitrary point $\vec{r}$. It then follows that we have a considerably simpler, scalar integral to use to determine the gravitational potential at some arbitrary location, $\vec{r}$, caused by a distribution of matter in the volume V:

$$\mathcal{G}(\vec{r}) = -G \int_V dV'\, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \equiv -G \int_V d\vec{r}\,'\, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \ , \tag{1.5}$$

where we are using, here, $d\vec{r}\,'$ to mean the same as dV', the 3-dimensional integral over the entire volume V, with points in it having locations $\vec{r}\,'$, and we have chosen our fixed reference point, $\vec{r}_s$, as the point at infinity, where of course the potential and the force vanish.

All actual distributions of matter of course have some volume, and therefore the mass density ρ is an appropriate way to describe them. Nonetheless, we sometimes like to make mathematical approximations that describe certain sorts of simpler systems. For instance, suppose that we have a volume that may be thought of in terms of some length or thickness that is really very small; then we could re-write small pieces of volume in terms of the area perpendicular to that thickness multiplied by the infinitesimal thickness: $dV = dAd\ell$, so that we could absorb the thickness into the (volume) density and re-describe things in terms of an areal density, σ, over an infinitesimally thin object:

$$dM = \rho\, dV = (\rho\, d\ell)\, dA \equiv \sigma\, dA \ . \tag{1.6a}$$

Since the thickness is now being ignored, the integral to determine the potential need only be a 2-dimensional integral:

$$\mathcal{G}(\vec{r}) = -G \int_A dA' \, \frac{\sigma(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \; . \tag{1.6b}$$

Alternatively, yet a different approach could be that the matter in question is simply long and very thin, i.e., at least approximately has negligible cross-sectional area, such as a long wire, or stick. In that case we re-describe it in terms of a linear density, λ, as follows:

$$dM = \rho \, dV = (\rho \, dA) \, d\ell \equiv \lambda \, d\ell \; , \tag{1.7a}$$

and then re-write our equation for the gravitational field it causes using only a 1-dimensional integral over its length:

$$\mathcal{G}(\vec{r}) = -G \int_L d\ell' \, \frac{\lambda(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \; . \tag{1.7b}$$

Which of these three approximative schemes will be useful clearly depends on the particular problem being considered. To see that more clearly, let us now look at some specific examples.

1. Consider an annular ring of matter, of radius a and of uniformly-dense material. Therefore, it is reasonable to describe in terms of its linear density, λ, which for simplicity we also take to be constant, since its density is uniform. What is the gravitational field that it causes at arbitrary points away from it? In fact this is a complicated question, and determination of the answer to that question is a large part of the purpose of these notes. However, let's answer for question for points along the axis perpendicular to its area. Therefore, let's set up the problem. We choose an origin and coordinates so that our ring simply corresponds to a circle, of radius a, in the x, y-plane, with the origin at its center. Then the $\hat{z}$-axis is perpendicular to our circle and goes through its center. At this point we only want to determine the field that it causes at any point along this $\hat{z}$-axis; i.e, we will only ask for $\mathcal{G}(\vec{r})$ when $\vec{r} = z\hat{z}$. Then we may use cylindrical coordinates to write the location vectors for the actual matter distribution, as

$$\begin{aligned} &\vec{r}\,' = a\hat{s} = a[\cos\phi \, \hat{x} + \sin\phi \, \hat{y}] \\ \Longrightarrow \quad &|\vec{r} - \vec{r}\,'| \quad \longrightarrow \quad |z\hat{z} - \vec{r}\,'| = \sqrt{z^2 + a^2} \; . \end{aligned} \tag{1.8a}$$

The evaluation of our integral then only requires summing up over all angles ϕ, since we may write $d\ell = a\,d\phi$:

$$\mathcal{G}(z\hat{z}) = -Ga\int_0^{2\pi} d\phi\,\frac{\lambda}{\sqrt{z^2+a^2}} = -G\frac{2\pi a\lambda}{\sqrt{z^2+a^2}} = -\frac{GM}{\sqrt{z^2+a^2}}\ , \tag{1.8b}$$

since the total mass is just the linear density, λ, multiplied by the distance around, the circumference, $2\pi a$.

2. One may now use this result to determine the gravitational field generated by a solid disk of matter, by thinking about that disk as being generated by a large number of very thin such annular rings. Each of these thin rings would have a mass dM, each at some radius r, where r varies from 0 up to the entire radius, a, of the disk. In that notion then the infinitesimal gravitational field from just one of our rings would be

$$d\mathcal{G}(z) = -\frac{G\,dM}{\sqrt{z^2+r^2}}\ . \tag{1.9a}$$

Since dM is the total mass "all the way around one ring," we may write it out in terms of an areal mass density, σ, as

$$dM = r\,dr\int_0^{2\pi} d\phi\,\sigma = 2\pi\sigma r\,dr\ , \qquad \sigma = \frac{M}{A} = \frac{M}{\pi a^2}\ , \tag{1.9b}$$

which then gives us the infinitesimal contribution of each ring, so that we may integrate over all the rings, and determine the field from the entire disk:

$$\begin{aligned} d\mathcal{G}(z) &= -\frac{G\,dM}{\sqrt{z^2+r^2}} = -\frac{2GM}{a^2}\frac{r\,dr}{\sqrt{z^2+r^2}}\ ,\\ \Longrightarrow\quad \mathcal{G}(z) = \int_{r=0}^{r=a} d\mathcal{G}(z) &= -\frac{GM}{a^2}\int_0^a \frac{2r\,dr}{\sqrt{z^2+a^2}} = -\frac{GM}{a^2}\int_0^{a^2}\frac{du}{\sqrt{z^2+u}}\\ &= -\frac{2GM}{a^2}\left[\sqrt{z^2+a^2}-|z|\right]\ . \end{aligned} \tag{1.9c}$$

3. If, instead of an entire solid disk as above, we had simply an annular ring, with matter uniformly distributed over it between an inner radius b and an outer radius a, then everything would go as above, except that the final integral should not begin at 0 but rather at b, which give us for that case

$$\text{for a thick annular ring, with } a>b,\ \ \mathcal{G}(z\hat{z}) = -2GM\frac{\sqrt{z^2+a^2}-\sqrt{z^2+b^2}}{a^2-b^2}\ . \tag{1.10}$$

4. As a last example for now, we consider a somewhat different one, which does require some ingenuity with the calculus, and, in fact, is the reason that Newton invented the calculus. We propose to determine the gravitational field caused by a spherically-symmetric, negligibly-thick shell of matter, of radius a and having total mass M. As before, since its thickness is infinitesimal, we need only perform our integral over its surface area and set up the surface density, σ, so that

$$M = 4\pi a^2 \sigma \ . \tag{1.11a}$$

Surely spherical coordinates are the best choice to perform the integral, where small patches of surface would have infinitesimal area given by $a^2 \sin\theta'\, d\theta'\, d\varphi'$. At an arbitrary, but fixed, point $\vec{r}$, we have the equation

$$\mathcal{G}(\vec{r}) = -G\sigma a^2 \int_0^{\pi} \sin\theta'\, d\theta' \int_0^{2\pi} d\varphi' \frac{1}{|\vec{r}-\vec{r}\,'|} \ . \tag{1.11b}$$

This integral appears fairly complicated, still, with the main problem being the fact that the cosine of the angle between the two vectors is a quite complicated function of our spherical coordinates. However, we can fix that since we are still at liberty to choose how our spherical coordinates are defined. We arrange them so that the North pole of our sphere lies directly below the special point with location $\vec{r}$. This is the same as saying that the $\hat{z}$-axis is chosen so that it lies in the same direction as $\vec{r}$, which makes the dot product in the integral only involve $\cos(\theta')$. Therefore, we may easily calculate the magnitude of the difference of 2 vectors that lies in the denominator of our integral, noting that the magnitude of $\vec{r}\,'$ is just a, since it lies on our spherical shell of matter:

$$|\vec{r}-\vec{r}\,'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + a^2} = \sqrt{r^2 - 2ra\cos\theta' + a^2} \ . \tag{1.11c}$$

Next we notice that the integral above, in Eq. (1.11b), over values of θ', allows us to write $\sin\theta'\, d\theta' = -d(\cos\theta')$, so that the integral may be re-written in the form

$$\begin{aligned} \mathcal{G}(\vec{r}) = & -G\sigma a^2 \int_{-1}^{+1} d(\cos\theta') \int_0^{2\pi} \frac{d\varphi'}{\sqrt{r^2+a^2-2ra\cos\theta'}} \\ & = -2\pi\sigma G a^2 \frac{\sqrt{r^2+a^2-2ar\cos\theta'}}{-ar}\Big|_{-1}^{+1} = -\frac{2\pi\sigma G a}{r}\left[|r+a| - |r-a|\right] \ . \end{aligned} \tag{1.11d}$$

We may simplify this by, first, inserting the value of σ in terms of M, and, secondly, by noticing that it has quite simple, but different, values depending on whether r is greater than or less than a; i.e., it has quite different values depending on whether we ask for the potential inside the spherical shell or outside it:

$$\begin{aligned} \mathcal{G}(\vec{r}) &= \begin{cases} -\dfrac{MG}{r} , & r \geq a , \\ -\dfrac{MG}{a} , & r \leq a . \end{cases} \\ \Longrightarrow \quad \vec{g}(\vec{r}) &= \begin{cases} -\dfrac{MG}{r^2}\vec{r} , & r \geq a , \\ \vec{0} , & r < a . \end{cases} \end{aligned} \tag{1.11e}$$

This is, perhaps, only a slight generalization of the standard result found in any freshman-physics text:

a.) that the potential from a point mass M would fall off like $1/r$,

b.) that the potential exterior to a spherically-symmetric, uniform distribution of matter, of mass M, would act just as if it were a point mass concentrated at the center, and

c.) that the potential interior to such a spherically-symmetric, uniform distribution of matter depends only on the amount of matter between it and the center—in this case, none so that the potential is a constant, because of the last expectation,

d.) that the potential is a continuous function of its arguments, while in this case the field is discontinuous since there is a discontinuity in the matter distribution, being infinitesimally thin.

II. Expansion of the Potential, for Points Outside the Source

I now want to return to the more general problem, as described in Eq. (1.5), and determine a way to evaluate the integral there exactly, but via an expansion in powers of r'/r, which we will require to be less than 1, by insisting that the evaluation point is outside the source of our field. Let me repeat Eq. (1.5) here, and re-consider its denominator:

$$\begin{aligned} \mathcal{G}(\vec{r}) = & -G \int_V dV' \, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} , \\ \frac{1}{|\vec{r} - \vec{r}\,'|} &= \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + (r')^2}} = \frac{1}{r} \frac{1}{\sqrt{1 - 2(r'/r)\cos\psi + (r'/r)^2}} , \end{aligned} \tag{2.1}$$

where ψ is the angle between $\vec{r}$ and $\vec{r}\,'$. We want to expand the denominator in a useful way; however, first it may be useful to simply recall the binomial theorem as it might apply to this problem, where we are considering the case that $r' < r$, i.e., the point of interest is **outside** the distribution of matter. In its simplest case we have simply

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + O(x^4) \ , \tag{2.2}$$

where it is not unreasonable to be dropping terms of high order in x provided x is sufficiently small. We are going to look, for now, only at calculations of the potential for locations outside of the matter distribution, i.e., for $r' < r$ for all r' inside the matter. Under those circumstances we may think of expanding our denominator, with the symbol x from Eq. (2.2) chosen as $(r'/r)[2\cos\psi - r'/r]$, which has terms of 1st order and also of 2nd order in our overall small parameter, r'/r. Inserting this value for x into the expansion above, we obtain the desired expansion for the integrand in our integral:

$$\begin{aligned}\frac{1}{\sqrt{1-2(r'/r)\cos\psi + (r'/r)^2}} &= 1 + \frac{r'}{r}\cos\psi + \left(\frac{r'}{r}\right)^2 \frac{3\cos^2\psi - 1}{2} \\ &\quad + \left(\frac{r'}{r}\right)^3 \frac{5\cos^3\psi - 3\cos\psi}{2} + O[(r'/r)^4] \ .\end{aligned} \tag{2.3}$$

Note that because our x has terms of both 1st and 2nd order, one must be careful in re-combining the various terms in order to acquire the expansion given in Eq. (2.3). However, looking at the terms in the expansion, or even just thinking about what sort of terms would come next, we can see that the various coefficients of powers of r'/r are polynomials in the variable $\cos\psi$. Legendre, already about 200 years ago, saw the usefulness of this expansion and said that he would give a name to the coefficients, for all powers, and then attempt to work out their properties. Therefore we may define the Legendre polynomials as a countable, infinite set of polynomials, in some variable, η, and denote them by $P_\ell(\eta)$, and define them by this expansion:

$$\frac{1}{\sqrt{1-2(r'/r)\cos\psi + (r'/r)^2}} \equiv \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\psi) \ , \tag{2.4}$$

where of course our physics problem wanted the arbitrary variable η to be $\cos\psi$, which therefore varies between -1 and $+1$. We can see immediately from the expansion we have already put into evidence the values of the first 4 of these polynomials:

$$P_0(\eta) = 1\ , \quad P_1(\eta) = \eta\ , \quad P_2(\eta) = \tfrac{1}{2}(3\eta^2 - 1)\ , \quad P_3(\eta) = \tfrac{1}{2}(5\eta^3 - 3\eta)\ . \tag{2.5}$$

I will say more about their properties, and also more about the higher-order ones later in these notes; however, notice, at least, that the ones shown are such that $P_\ell(\eta)$ is an ℓ-th order polynomial in η, and that $P_\ell(1) = 1$, while the polynomial is either an even polynomial—all even powers—or an odd polynomial—all odd powers—depending on whether ℓ is an even or an odd integer.

I now insert this expansion into our equation for the gravitational potential:

$$\mathcal{G}(\vec{r}) = -\frac{G}{r}\int_V dV'\,\rho(\vec{r}\,')\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\psi)\ . \tag{2.6}$$

We can of course next insert, at least, the terms where we already know the values of P_ℓ and work on the integrals that arise:

$$\mathcal{G}(\vec{r}) = -\frac{G}{r}\int_V dV'\,\rho(\vec{r}\,') - \frac{G}{r^2}\int_V dV'\,\rho(\vec{r}\,')r'\,\cos\psi - \frac{G}{r^3}\int_V dV'\,\rho(\vec{r}\,')(r')^2\,P_2(\cos\psi) + \ \ldots\ \ . \tag{2.7}$$

The first of these integrals is very easy: it is just the integral of the mass density over the entire volume, which gives the total mass, M. The second one is also fairly easy to determine. Using $\hat{r}$ as the unit vector in the direction of the point where we are trying to determine the field, that integral is

$$\int_V dV'\,\rho(\vec{r}\,')\,r'\cos\psi = \int_V dV'\,\rho(\vec{r}\,')\,\vec{r}\,'\cdot\hat{r} = \hat{r}\cdot\int_V dV'\,\rho(\vec{r}\,')\,\vec{r}\,' = M\hat{r}\cdot\vec{R}_{\rm cm}\ , \tag{2.8}$$

where we have recognized the last form of the integral as that which determines the center of mass of our distribution. Therefore we will now summarize that information that we have so far:

$$\begin{aligned}\mathcal{G}(\vec{r}) = &\ -\frac{M\,G}{r} - \frac{MG}{r^2}\,\hat{r}\cdot\vec{R}_{\rm cm}\\ &\ - \frac{G}{r^3}\int_V dV'\,\rho(\vec{r}\,')(r')^2\,P_2(\cos\psi) + \ O(1/r^4)\ ,\end{aligned} \tag{2.9}$$

and we also note that if we would take the origin of coordinates at the center of mass of our distribution—surely a fairly normal approach—then the second term would vanish, leaving us with only the total mass term, the second-order ones, and the yet-higher-order ones that we are ignoring as being too small.

The next several terms are somewhat more difficult to interpret on the basis of your past experience. In fact the very next one involves the moments of inertia of the distribution of mass, and we will study it more than once. In connection with that I would also like to point out that the general sort of integrals of this sort, where one multiplies a density—in this case a mass density—by some other variable quantities and then integrates is, generically, referred to as *a moment of the mass distribution.* Notice that the mass times the center of mass location is the *first (position) moment of the mass distribution*, also of course known as the center of mass, multiplied by the total mass. Now we go ahead and consider the best way to calculate the second moments, those that constitute the proportionality factor for $(r'/r)^2$. To begin, consider that part of the integrand that does not include the density, and re-write it in terms of Cartesian coordinates:

$$\vec{r}\cdot\vec{r}\,' = r(r')\cos\psi = xx' + yy' + zz' \ ,$$

$$\begin{aligned}(r')^2 P_2(\cos\psi) =& \tfrac{1}{2}(r')^2[3\cos^2(\psi) - 1] = \tfrac{1}{2}[3(\vec{r}\cdot\hat{r}\,')^2 - (r')^2] \\ =& \tfrac{1}{2}\left[\frac{3}{r^2}(xx' + yy' + zz')^2 - (x')^2 - (y')^2 - (z')^2\right] \\ =& \tfrac{1}{2}\Big\{ [3(x/r)^2 - 1](x')^2 + [3(y/r)^2 - 1](y')^2 + [3(z/r)^2 - 1](z')^2 \\ & \qquad + 3\frac{xy}{r^2}x'y' + 3\frac{xz}{r^2}z'x' + 3\frac{yz}{r^2}y'z' \Big\}\end{aligned} \tag{2.10}$$

We have replaced that single, difficult integral with some 6 different integrals, each of which is actually considerably easier. There are clearly two different types of integrals here, those that require integration over the square of some (Cartesian) coordinate, and those that require integration over a product of two different such coordinates—all of them of course *second moments* of the mass distribution. The ones involving squares are actually related to the usual (standard) definition of the moment of inertia for rotation about an axis $\hat{n}$, which is given by

$$I_{\hat{n}} = \int_V dV'\,\rho(\vec{r}\,')\,(\vec{r}\,'\times\hat{n})^2 \ , \tag{2.11}$$

where of course the weighting factor $(\vec{r}' \times \hat{n})^2$ is the square of the (perpendicular) distance from the axis of rotation to that particular small piece of the matter which is at the location $\vec{r}'$. To see the relationship of this to the integrals we must do, we consider the three independent cases when we choose $\hat{n}$ to be along the $\hat{x}$- or $\hat{y}$- or $\hat{z}$-directions. If the axis of rotation is the $\hat{z}$-direction, for instance, then the distance to that axis is just the length of a perpendicular drawn to it, which is $\sqrt{x^2+y^2}$, ignoring the z-coordinate. Therefore we can define three so-called *principal moments of inertia*:

$$\begin{aligned} I_1 \equiv I_{\hat{x}} &= \int_V dV'\,\rho(\vec{r}')\,(y'^2+z'^2)\;, \\ I_2 \equiv I_{\hat{y}} &= \int_V dV'\,\rho(\vec{r}')\,(x'^2+z'^2)\;, \\ I_3 \equiv I_{\hat{z}} &= \int_V dV'\,\rho(\vec{r}')\,(x'^2+y'^2)\;. \end{aligned} \tag{2.12}$$

These are the same sorts of integrals as we have already found are needed to describe the second-order terms in the expansion for the gravitational field, the two sets being related one to the other as follows:

$$\begin{aligned} \int_V dV'\,\rho(\vec{r}')\,x'^2 &= \tfrac{1}{2}(I_3+I_2-I_1)\;, \\ \int_V dV'\,\rho(\vec{r}')\,y'^2 &= \tfrac{1}{2}(I_3+I_1-I_2)\;, \\ \int_V dV'\,\rho(\vec{r}')\,z'^2 &= \tfrac{1}{2}(I_1+I_2-I_3)\;. \end{aligned} \tag{2.13}$$

With this information, we next note that the coefficients of these three integrals in Eq. (2.7) are each of the form of a second-order Legendre polynomial, $P_2(\eta)$, for various different choices of the variable η. In particular we can re-write the second-order contribution to the field, i.e., the term involving G/r^3 as follows:

$$\begin{aligned} &-\frac{G}{r^3}\int_V dV'\,\rho(\vec{r}')(r')^2 P_2(\cos\psi) = -\frac{G}{r^3}\int_V dV'\,\rho(\vec{r}')\,[\text{the 6 terms shown in Eq. (2.9).}] \\ &\quad = -\frac{G}{r^3}\Bigg\{P_2(x/r)\int_V dV'\,\rho(\vec{r}')\,x'^2 + P_2(y/r)\int_V dV'\,\rho(\vec{r}')\,y'^2 + P_2(z/r)\int_V dV'\,\rho(\vec{r}')\,z'^2 \\ &\quad + \frac{3}{2}xy\int_V dV'\,\rho(\vec{r}')\,x'y' + \frac{3}{2}zy\int_V dV'\,\rho(\vec{r}')\,z'y' + \frac{3}{2}xz\int_V dV'\,\rho(\vec{r}')\,x'z'\Bigg\} \\ &\quad . \end{aligned} \tag{2.14}$$

As already noted there are two quite different types of integrals above, and the ones involving squares of coordinates may be re-written in terms of the moments of inertia given above. I now want to insert the values in terms of the three moments of inertia for those first three integrals, which gives us the following, and I have, so far, ignored the other three terms:

$$
\begin{gathered}
-\frac{G}{r^3}\left\{P_2(x/r)\int_V dV'\,\rho(\vec{r}\,')\,x'^2 + P_2(y/r)\int_V dV'\,\rho(\vec{r}\,')\,y'^2 + P_2(z/r)\int_V dV'\,\rho(\vec{r}\,')\,z'^2\right\} \\
= -\frac{G}{2r^3}\left[P_2(x/r)(I_2+I_3-I_1) + P_2(y/r)(I_3+I_1-I_2) + P_2(z/r)(I_1+I_2-I_3)\right]\ ; \\
x/r = \sin\theta\,\cos\varphi\ , \quad y/r = \sin\theta\,\sin\varphi\ , \quad z/r = \cos\theta\ ,
\end{gathered}
\tag{2.15}
$$

where I have explicitly inserted the values of the 3 arguments of the 3 Legendre polynomials in terms of the usual spherical coordinates.

The other three terms are somewhat more complicated, but are determined by something called the *inertia tensor*, which we will discuss later in the course, in Ch. 10. There we will show that one can always choose directions for the Cartesian axes of the mass distribution so that those three additional integrals always vanish. However, at the moment, while everything so far has been completely general, now we will simplify things substantially by looking only at the special case when the mass under study has an axis of cylindrical symmetry, which will make the choice of proper direction for the Cartesian axes very obvious.

III. Fields of Cylindrically-Symmetric Objects

An axis of symmetry is one such that if we rotate the matter around that axis, we see no change in its distribution. Obviously the $\hat{z}$-axis used above for an annular ring is an example of such an axis; again any axis at all, through its origin, is an axis of symmetry for our spherical shell. More generally this is an axis that makes cylindrical coordinates seem appropriate, and we will then choose the $\hat{z}$-axis along that axis of symmetry. In these coordinates a rotation about the axis of symmetry changes the values for the azimuthal angle ϕ. Since the distribution is unchanged under this rotation, it follows that the density cannot depend on that angle; i.e., ρ does not depend on ϕ', but, at most, on the other cylindrical coordinates s' and z'. This fact makes our three troublesome integrals vanish, and also makes two others equal. To show this we need to re-write the integrals one more time, using cylindrical coordinates so that we can make explicit what the dependence is on ϕ. We will use the relationship between Cartesian and cylindrical coordinates:

$$x = s\cos\phi\ , \quad y = s\sin\phi\ , \quad z = z\ . \tag{3.1}$$

Then let's first re-write one of the integrals we want to vanish:

$$\int_V dV'\,\rho(s',z')\left(3\frac{xy}{r^2}x'y'\right) = 3\frac{xy}{r^2}\int s'\,ds'\,\int dz'\rho(s',z')s'^2\int_0^{2\pi} d\phi'\,\cos\phi'\,\sin\phi' = 0\ . \tag{3.2}$$

Because the mass density did not depend on ϕ' we could go ahead and calculate the $d\phi'$ integral, not knowing anything more about the density. But $\int_0^{2\pi} d\phi'\,\sin\phi'\,\cos\phi'$ is simply zero. By the same token we can see that the two other integrals of this same nature will involve

$$\int_0^{2\pi} d\phi'\,\cos\phi' = 0 = \int_0^{2\pi} d\phi'\,\sin\phi'\ .$$

Therefore those 3 integrals are zero, in this situation of cylindrical symmetry. Now, let us look at little bit more at the three moments of inertia as defined in Eqs. (2.12) under these conditions of cylindrical symmetry:

$$\begin{aligned}
I_1 \equiv I_{\hat{x}} = &\int_V dV'\,\rho(\vec{r}\,')\,(y'^2+z'^2)\\
&= \int_V s'\,ds'\,dz'\,\rho(s',z)\,\int_0^{2\pi} d\phi'\,\left(s'^2\sin^2\phi' + z'^2\right)\\
&\quad = 2\pi\int_V s'\,ds'\,dz'\,\rho(s',z')\,(\tfrac{1}{2}s'^2+z'^2)\ ;\\
I_2 \equiv I_{\hat{y}} = &\int_V dV'\,\rho(\vec{r}\,')\,(x'^2+z'^2)\\
&= \int_V s'\,ds'\,dz'\,\rho(s',z')\,\int_0^{2\pi} d\phi'\,\left(s'^2\cos^2\phi' + z'^2\right)\\
&\quad = 2\pi\int_V s'\,ds'\,dz'\,\rho(s',z')\,(\tfrac{1}{2}s'^2+z'^2)\ ,\\
I_3 \equiv I_{\hat{z}} = &\int_V dV'\,\rho(\vec{r}\,')\,(x'^2+y'^2)\\
&= \int_V s'\,ds'\,dz'\,\rho(s',z')\,s'^2\int_0^{2\pi} d\phi'\\
&\quad = 2\pi\int_V s'\,ds'\,dz'\,\rho(s',z')\,s'^2\ .
\end{aligned}\tag{3.3}$$

We see that our cylindrical symmetry requirement tells us that $I_1 = I_2$; this is not surprising since the difference between x and y only amounts to a difference in where one sets an initial value for ϕ and we have required our mass distribution not to depend on that.

Therefore we may simplify the form of the second-order contributions to the gravitational field, as given in Eq. (2.15), in the case of cylindrical symmetry

$$\begin{aligned} &-\frac{G}{2r^3}\left[P_2(x/r)I_3 + P_2(y/r)I_3 + P_2(z/r)(2I_1 - I_3)\right] \\ &\qquad = -\frac{G}{2r^3}\left[P_2(x/r)I_3 + P_2(y/r)I_3 + P_2(z/r)I_3 + 2P_2(z/r)(I_1 - I_3)\right] \ , \end{aligned} \tag{3.4}$$

where I have added and subtracted terms involving I_3 so as to obtain a sum of three terms involving Legendre polynomials of 3 different arguments, which I now show is zero:

$$\begin{aligned} P_2(x/r) + P_2(y/r) + P_2(z/r) &= \frac{1}{2}\left[3(x/r)^2 - 1 + 3(y/r)^2 - 1 + 3(z/r)^2 - 1\right] \\ &= \frac{1}{2}\left[3\frac{x^2+y^2+z^2}{r^2} - 3\right] = 0 \ . \end{aligned} \tag{3.5}$$

Choosing our origin at the center of mass of our gravitating body, which we have assumed is cylindrically symmetrical, the entire equation for the gravitational field it produces is then given by

$$\mathcal{G}(\vec{r}) = -G\left\{\frac{M}{r} + \frac{I_1 - I_3}{r^3}P_2(\cos\theta) + O(1/r^4)\right\} \ , \tag{3.6}$$

where I have rewritten z/r in terms of the angle θ, used in spherical coordinates.

We can make this equation appear somewhat more physically-motivated by introducing the so-called *oblateness parameter* or *deformation parameter*, which measures the difference between the two remaining moments of inertia. Were our object to have a mass distribution that was spherically symmetric, then the two moments would be the same, since any axis would look the same as any other, and of course there would be no higher-order terms in the gravitational field. So this parameter is the very first parameter deviations from sphericity for cylindrically-symmetric objects. To understand the definition it is worthwhile to first evaluate these two moments. The moment I_3 is a mass-weighted measure of distance away from the $\hat{z}$-axis, i.e., a mass-weighted sum of locations $x^2 + y^2$, so that we could say that it concerns itself with what one might call the *equatorial radius*, $R_{_E}$, at least on the Earth. On the other hand the moment I_1 is concerned with summing up values of, say, x^2+z^2; on the Earth we can visualize that as the circumference of a circle that goes along some line of constant longitude, i.e., a line of constant ϕ, say through Albuquerque, up to the North pole, down around the other side of the Earth to the South Pole, and back up to Albuquerque again. Call that an azimuthal radius, and denote it by $R_{_P}$ since it is a measurement of the radius of a circle that goes through the poles. If we knew the object had a constant mass density, then it would be simply a mathematical calculation to determine these two moments, which would in fact give us

$$I_3 = \tfrac{2}{5}MR_{_E}^2\ , \qquad I_1 = I_2 = \tfrac{1}{5}M(R_{_E}^2 + R_{_P}^2)\ , \quad \text{for constant mass density.} \tag{3.7}$$

For the Earth these two radii are $R_{_E} = 6378.20$ km and $R_{_P} = 6356.79$ km. Therefore, we now define two different versions of the oblateness parameter:

$$\epsilon \equiv \frac{I_3 - I_1}{I_3} = 1 - \frac{I_1}{I_3}\ , \qquad \epsilon' \equiv \frac{R_{_E}^2 - R_{_P}^2}{2\,R_{_E}^2}\ , \tag{3.8}$$

where ϵ' is the value that the parameter, ϵ, would have if its density were constant. For the Earth, for instance, we may calculate that $\epsilon' = 1/298.41$. On the other hand, as we will see soon, one can experimentally measure ϵ for the Earth, using this gravitational field data, and it is found that $\epsilon = 1/367$, which is of course completely reasonable since we know that the mass density of the earth is surely not uniform, although there is good reason to believe that it is pretty close to being cylindrically symmetric.

We see then how one might measure this parameter when we insert it into our equation for the gravitational field, Eq. (3.6), which now looks like

$$\mathcal{G}(\vec{r}) = -\frac{MG}{r}\left\{1 - \frac{2}{5}\epsilon\left(\frac{R_{_E}}{r}\right)^2 P_2(\cos\theta) + O[(R_{_E}/r)^3]\right\}\ . \tag{3.9}$$

IV. Use of the Series Expansion to Determine the Field Everywhere

We have so far simply talked about specific ways to determine the exact coefficients of the first three terms of the expansion—the most important terms usually, to be sure. However, in principle it would be rather nicer if one could determine all of the coefficients of $(r'/r)^\ell$, for all values of ℓ. In cases with azimuthal symmetry there are sometimes quite interesting and straightforward ways to find all of them. To see this let us now re-write Eq. (2.6) slightly, using spherical coordinates to describe things:

$$\mathcal{G}(r,\theta,\varphi) \equiv \mathcal{G}(\vec{r}) = -\frac{G}{r}\sum_{\ell=0}^{\infty}\frac{1}{r^\ell}\int_V dV'\,\rho(\vec{r}\,')\,r'^\ell\,P_\ell(\cos\psi)\ . \tag{4.1}$$

The integral which is the coefficient of $r^{-\ell}$ depends on the field point only through the dependence of $\cos\psi$ on that field point. In general, a good geometrical picture would allow you to describe each vector by its spherical angles, and then to show how to put them together to determine the angle between the two vectors, which is given by

$$\cos\psi = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\varphi-\varphi')\ , \tag{4.2}$$

so that in fact the integral in question does actually depend on both θ and φ. However, in the case of cylindrical symmetry, we know that the density does not depend on φ'. Therefore, we can see that the final result, $\mathcal{G}(\vec{r}) = \mathcal{G}(r,\theta,\varphi)$, also cannot depend on φ by the following scheme. Suppose we determine the value of $\mathcal{G}$ for some particular value of (r,θ,φ). Then we move our observation point to a different value of φ, maintaining the same values for r and θ, which amounts to a rotation about the $\hat{z}$-axis, the axis of symmetry of the matter distribution. Now, since the matter distribution is invariant under rotations of it, we rotate the matter distribution by the same amount; nothing can have changed in it, but the relation of the field point to the matter is now the same as it was before the rotation. Therefore the field, $\mathcal{G}$, also cannot have changed. This is then the statement that for a cylindrically-symmetric matter distribution its gravitational field at some arbitrary point outside it cannot depend on the values of ϕ, so that in fact we have only that $\mathcal{G}$ depends on r and θ. How to find that dependence? Notice that in the calculations we did above, for the coefficients of $1/r$, $1/r^2$, and $1/r^3$, the final dependence on θ—in the cylindrically-symmetric case—were simply $P_0(\eta) = 1$, $P_1(\eta) = \eta = \cos\theta$, and $P_2(\eta) = \frac{1}{2}(3\cos^2\theta - 1)$, i.e., we found that the integral

$$\int_V dV'\,\rho(\vec{r}\,')\,r'^\ell\,P_\ell(\cos\psi) = P_\ell(\cos\theta)\,M\,J_\ell\ , \tag{4.3}$$

where J_ℓ was a constant depending completely on the distribution of the matter, but not on the location of the field point, $\vec{r}$. Again, for the three cases we have studied, we found that $J_0 = 1$, $J_1 = R_{\rm cm}$ and $J_2 = (I_1 - I_3)/M$. The statement is in fact true in general, although I will, again, not provide quite the complete proof of that. Instead, let us consider how to determine them all through a clever trick, at least for sufficiently simple gravitating masses.

We begin this undertaking by going back and looking at the very simplest case we studied earlier, the annular ring of radius a. Our discussion above tells us that there are some-yet-unknown coefficients, depending on the mass distribution and shape of our ring, such that at field points outside of it the gravitational field is given by

$$\mathcal{G}(r,\theta) = -\frac{GM}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} J_\ell\, P_\ell(\cos\theta) \ . \tag{4.4}$$

The problem is only that we do not quite know what are the values of the various quantities, J_ℓ, beyond the first 3 values for ℓ. However, if we evaluate it at $\theta = 0$, which puts the field point along the positive $\hat{z}$-axis, so that we may also substitute z for r, we have

$$\mathcal{G}(r,0) = -\frac{GM}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} J_\ell \ , \tag{4.5}$$

since $P_\ell(1 = \cos 0) = 1$. However, earlier, by quite a different scheme we were able to determine the values for the gravitational field of the annular ring along the $\hat{z}$-axis, noted in Eq. (1.8b). We now use the binomial theorem to expand this form in powers of $1/z$, which gives

$$\begin{aligned}\mathcal{G}(z\hat{z}) = \; & -\frac{GM}{z}\left[1+(a/z)^2\right]^{-1/2} = -\frac{GM}{z}\left[1-\tfrac{1}{2}(a/z)^2+\tfrac{3}{8}(a/z)^4-\tfrac{5}{16}(a/z)^6+\ \ldots\,\right] \\ & = -\frac{GM}{z}\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2m-1)}{2^m}\left(\frac{a}{z}\right)^{2m}\right] \ .\end{aligned} \tag{4.6}$$

From this sum we can read off the values of the various coefficients J_ℓ for the annular ring, for all values of ℓ:

$$J_\ell = \begin{cases} 1 & , \quad \ell = 0 \ , \\ 0 & , \quad \ell = 2m+1\,, \quad \text{an odd integer}, \\ \dfrac{(-1)^m}{m!}\dfrac{1\cdot 3\cdot 5\cdot\ldots\cdot(2m-1)}{2^m}a^{2m} & , \quad \ell = 2m\,, \quad \text{an even integer.}\end{cases} \tag{4.7}$$

We may now insert these values back into the general equation, with Legendre polynomials, Eq. (4.5), to find the exact expression for the gravitational field of our annular ring, for distances away from it larger than its own radius and in any direction, albeit as an infinite series:

$$\mathcal{G}(r,\theta) = -\frac{GM}{r}\left\{1+\sum_{m=1}^{\infty}(-1)^m\left(\frac{a'}{r}\right)^{2m}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2m-1)}{2^m\,m!}P_{2m}(\cos\theta)\right\}\ ,\ a<r\ . \tag{4.8}$$

V. Response of the Surface of the Earth to its own Field, and its Rotation

As, I trust, a particular very interesting application of the calculations made above in Section III, let us understand what is the shape of the surface of our Earth. We first consider a small mass on the surface of the Earth, rotating with it. It feels the gravitational attraction just calculated, and also a repulsive centrifugal force, so that it feels a total acceleration $\vec{a}(\vec{r})$ generated by a "potential" of the form

$$\mathcal{G}^*(\vec{r}) = \mathcal{G}(\vec{r}) - \tfrac{1}{2}\Omega^2 r\,\sin^2\theta\ , \tag{5.1}$$

where the last term is the potential for the centrifugal force. Since the centrifugal force is somewhat important in all this discussion it is quite useful to define another dimensionless parameter that gives us an appropriate feel for how important it is, relative to just the simple gravitational force due to a perfectly spherical, non-rotating Earth. Therefore we define

$$\lambda \equiv \frac{\omega^2 R_{_E}}{GM/R_{_E}^2} \approx 1/(290.7)\ , \tag{5.2}$$

which is the ratio of the centrifugal force contribution where it is strongest, at the equator, to the purely gravitational force, also at the equator. We see that it is of the same order of magnitude as the contribution due to the lack of sphericity of the Earth, so that it is important to keep both of them in any problem where we want to find the effects of either one:

$$\mathcal{G}^*(\vec{r}) = -\frac{GM}{r}\left\{1-\tfrac{2}{5}\epsilon\left(\frac{R_{_E}}{r}\right)^2 P_2(\cos\theta)+\tfrac{1}{2}\lambda\left(\frac{r}{R_{_E}}\right)^3\sin^2\theta+O(1/r^3)\right\}\ . \tag{5.3}$$

We see that the two additional terms have different dependence on both r and on θ, so that we ought surely to be able to measure both contributions, by measurements at enough locations. On the other hand, of course, this is just a potential; it is really the effect of the field that we measure, which is just the acceleration, more or less downward, that is easily observed. Therefore we should take the negative gradient of our potential, to determine the locally-measured acceleration on our Earth at an arbitrary location, denoted by r and θ, where we use $g_{_E}$ to denote the usual acceleration that we would have if we were located at sea level on a spherical, non-rotating Earth:

$$\vec{a}(\vec{r}) = -\nabla\mathcal{G}^*(\vec{r}) = \; -\, g_{_E}\left\{ \left[\left(\frac{R_{_E}}{r}\right)^2 - \tfrac{6}{5}\epsilon \left(\frac{R_{_E}}{r}\right)^4 P_2(\cos\theta) - \lambda \frac{r}{R_{_E}}\, \sin^2\theta \right] \hat{r} \right. \\ \left. + \sin\theta\,\cos\theta\, \left[\lambda\frac{r}{R_{_E}} + \tfrac{6}{5}\epsilon \left(\frac{R_{_E}}{r}\right)^4 \right] \hat{\theta} \right\}\,, \quad g_{_E} \equiv \frac{GM}{R_{_E}^2}\,. \tag{5.4}$$

We would like to use this to measure both ϵ and also ϵ', to get an idea how different they are from each other, assuming, as was true not all that long ago, that we did not have really precise numbers for the radii of the earth. We will need two equations to do this, as we have a good knowledge of λ. We first determine the difference in the accelerations at the poles and at the equator, using Eq. (4.4) just above. That acceleration has no tangential component at either place, making the calculation somewhat simpler. We simply determine the ratio of the difference of the radial components over the acceleration denoted by $g_{_E}$:

$$\frac{\Delta a}{g_{_E}} = \frac{a_r(\theta = 0^\circ) - a_r(\theta = 90^\circ)}{g_{_E}} = \left[-\left(\frac{R_{_E}}{R_{_P}}\right)^2 + \tfrac{6}{5}\epsilon\left(\frac{R_{_E}}{R_{_P}}\right)^4 \right] - \left[-1 - \tfrac{3}{5}\epsilon + \lambda \right]\,. \tag{5.5a}$$

We go ahead and simplify this difference; however, it is clear that it involves the ratio of $(R_{_E}/R_{_P})^2 = 1/(1-2\epsilon')$; therefore, I simplify it further by replacing this ratio of the radii by just the small parameter ϵ', which gives me

$$\frac{\Delta a}{g_{_E}} = \frac{-2\epsilon' + \frac{9}{5}\epsilon - \frac{6}{5}\epsilon\epsilon'}{1 - 2\epsilon'} - \lambda \;\approx\; \tfrac{9}{5}\epsilon - 2\epsilon' - \lambda + O(\epsilon\epsilon') \approx \frac{1}{190}\,, \tag{5.5b}$$

where in the last term I have approximated the result to lowest order in these 3 small quantities, and then also inserted the measured value. Although we are using this to devise a scheme to measure these numbers, we can already note that the measured numbers I have told you tell us that the earth cannot be exactly spherical, nor even oblate with constant density.

To determine the second equation, we have to think about the fact that the earth's surface moves fairly easily, at least on the scale of hundreds of thousands of years; therefore, since there are these tangential forces caused by the rotation, the surface must have already moved in such a way as to ensure that those forces vanish. A different phraseology is that the Earth's surface must be an equipotential surface, i.e., the surface must now be in such a position that the potential, $\mathcal{G}^*(\vec{r})$, has the same value at every point on the surface. We can determine this constant value by evaluating the potential anywhere: the pole has the advantage there is no centrifugal force there, so that we write the following, where we again replace the ratio R_E/R_P by its value in terms of ϵ':

$$\mathcal{G}^*(\theta = 0) = -\frac{GM}{R_P}\left\{1 - \tfrac{2}{5}\epsilon\left(\frac{R_E}{R_P}\right)^2\right\} = -\frac{GM}{R_P}\left\{\frac{1 - 2\epsilon' - \frac{2}{5}\epsilon}{1 - 2\epsilon'}\right\} \equiv -\frac{GM}{R_P}C; . \tag{5.6}$$

Given this value we can set it equal to the value of the potential anywhere, which gives us **an equation for the shape of the Earth**, giving r as a function of θ, and of course all invariant under changes of ϕ, since it is being assumed to be cylindrically symmetric:

$$\frac{R_P}{r}\left\{1 - \tfrac{2}{5}\epsilon\left(\frac{R_E}{r}\right)^2 P_2(\cos\theta) + \tfrac{1}{2}\lambda\frac{R_P}{R_E}\left(\frac{r}{R_E}\right)^2\sin^2\theta\right\} = C = \frac{1 - 2\epsilon' - \frac{2}{5}\epsilon}{1 - 2\epsilon'} = 0.9989\ , \tag{5.7}$$

where of course the numerically-measured values of ϵ, ϵ' and λ must be inserted, and those values determined below—from our two equations—then give the numerical result shown. On the other hand, to finish up our discussion of how to measure ϵ and e', we now evaluate this equation at the equator, inserting the relationship between R_P/R_E and ϵ' given above:

$$\begin{aligned} \sqrt{1 - 2\epsilon'}(1 + \tfrac{1}{5}\epsilon + \tfrac{1}{2}\lambda) &= \frac{1 - 2\epsilon' - \frac{2}{5}\epsilon}{1 - 2\epsilon'} \\ \Longrightarrow \quad 1 + \tfrac{1}{5}\epsilon + \tfrac{1}{2}\lambda &= \frac{1 - 2\epsilon' - \frac{2}{5}\epsilon}{(1 - 2\epsilon')^{3/2}} = 1 + \epsilon' - \tfrac{2}{5}\epsilon + O(ee') \\ \Longrightarrow \quad \epsilon' - \tfrac{3}{5}\epsilon - \tfrac{1}{2}\lambda &= 0\ . \end{aligned} \tag{5.8}$$

With this equation, and the other one, and already knowing λ, we can now compute ϵ' and e'. This was already done first by Newton, whose numbers were not quite as accurate as those today. We acquire from these equations the following:

$$\begin{aligned} \epsilon &= \tfrac{5}{3}(2\lambda + \Delta a/g_E) = \tfrac{5}{3}(2/290 - 1/190) = \frac{1}{367}\ , \\ \epsilon' &= \tfrac{5}{2}\lambda + (\Delta a/g_E) = \tfrac{5}{2}\tfrac{1}{290} - \tfrac{1}{190} = \frac{1}{298}\ . \end{aligned} \tag{5.9}$$

VI. A Few More Properties of the Legendre Polynomials

We recall that the polynomials are defined via the power-series expansion of a particular inverse square root:

$$G(r,\eta) \equiv \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{1+r^2-2r\eta}} \equiv \sum_{\ell=0}^{\infty} r^\ell \, P_\ell(\eta) \; , \qquad r<1 \; , \quad |\eta| \le 1 \; , \tag{6.1}$$

where a function G which has this property of "generating" the desired new set of functions via some known method, as, in this case, a power series expansion, is often referred to as a *generating function.*

The first thing I will use this generating function for is to determine some of the special values of these polynomials; first let's show that the function $P_\ell(\eta)$ is an even or odd function of η according as to whether ℓ is an even or odd integer:

$$\begin{aligned} \sum_{\ell=0}^{\infty} r^\ell P_\ell(-\eta) &= \frac{1}{\sqrt{1+r^2+2r\eta}} = \frac{1}{\sqrt{1+(-r)^2-2(-r)\eta}} \\ &= \sum_{\ell=0}^{\infty}(-r)^\ell P_\ell(+\eta) = \sum_{\ell=0}(+r)^\ell \left[(-1)^\ell P_\ell(\eta)\right] \; , \end{aligned} \tag{6.2a}$$

from which we conclude the desired statement of evenness or oddness:

$$P_\ell(-\eta) = (-1)^\ell P_\ell(+\eta) \; . \tag{6.2b}$$

Next we show that the value of every one of the polynomials is $+1$ when its argument is $+1$; since its argument is the cosine of the usual polar angle (in spherical coordinates) this is the same as saying that they all take the value $+1$ at the North pole:

$$\sum_{\ell=0}^{\infty} r^\ell P_\ell(+1) = \frac{1}{\sqrt{1+r^2-2r}} = \frac{1}{1-r} = 1 + r + r^2 + r^2 + r^3 + r^4 + r^5 + \; \ldots \quad , \tag{6.3}$$

where of course the generating function is only well-defined, i.e., the series only converges, for values of $r<1$. From this, however, we easily see the desired statement, and then, using the evenness property above, we also have the appropriate values at the South pole:

$$P_\ell(+1) = +1 \; , \qquad P_\ell(-1) = (-1)^\ell \; . \tag{6.4}$$

Next I will derive a so-called *recursion relation*, a very useful relationship between polynomials with different indices, i.e., different values for ℓ. Given the values of the polynomials for two adjacent values of ℓ, it allows you to determine the next one. We begin the derivation by using the derivative, with respect to r, of the generator:

$$\begin{aligned} \frac{\partial}{\partial r}G(r,\eta) = (\eta - r)G^3(r,\eta) \quad &\Longrightarrow \quad (1-2r\eta+r^2)\frac{\partial}{\partial r}G(r,\eta) = (\eta-r)G(r,\eta) \\ \Longrightarrow \quad \sum_{\ell=0}^{\infty}\left[\ell r^{\ell-1} - 2\eta\ell r^{\ell} + \ell r^{\ell+1}\right]P_\ell(\eta) &= \sum_{\ell=0}^{\infty}\left[zr^{\ell} - r^{\ell+1}\right]P_\ell(\eta) \end{aligned} \tag{6.5}$$

We now take the first sum on the left-hand side and re-write it, setting $m \equiv \ell - 1$ and nonetheless executing that sum from $m = 0$ to infinity, since the term where m might have had the value -1 doesn't really count since that term was multiplied by $\ell = 0$; therefore that first term can be re-written as

$$\sum_{\ell=0}^{\infty}\ell r_\ell^{\ell-1}(\eta) = \sum_{m=0}^{\infty}(m+1)r^m P_m(\eta)\ . \tag{6.6a}$$

Next we take the two terms in the equation which involve $r^{\ell+1}$ and bring them together to give a composite term:

$$\begin{aligned} -\sum_{\ell=0}^{\infty} r^{\ell+1}P_\ell(\eta)\ - \sum_{\ell=0}^{\infty}\ell r^{\ell+1}P_\ell(\eta) &= -\sum_{\ell=0}^{\infty}(\ell+1)r^{\ell+1}P_\ell(\eta) \\ &= -\sum_{m=1}^{\infty} m r^m P_{m-1}(\eta) = -\sum_{m=0}^{\infty} m r^m P_{m-1}(\eta)\ , \end{aligned} \tag{6.6b}$$

where we have set $\ell \equiv m - 1$ and in the last equality we have added in the term $mr^m P_{m-1}(\eta)$ for the case $m = 0$ since, after all, it is just zero and I can add it in if I want to. Having done all this two terms still remain which we have not "fooled with"; we take those and simply replace ℓ by m, so that all terms now contain a simple factor r^m, and our sum takes on the form

$$0 = \sum_{m=0}^{\infty} r^m\left[(2m+1)\eta P_m(\eta) - (m+1)P_{m+1}(\eta) - mP_{m-1}(\eta)\right]\ . \tag{6.7}$$

Since, again, this must be true for all values of r, it follows that the coefficient of r^m must vanish for all values of m, which gives us the desired recursion relation:

$$(2\ell+1)\eta P_\ell(\eta) = (\ell+1)P_{\ell+1}(\eta) + \ell P_{\ell-1}(\eta)\ , \qquad \ell \geq 1\ . \tag{6.8}$$

We now recall that it is rather easy to know that $P_0(\eta) = 1$ and $P_1(\eta) = \eta$. Therefore, we may use this equation to generate higher-order ones:

$$
\begin{aligned}
&\text{set } \ell = 1: \;\; 2P_2 = 3\eta P_1 - P_0 = 3\eta\eta - 1 \quad \Longrightarrow \quad P_2 = \tfrac{1}{2}(3\eta^2 - 1)\ , \\
&\text{set } \ell = 2: \;\; 3P_3 = 5\eta P_2 - 2P_1 = 5\eta\frac{3\eta^2 - 1}{2} - 2\eta \quad \Longrightarrow \quad P_3 = \tfrac{1}{2}\eta(5\eta^2 - 3)\ , \\
&\text{set } \ell = 3: \;\; 4P_4 = 7\eta P_3 - 3P_2 = 7\eta^2\frac{5\eta^2 - 3}{2} - 3\frac{3\eta^2 - 1}{2} \quad \Longrightarrow \quad P_4 = \tfrac{1}{8}(35\eta^4 - 30\eta^2 + 3)\ .
\end{aligned}
\tag{6.9}
$$

There are several more interesting, or useful, relationships between these polynomials. I will only note one quite important differential relation and one integral relation, both of which I will simply state without proofs. As a differential relation, I note the standard ordinary differential equation to which these polynomials are a solution:

$$
\left\{(1-\eta^2)\frac{d^2}{d\eta^2} - 2\eta\frac{d}{d\eta} + \ell(\ell+1)\right\} P_\ell(\eta) = 0\ , \qquad \ell \;\text{ an integer.} \tag{6.10}
$$

An integral relation between them is often phrased as “they are orthogonal” which is a statement about integrals of products:

$$
\int_{-1}^{+1} d\eta\, P_\ell(\eta)\, P_m(\eta) = \begin{cases} \quad 0 \quad , & \ell \neq m\ , \\ \dfrac{2}{2\ell+1}, & \ell = m\ . \end{cases} \tag{6.11}
$$