

Physics 304

Exact Motion of an undamped Pendulum with an introduction to Jacobi elliptic functions

I. The physical pendulum

We model the pendulum, as usual, as a point mass, m , on the end of a massless rod of length ℓ , mounted on a frictionless pivot located at the other end of the rod. The pivot is such that the rod can swing only in a single (vertical) plane, but can swing “all the way around,” when it has sufficient initial velocity to do so. This is a one-dimensional system which can be described using only a single generalized coordinate, which we take as the angle, θ , that the rod makes with the vertical direction.

There are several, equivalent ways to write down differential equations that describe the behavior of the pendulum, given two independent initial conditions. A reasonable one is to first determine the kinetic and potential energies, and note that it is a conservative system—no friction in any of the bearings—so that the energy is conserved:

$$E = T + U = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1 - \cos\theta), \quad (1.1)$$

where we have chosen to measure potential energy so that it is zero when the pendulum hangs straight down. A different way would have been to determine the (Newtonian) forces acting on the system, which would have given us the second-order differential equation

$$\ddot{\theta} = -\frac{g}{\ell} \sin\theta \equiv -\omega_0^2 \sin\theta, \quad (1.2)$$

and we have used the combination g/ℓ , which has dimensions of seconds⁻² to define an angular frequency appropriate to the system.

These two equations are related to one another via differentiation of the first to acquire the second, or integration of the second to acquire the first, with an appropriate choice of the constant of integration. It is of course also relevant to note that if we had written down the Lagrangian for the pendulum, using our knowledge of the kinetic and potential energies above, and then

determined the Lagrange equations for the motion, we would also have obtained this same second order differential equation, Eq. (1.2).

To determine the actual time dependence of the motion of the pendulum, we must integrate one of these equations. As the energy equation is already the first integral of the force equation, we begin there. Quite analogous to the questions as to whether a gravitational orbit, around the sun, say, is bounded or unbounded, it turns out to be quite useful to divide the question of the motion of the pendulum into two distinct parts, as noted below. As well, I go ahead and give now a preview of the exact solutions to the equations for those two cases, which involve the elliptic sine function. The function of the remainder of these notes will then be to explain why these are the solutions, and also to provide more information concerning these functions new for you:

- a. Whether the motion has “turning points,” i.e., whether there is a maximum angle to which it moves, stops momentarily there, and then returns back toward smaller angles; this is a bounded motion between two maximum angles, $\pm\theta_{\max}$. In this case the total energy is less than the potential energy at the top, and in fact is simply such that $E = mgl(1 - \cos\theta_{\max})$. The angle is then given by

$$\theta(t) = 2 \sin^{-1}[k \operatorname{sn}(\omega_0 t - \delta; k)] , \quad k^2 \equiv \sin^2(\theta_{\max}/2) = \frac{E}{2mgl} \leq 1 . \quad (1.10)$$

- b. Or whether the motion goes all the way around the pivot, slowing down but never stopping as it comes to where the rod points “straight up,” and then speeding up again as it continues its turning down toward the bottom again; this is an unbounded motion. In this case the energy E is larger than the potential energy at the top, i.e., $E > mgl(1 - \cos\theta)|_{\theta=\pi} = 2mgl$, and we have

$$\sin(\theta/2) = \operatorname{sn}(\omega_0 t/k - \delta; k) , \quad k^2 \equiv 2mgl/E < 1 . \quad (4.3)$$

We will first consider the case of bounded motion. Since it occurs when $E \leq 2mgl$, in this case it is useful to divide the energy by that maximum value it could have, giving us a parameter that is surely less than or equal to 1, and to then re-write the energy equation in the following way:

$$\frac{1}{2}(1 - \cos\theta_{\max}) = \frac{E}{2mgl} = \frac{1}{2}(1 - \cos\theta) + \frac{\ell}{g}(\dot{\theta}/2)^2 . \quad (1.3)$$

To simplify this still more, we use the trigonometric identity for half-angles, and give a name to the half-angle involved:

$$\begin{aligned}
\zeta &\equiv \theta/2, & \sin^2 \zeta &= \sin^2(\theta/2) = \frac{1}{2}(1 - \cos \theta), \\
\frac{E}{2mg\ell} &= \frac{1}{2}(1 - \cos \theta_{\max}) = \sin^2 \zeta_{\max}, \\
\implies \dot{\zeta}^2 &= \omega_0^2(\sin^2 \zeta_{\max} - \sin^2 \zeta), \\
\implies \omega_0 dt &= \pm \frac{d\zeta}{\sqrt{\sin^2 \zeta_{\max} - \sin^2 \zeta}} = \pm \frac{d(\zeta/\sin \zeta_{\max})}{\sqrt{1 - (\sin \zeta/\sin \zeta_{\max})^2}},
\end{aligned} \tag{1.4}$$

where the last equation has been obtained by taking the square root of both sides of the previous line, and then separating the variables in the first-order differential equation. Note that the plus or minus sign in front of the square root is actually the sign of $d\zeta/dt$, since we began with the square of that angular velocity of the pendulum.

The next step is to determine **limits of integration** for these two integrals. This requires using some initial conditions. We of course know that, from our physical understanding of the problem, that it will surely oscillate between the two, symmetric values of its maximum allowed angle, $\pm\zeta_{\max}$, and that it will be momentarily at rest at those two places. Therefore, if we choose to begin measuring time, i.e., to set $t = 0$, when the pendulum is at the very bottom of its swing, i.e., when $\theta = 0$, and to presume that we take the case where it is headed upward toward positive values of θ , and therefore $\zeta = \theta/2$ is also positive. Therefore, we have $d\zeta/dt > 0$, and we should use the positive sign in the plus-or-minus sign above, which gives us the following:

$$\omega_0 t = \omega_0 \int_0^t dt = + \int_0^\zeta \frac{d(\zeta/\sin \zeta_{\max})}{\sqrt{1 - (\sin \zeta/\sin \zeta_{\max})^2}}. \tag{1.5}$$

If we could evaluate this integral, the relationship would give us the time as a function of the angle, which could then be inverted to determine the angle as a function of time; i.e., this would give us the result that was actually wanted. **However, it is usually at this point** that more elementary discussions of the behavior of a pendulum say that we do not know enough mathematics to actually calculate the integral above, but that the integral may be approximated reasonably well, provided we only consider the rather-small energy situations where θ_{\max} is rather small, so that the sine function may be approximated.

The actual purpose of these notes is to provide you with sufficient new mathematics to determine this integral exactly. However, I believe it will nonetheless be quite useful to go ahead from this point and, first, perform this approximation and see how the mathematics goes, onward from this point, in the small-angle approximation; it will be a useful guide for us to follow when looking at the exact case. Therefore, **we now consider the special case** when ζ_{\max} is sufficiently small that $\sin \zeta_{\max} \approx \zeta_{\max}$. This is of course the same as saying that the next term in the Taylor series for the sine function is smaller than our measurement capabilities (or desires), namely that $\frac{1}{6}\zeta_{\max}^3 \ll 1$. Since this is the maximum value for ζ , as involved in the integration, it follows that this would also be true for all the values over which we are integrating, so that we may approximate the sine function throughout the integral by just its argument.

$$\omega_0 t = + \int_0^\zeta \frac{d\zeta/\zeta_{\max}}{\sqrt{1 - (\zeta/\zeta_{\max})^2}} = \int_0^w \frac{dw}{\sqrt{1 - w^2}} = \sin^{-1} w , \quad w \equiv \zeta/\zeta_{\max} = \theta/\theta_{\max} . \quad (1.6)$$

In this approximation of small angles, here then is the above-promised demonstration of the functional form of the time, $t = t(\theta)$, as a function of the angle. As noted there, the next step is to invert this function, giving us

$$\theta = \theta_{\max} \sin(\omega_0 t) , \quad \text{for } \theta_{\max} \text{ sufficiently small.} \quad (1.7)$$

I now want to go backwards in our discussion and take the same approach to determine the exact integral corresponding to Eq. (1.5). Therefore, let us re-formulate that integral to make it look as straightforward as possible:

$$\begin{aligned} u &\equiv \frac{\sin \zeta}{\sin \zeta_{\max}} \quad \text{and define} \quad k \equiv \sin \zeta_{\max} & (1.8) \\ \Rightarrow \quad \frac{d\zeta}{\sin \zeta_{\max}} &= \frac{du}{\cos \zeta} = \frac{du}{\sqrt{1 - \sin^2 \zeta}} = \frac{du}{\sqrt{1 - (\sin \zeta_{\max})^2 u^2}} = \frac{du}{\sqrt{1 - k^2 u^2}} , \\ &\Rightarrow \quad \omega_0 t = \int_0^u \frac{du}{\sqrt{1 - k^2 u^2}} \frac{1}{\sqrt{1 - u^2}} . \end{aligned}$$

This equation is the same as before, except we have changed the names of the integration variables to eliminate the trigonometric functions. Just as did Eq. (1.6) it gives us t as a function of u , which, this time, also depends on a parameter $k \equiv \sin(\theta_{\max}/2)$, which clearly varies from 0 to

1. As before if we knew the functional form of this integral, we could invert it and find u , and therefore θ , as a function of t .

I claim that the answer is that the integral is the *inverse elliptic sine* of u , usually denoted as $\text{sn}^{-1}(u; k)$, since of course it also depends on the value of k . At the moment that is of course just a name; however, let us suppose that in fact it was so, and we use this definition to determine some of the properties of that newly-defined function. Before we do that I will go ahead and use the fact that it is true, and use Maple to make some plots of the exact solution of the question of the motion of the pendulum, still remembering that we are currently considering only the case where there is a maximum angle. Given the definition, we may re-write the last equation of Eq. (1.8) and then resolve it for the desired angle, as a function of time:

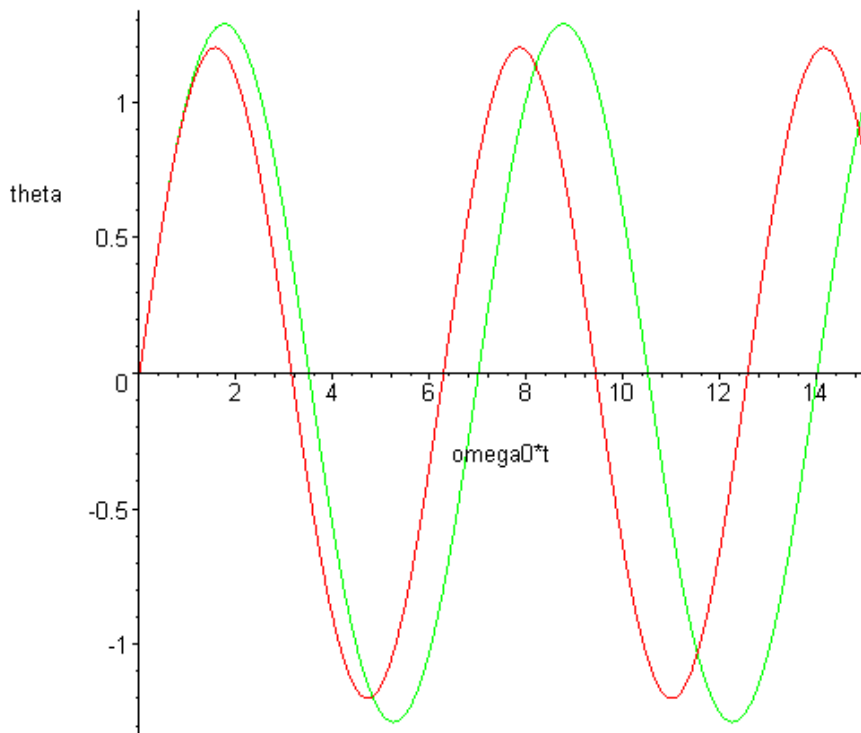
$$\begin{aligned} \omega_0 t = \text{sn}^{-1}(u; k) &\iff u = \text{sn}(\omega_0 t; k), \quad ku \equiv \sin(\theta/2), \quad k \equiv \sin(\theta_{\max}/2), \\ \sin(\theta/2) = k \text{sn}(\omega_0 t; k) &\iff \theta(t) = 2 \sin^{-1}[k \text{sn}(\omega_0 t; k)]. \end{aligned} \tag{1.9}$$

This form has set $t = 0$ when the pendulum hangs straight down. If, instead, straight down occurs at an arbitrary time t_0 , then we must replace t above by $t - t_0$. Setting the constant $\omega_0 t_0 \equiv \delta$, we then have the general form:

$$\theta(t) = 2 \sin^{-1}[k \text{sn}(\omega_0 t - \delta; k)] \quad k^2 \equiv \sin^2(\theta_{\max}/2) = \frac{E}{2mgl} \leq 1. \tag{1.10}$$

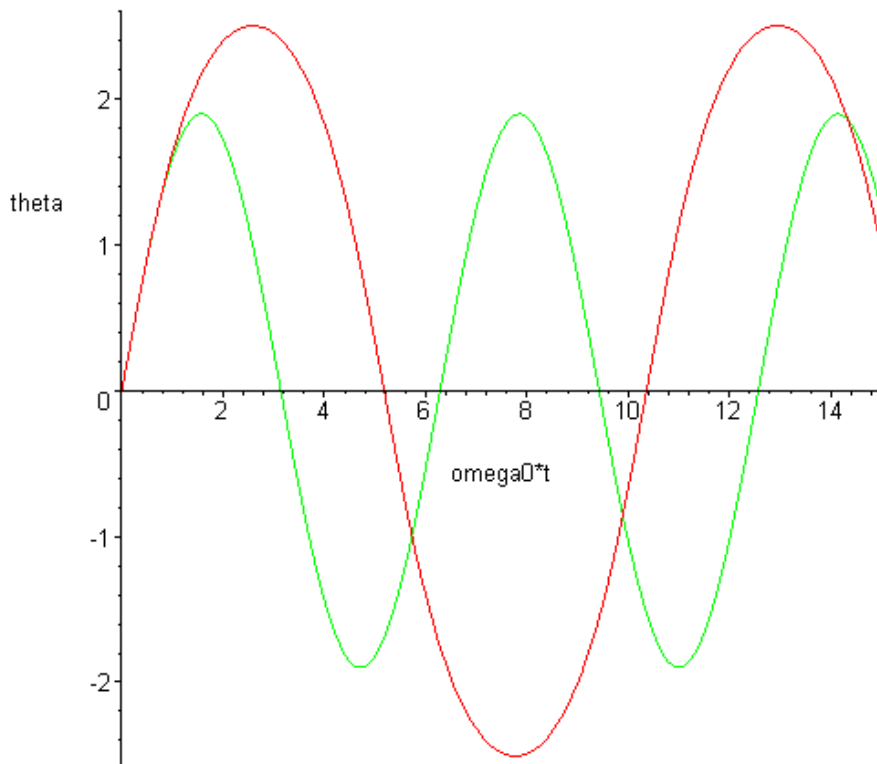
Using the fact that Maple refers to the elliptic sine by the name `JacobiSN(u,k)`, I go ahead and show two plots of the exact solution for $\theta(t)$, given just above by Eq. (1.9), and, on the same plot, the approximate, linearized solution, given by Eq. (1.6).

true pendulum for max angle=74 degrees
 versus linear approx. (the one with shorter period)



Plot shown for $k = 0.6$, which means that $\theta_{\max} = 73.74^\circ = 1.287$ radians.

true pendulum for max angle=143 degrees
 versus linear approx. (the one with shorter period)



Plot shown for $k = 0.95$, which means that $\theta_{\max} = 143.6^\circ = 2.506$ radians.

II. Introduction to the elliptic sine function, as defined by Jacobi

We now want to give some more complete introduction to the properties of the elliptic sine function and the other elliptic functions closely associated with it, all originally defined by Jacobi, in the early to middle 1800's. To do this it is best to first give a simpler name to $\omega_0 t$ for the moment, just calling it y , and we re-write the last line of Eq. (1.8) above as follows:

$$y = \int_0^u \frac{du}{\sqrt{(1-k^2u^2)(1-u^2)}} \equiv \text{sn}^{-1}(u; k) \iff u = \text{sn}(y; k) . \quad (2.1)$$

This definition, via the integral, allows us to determine the differential equation which has the elliptic sine as its solution. However, before doing that, let me note that the value of the elliptic sine depends also on the value of the constant k , which may take on any value between 0 and 1, since it is defined as the sine of half the maximum-allowed angle of rotation:

- a. When $k = 0$ the additional square root reduces to simply the constant value 1, so that the integral is just the integral for the inverse sine, so that the elliptic sine with $k = 0$ is just the ordinary trigonometric sine function; i.e.,

$$\text{sn}(y; 0) = \sin y . \quad (2.2a)$$

- b. Then, when $k = 1$ the two square roots in the denominator are the same, so that $\sqrt{(1-u^2)^2} = 1-u^2$, and our integral is now easily computed, giving the inverse hyperbolic tangent; i.e.,

$$\text{sn}(y; 1) = \tanh y . \quad (2.2b)$$

This allows us to have some mental picture of the family of elliptic functions as an intermediate set that vary between the trigonometric ones and the hyperbolic ones.

However, now let us go ahead and determine that differential equation. Return to Eq. (2.1) and differentiate both sides with respect to the variable u , where I re-write the variable of integration as u' , a more rigorously correct approach, so that we can see that the only u -dependence is in the limit of integration:

$$\frac{dy}{du} = \frac{d}{du} \int_0^u \frac{du'}{\sqrt{(1-u'^2)(1-k^2u'^2)}} = \frac{1}{\sqrt{(1-u^2)(1-k^2u^2)}} . \quad (2.3)$$

Remembering that du/dy is just the reciprocal of dy/du , we may now re-write this as

$$\begin{aligned} \frac{du}{dy} &= \sqrt{(1-u^2)(1-k^2u^2)}, \\ &\text{or, more commonly, as} \\ \left(\frac{du}{dy}\right)^2 &= (1-u^2)(1-k^2u^2), \end{aligned} \tag{2.4}$$

where of course now we are treating u as a function of y . Replacing u with its claimed value, we may write our differential equation:

$$\frac{d}{dy} \operatorname{sn}(y; k) = \sqrt{1 - \operatorname{sn}^2(y; k)} \sqrt{1 - k^2 \operatorname{sn}^2(y; k)}. \tag{2.5}$$

To try to begin to better understand this, remember that when $k = 0$ this is just the usual trigonometric sine function; i.e., it says that

$$\begin{aligned} \text{when } k = 0, \text{ this says } \frac{d}{dy} \sin(y) &= \sqrt{1 - \sin^2(y)} \equiv \cos(y), \\ &\text{and} \\ \text{when } k = 1, \text{ this says } \frac{d}{dy} \tanh(y) &= 1 - \tanh^2(y) = \frac{1}{\cosh^2(y)} \equiv \operatorname{sech}^2(y). \end{aligned} \tag{2.6}$$

where the cosine function, and the hyperbolic cosine function, have been inserted using their usual definitions in terms of the sine function, and the hyperbolic tangent function. Of course these derivatives agree with what we know is the usual derivative for the sine and the tanh functions.

To make the differential equation for the elliptic sine look more like those for the trigonometric and/or hyperbolic functions, I will first define the two elliptic functions that are “associated” with the elliptic sine, in the same way as the cosine is associated with the sine and the sech is associated with the tanh function. We define the elliptic cosine function, denoted $\operatorname{cn}(y; k)$, and also an additional function, denoted by $\operatorname{dn}(y; k)$ such that:

$$\begin{aligned} \operatorname{cn}^2(y; k) &\equiv 1 - \operatorname{sn}^2(y; k), \\ \operatorname{dn}^2(y; k) &\equiv 1 - k^2 \operatorname{sn}^2(y; k), \\ \implies \frac{d}{dy} \operatorname{sn}(y; k) &= \operatorname{cn}(y; k) \operatorname{dn}(y; k), \end{aligned} \tag{2.8}$$

where the last line is just the derivative equation, Eq. (2.5), re-written in terms of these new functions, making that equation now look much more like the other, simpler ones we already

know.

Just like the relationship between the sine and the cosine, since the defining relationships involve the squares of the relevant functions, one must make the proper choice of a plus sign or a minus sign, usually in such a way as to ensure that the functions so defined are continuous. We make this choice by taking the $\text{dn}(y; k)$ function to always be positive, and then fix the sign of $\text{cn}(y; k)$ by insisting that the differential equation above determine the sign. Therefore, for instance, when the pendulum is swinging upwards from the bottom, toward larger, positive values of θ , then the derivative of $\text{sn}(y; k)$ should be positive, and therefore so should be the $\text{cn}(y; k)$. However, on the way back down from that maximum value, the $\text{sn}(y; k)$ is of course still positive, but its derivative is negative, so that the $\text{cn}(y; k)$ should also be negative.

We next note that the definition of the elliptic sine, via the integral in Eq. (2.1), makes it clear that it is *an odd function* of its principal argument. Of course the fact that it is odd makes it clear that it has value zero when its argument is zero; i.e., $\text{sn}(0; k) = 0$. One can of course also see this by looking at the integral definition, since the integral from zero to zero should surely be zero. Since the elliptic cosine is defined in terms of the square of the elliptic sine, it is possible to choose the signs, relative to the square root, so that it is an even function, and its value at zero must be just $+1$. It then follows from the differential equation above that the dn function is also an even function, and from the definition that its value at zero is also $+1$.

Now we want to find out the maximum value it can have. There are two different square roots under the integral sign; the integral only makes sense—at least as a description of the behavior of a pendulum—when the quantities under the integral sign are real-valued. In this case, since $k^2 \leq 1$, the integrand is real only when u is between -1 and $+1$. Since u is actually $\text{sn}(y; k)$, we see that the elliptic sine can in fact only vary between -1 and $+1$, at least when it is real-valued! However, we also know that our pendulum is swinging back and forth in a periodic way; in particular, since we started it at the bottom of its swing, it goes up to its maximum value, which for sn is just $+1$, then swings back down to zero, then goes on up on the other side, where sn is -1 , and then again back to zero; therefore, there are just 4 distinct parts to the swing, and then it repeats. This is just exactly like the 4 quadrants of an angle. However, since the elliptic

sine is an odd function, its behavior as it swings up “the back side,” where the angles are negative, is just the same as when it goes up the positive side, except for the desired minus sign. Then as it comes back down from having gone up to the top, it just comes back from +1 down to zero again. So, there are 4 quadrants here too, and we may think of the period as 4 times the value all the way from the bottom up to the top of the swing. We therefore define a quarter-period, which depends on k of course; it is the value of our integral from 0 up to +1, i.e., up to the top of the swing of the pendulum:

$$K(k) \equiv \int_0^1 \frac{du}{\sqrt{(1-k^2u^2)(1-u^2)}} . \quad (2.10)$$

Remembering that this is just the quarter-period of the (trigonometric) sine function when $k = 0$, we realize that $K(0) = \pi/2$. However, it is not completely clear what happens when $k \rightarrow 1$, since the hyperbolic tangent is not a periodic function. On the other hand, when $k = 1$, it is clear that the integral for $K(1)$ diverges, i.e., it is infinite; more precisely it is in fact $\tanh^{-1}(1) = +\infty$. To acquire more precise behavior, for small k , one can expand the square root in the denominator in powers of k , using the binomial theorem, and then integrate each term; this gives us

$$K(k) = \frac{\pi}{2} \left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + O(k^8) \right] . \quad (2.11)$$

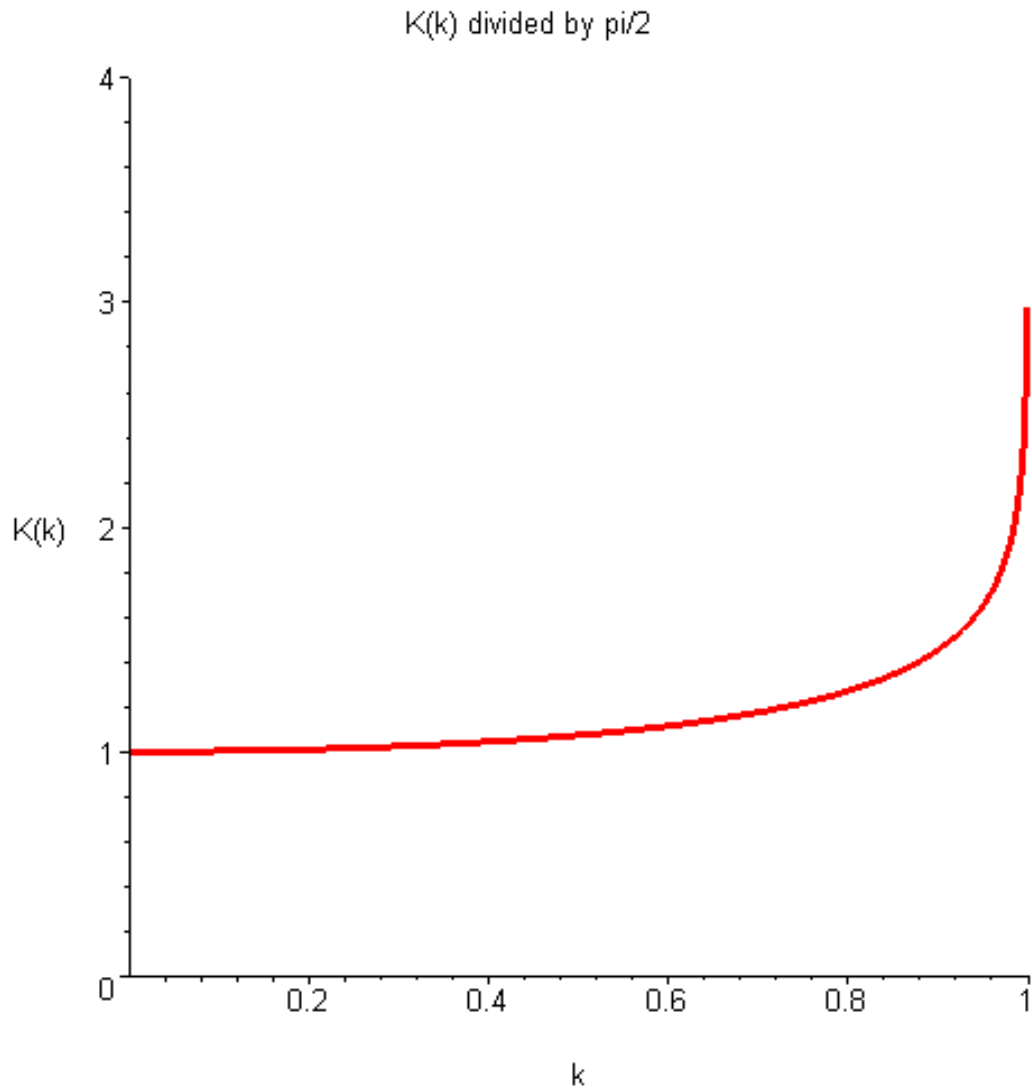
On the other hand, as k approaches 1, it diverges like a logarithm, since \tanh^{-1} and logarithms are closely related. However, to explain in somewhat more detail just how it behaves as $k \rightarrow 1$, it is very useful to create the commonly used *complementary modulus*,

$$k' \equiv \sqrt{1-k^2} . \quad (2.12)$$

It is clear that as k approaches 1, the complementary modulus, k' approaches 0. Then we can express the behavior of $K(k)$ as k approaches 1 by giving it as a power series in k' , which is then approaching zero. The answer is that

$$K(k) = \ln(4/k') \left[1 + \frac{1}{4}k'^2 + \frac{9}{64}k'^4 + \dots \right] - \frac{1}{4}k'^2 - \frac{13}{128}k'^4 + O(k'^6) . \quad (2.13)$$

The curve below presents a more graphical answer to the question as to how the quarter-period $K(k)$ changes with $k = \sin(\theta_{\max}/2)$, where $K(k)/(\pi/2)$ is plotted:



This allows us to see that, **yes**, the period stays close to constant for quite a little while as the maximum angle increases, still staying small; on the other hand, as the maximum angle approaches π , i.e., the pendulum is becoming almost straight up, the time required to get to that maximum angle is approaching infinity.

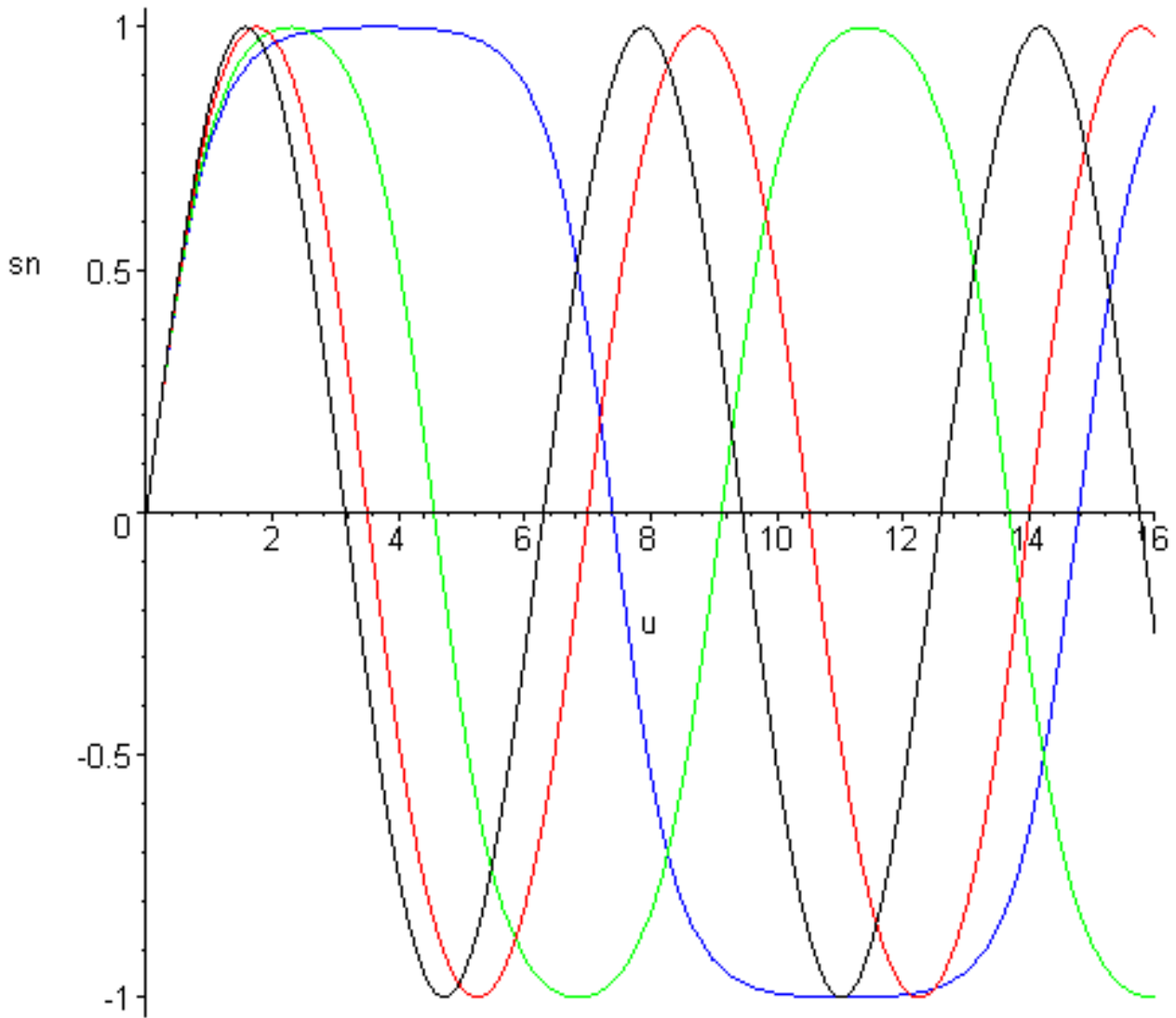
Quite a different approach to understanding the behavior of the period is to go into a computer algebra program, and look at its behavior there; in Maple this function is referred to as `EllipticK(k)`, which is of course where the graph shown was obtained.

Having understood the behavior of the quarter period one should then look for the behavior of the various elliptic functions with that quarter period. Thinking back to the behavior of the physical pendulum, we know that it begins with value 0 at its argument, $\omega_0 t$ is zero; when its

argument is $K(k)$ then it has value $+1$; when its argument has increased onward to $2K(k)$ then the elliptic sine has value zero again; when its argument is $3K(k)$ then its argument is -1 , while when it gets to $4K(k)$ then it has again gone back to zero. As $4K(k)$ is the full period, it then repeats over and over again. We can summarize these few points in the equations below, and then we give a graph for 4 choices of k , namely for $k = 0.1$, $k = 0.6$, $k = 0.9$, and, lastly, for $k = 0.995$. Notice that the period increases along with k , although rather slowly until k becomes rather close to 1; however, in addition, notice that as the period increases the function becomes flatter and flatter when it is approaching its maximum (or minimum) possible value of 1 (or -1).

$$\begin{aligned}
 \operatorname{sn}(0; k) = 0, \quad \operatorname{sn}(K; k) = +1, \quad \operatorname{sn}(2K; k) = 0, \quad \operatorname{sn}(4K; k) = 0, \\
 \operatorname{sn}(y + 4K; k) = \operatorname{sn}(y; k).
 \end{aligned}
 \tag{2.14}$$

sn(u;k)
for k=0.1, 0.6, 0.9, 0.995



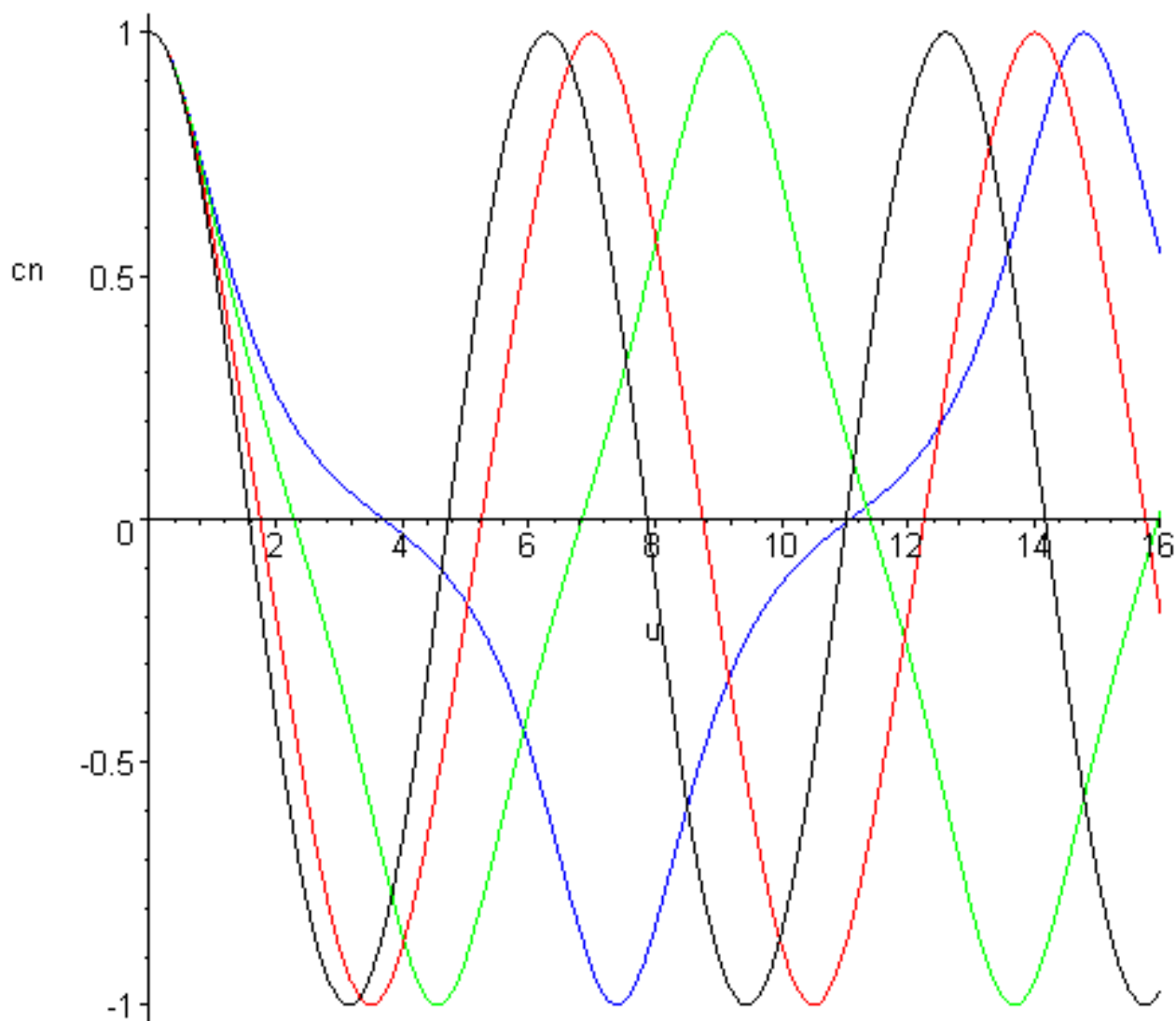
One should also have power series expansions for small arguments, just as one does for the usual trigonometric and hyperbolic functions, as well as series expansions in k , showing the deviations from the (circular) functions, and also series expansions in k' , showing the deviations from the hyperbolic functions:

$$\begin{aligned}
 \operatorname{sn}(y; k) &= y - \frac{1}{3!}(1 + k^2)y^3 + \frac{1}{5!}(1 + 14k^2 + k^4)y^5 + O(y^7) , \\
 \operatorname{sn}(y; k) &= \sin y - \frac{1}{4}k^2(y - \sin y \cos y) \cos y + O(k^4) , \\
 \operatorname{sn}(y; k) &= \tanh y + \frac{1}{4}k'^2(\sinh y \cosh y - y) \operatorname{sech}^2 y + O(k'^4) .
 \end{aligned}
 \tag{2.15}$$

Using their definitions above, we can also now discuss the other two elliptic functions. In particular they clearly all have $4K(k)$ as a complete period. However, while $\text{sn}(2K; k)$ is again zero as it was when its argument was zero, $\text{cn}(0; k) = +1$ while $\text{cn}(2K; k) = -1$, which we know from the derivative equation. Therefore here are some typical values for $\text{cn}(y; k)$, and, below, a set of graphs for the same values of k as before, namely $k = 0.1$, $k = 0.6$, $k = 0.9$, and $k = 0.995$.

$$\begin{aligned} \text{cn}(0; k) = 1, \quad \text{cn}(K; k) = 0, \quad \text{cn}(2K; k) = -1, \quad \text{cn}(4K; k) = 1, \\ \text{cn}(y + 4K; k) = \text{cn}(y; k). \end{aligned} \tag{2.16}$$

$\text{cn}(u; k)$
for $k=0.1, 0.6, 0.9, 0.995$

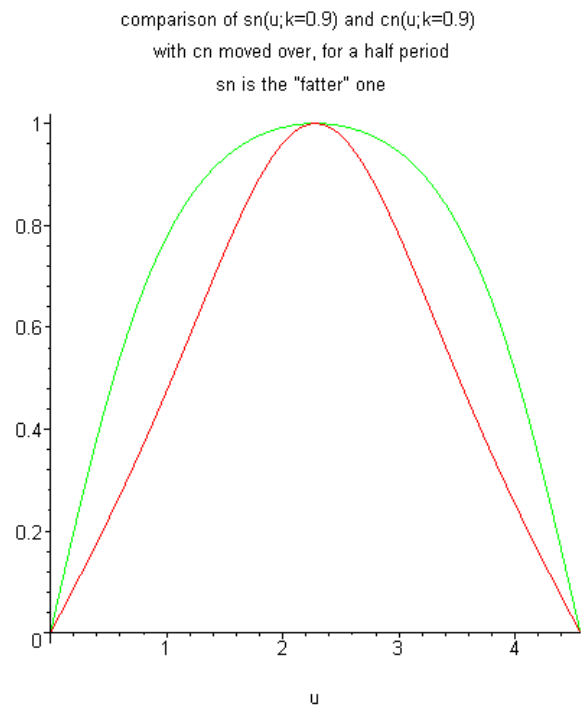
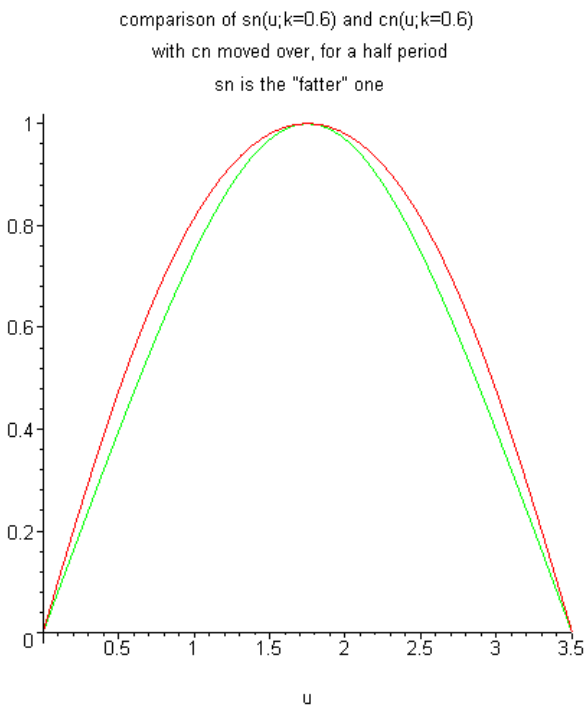


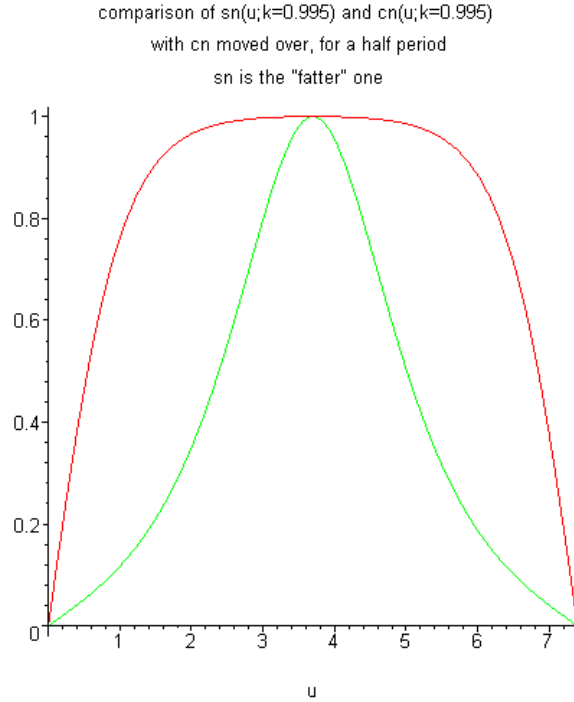
And the power series expansions for small arguments are just the following:

$$\begin{aligned}
 \operatorname{cn}(y; k) &= 1 - \frac{1}{2!}y^2 + \frac{1}{4!}(1 + 4k^2)y^4 - \frac{1}{6!}(1 + 44k^2 + 16k^4)y^6 + O(y^8), \\
 \operatorname{cn}(y; k) &= \cos y + \frac{1}{4}k^2(y - \sin y \cos y) \sin y + O(k^4), \\
 \operatorname{cn}(y; k) &= \operatorname{sech} y - \frac{1}{4}k'^2(\sinh y \cosh y - y) \tanh y \operatorname{sech} y + O(k'^4).
 \end{aligned}
 \tag{2.17}$$

We recall that the shape of the usual trigonometric cosine is really just the same as that of the sine function except that one must change its phase, i.e., $\cos(u - \pi/2) = \sin(u)$. However, this is not the case for their elliptic counterparts; to make this more obvious I show below graphs of $\operatorname{sn}(y; k)$ during its first half period and $\operatorname{cn}(y - K; k)$ over the same range, this time for three

different values of k , namely $k = 0.6$, $k = 0.9$, and, on the next line, for $k = 0.995$:

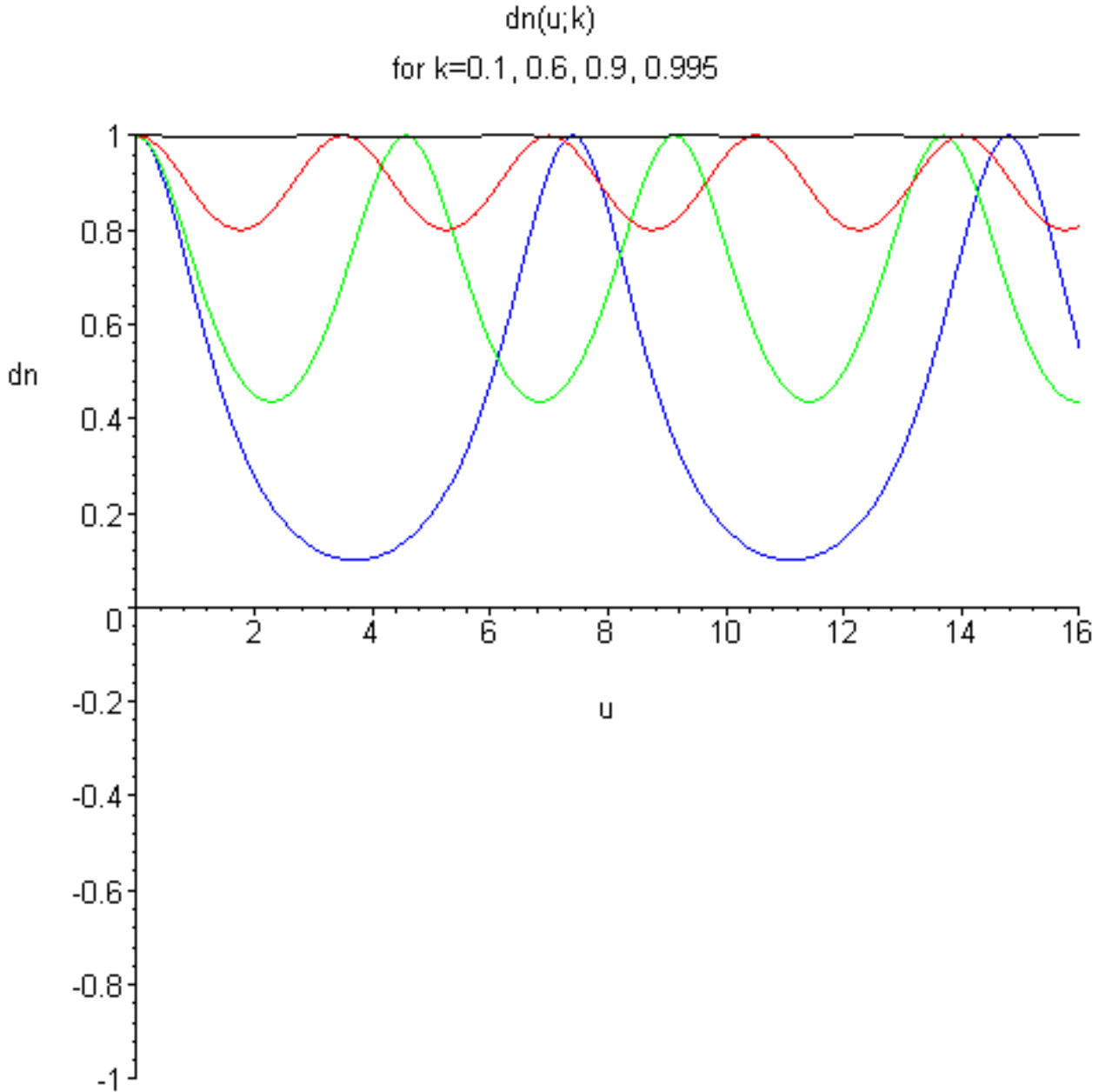




Having already commented that $\text{sn}(y; k)$ became more and more flat near its peaks as k increased, we can see that, contrariwise, $\text{cn}(y; k)$ becomes more and more pointed near its peaks as k increases.

Lastly we should consider the (even) function $\text{dn}(y; k)$, which does not have a very clear analogue among the trigonometric functions—when $k = 0$, $\text{dn}(y; k)$ is just the constant $+1$ —nor among the hyperbolic ones—for $k = 1$, $\text{dn}(y; 1)$ is just the same as $\text{cn}(y; 1)$, which equals $\text{sech}(y)$. Nonetheless, it is needed in this more general case, and we have the following values, including the fact that it actually has a period of $2K$ rather than the $4K$ like the other two—in this way it is like the trigonometric tangent, which also has a period of π rather than the 2π periods for the sine and cosine—as well as the graph shown below:

$$\text{dn}(0; k) = 1, \quad \text{dn}(K; k) = \sqrt{1 - k^2} \equiv k', \quad \text{dn}(2K; k) = 1, \quad \text{dn}(y + 2K; k) = \text{dn}(y; k). \quad (2.18)$$



The graph again shows all 4 values of k ; however, the one for $k = 0.1$ is so close to just the constant value of 1—since it varies between 1 and $\sqrt{1 - .0001}$ —that it cannot easily be distinguished on this graph.

And the power series expansions for small arguments is just the following:

$$\begin{aligned}
 \operatorname{dn}(y; k) &= 1 - \frac{1}{2!}k^2y^2 + \frac{1}{4!}(4 + k^2)k^2y^4 - \frac{1}{6!}(16 + 44k^2 + k^4)k^2y^6 + O(y^8), \\
 \operatorname{dn}(y; k) &= 1 - \frac{1}{2}k^2 \sin^2 y + O(k^4), \\
 \operatorname{dn}(y; k) &= \operatorname{sech} y + \frac{1}{4}k'^2(\sinh y \cosh y + y) \tanh y \operatorname{sech} y + O(k'^4).
 \end{aligned}
 \tag{2.19}$$

III. Differential Equations Relevant for Jacobi Elliptic Functions

With very little effort from our first differential equation, for $\text{sn}(y; k)$, we may also find first-order differential equations satisfied by our other two elliptic functions. As the parameter (or modulus) k stays constant during those differential equations, we simplify the notation below by not writing it:

$$\begin{aligned}\frac{d}{dy} \text{sn } y &= \text{cn } y \text{ dn } y , \\ \frac{d}{dy} \text{cn } y &= -\text{sn } y \text{ dn } y , \\ \frac{d}{dy} \text{dn } y &= -k^2 \text{sn } y \text{ cn } y .\end{aligned}\tag{3.1}$$

It is also worth noting that these are first order equations where the independent variable does not appear explicitly. Therefore the general solution is not, for instance $\text{sn}(y; k)$, but rather $\text{sn}(y - c; k)$ for any constant c , simply exemplifying the fact that there should be a single constant of integration for a first order equation.

The equations above are of course just the derivatives of the three functions; however, just as we used those derivatives to show that the solution to a particular differential equation was the elliptic sine, we can relate each of these to an associated equation involving square roots of polynomials:

$$\begin{aligned}\left(\frac{du}{dy}\right)^2 &= (1 - u^2)(1 - k^2 u^2) \implies u = \text{sn}(y - y_0; k) , \\ \left(\frac{du}{dy}\right)^2 &= (1 - u^2)(1 - k^2 + k^2 u^2) \implies u = \text{cn}(y - y_0; k) , \\ \left(\frac{du}{dy}\right)^2 &= (1 - u^2)(u^2 - 1 + k^2) \implies u = \text{dn}(y - y_0; k) ,\end{aligned}\tag{3.2}$$

where the two lower ones were obtained from the two lower ones in Eqs. (3.1) by re-writing each of the other two elliptic functions on the right-hand side in terms of square roots of the one on the left-hand side.

I can also simply note what are the integrals of each of these three functions; many more are

of course given in tables:

$$\begin{aligned}
k \int dy \operatorname{sn}(y; k) &= \ln[\operatorname{dn}(y; k) - k \operatorname{cn}(y; k)] , \\
k \int dy \operatorname{cn}(y; k) &= \cos^{-1}[\operatorname{dn}(y; k)] , \\
\int dy \operatorname{dn}(y; k) &= \sin^{-1}[\operatorname{sn}(y; k)] .
\end{aligned} \tag{3.3}$$

Another quite useful thing to have for elliptic functions are their addition theorems; i.e., how to evaluate them when the argument is a sum of quantities. I simply quote three of them here from the literature, suppressing the modulus k :

$$\begin{aligned}
\operatorname{sn}(u + v) &= \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{cn} u \operatorname{dn} u \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} , \\
\operatorname{cn}(u + v) &= \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} , \\
\operatorname{dn}(u + v) &= \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} ,
\end{aligned} \tag{3.4}$$

As a very simple example of how they might be used, consider the following:

$$\operatorname{sn}(u - K; k) = \frac{\operatorname{sn} u \operatorname{cn} K \operatorname{dn} K - \operatorname{cn} u \operatorname{dn} u \operatorname{sn} K}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 K} = \frac{-\operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u} = -\frac{\operatorname{cn} u}{\operatorname{dn} u} . \tag{3.5}$$

The physical value of this comes from the fact that we have normalized our elliptic sine so that it vanishes when its argument is zero. However, we might often want to use it to describe a pendulum that was released from rest at some place other than the bottom, i.e., to be released from rest at its maximum angle. This of course is already accounted for in our insertion of δ into the argument of our solution; nonetheless, this is an alternative approach to considering that. In particular if we begin a pendulum from some maximum angle, from rest, then it will take a quarter period to reach the bottom, which means that $\delta = K(k)$ in our equations above. Therefore we could use Eq. (3.5) above to re-write the form so that, in terms of other choices of functions to be sure, we need not concern ourselves with δ . [Recall that this is the same thing we do when we consider trigonometric functions that describe oscillations. We use sine if the oscillator begins from equilibrium with a “kick,” but cosine if it begins from rest at some point away from equilibrium.]

IV. The Pendulum for the Case when It goes “over the top”

Remembering back to the original (energy) equation for the motion of the pendulum, we so far have only discussed the case when $E \leq 2mg\ell$. However, when the energy is greater than this the pendulum has enough total energy to swing over the top of its pivot point and come back down the other side, coming down to the bottom, swinging onward still, back up and over the top, and, therefore, round and round and round, slowing as it goes over the top and speeding up as it comes back down toward the bottom. In this case there is no maximum angle, and the details of the analysis above do not work; however, it is not very difficult to re-perform the calculations above and obtain the correct answer for this other case, again with the use of the Jacobi elliptic sine function.

We retreat back to Eq. (1.4), which is repeated below. However, when we were considering the case $E/2mg\ell < 1$, we used it to define a maximum angle, and also set it equal to k^2 , the modulus of the elliptic function. In this new case, where we have $E/2mg\ell > 1$, and there is no maximum angle, we may still use it to define the elliptic modulus, but in the inverse mode, i.e., we will set $k^2 \equiv 2mg\ell/E < 1$:

$$\begin{aligned} \zeta &\equiv \theta/2, & \sin^2 \zeta &= \sin^2(\theta/2) = \frac{1}{2}(1 - \cos \theta), & \frac{E}{2mg\ell} &= 1/k^2 \\ \implies \dot{\zeta}^2 &= \omega_0^2(1/k^2 - \sin^2 \zeta) = \left(\frac{\omega_0}{k}\right)^2 (1 - k^2 \sin^2 \zeta), & & & & (4.1) \\ \implies \frac{\omega_0}{k} dt &= \pm \frac{d\zeta}{\sqrt{1 - k^2 \sin^2 \zeta}}. \end{aligned}$$

This equation now looks somewhat familiar, i.e., similar to the integral that needed to be performed earlier. We will therefore proceed in (almost) the same way as before, by assigning a new variable name, $v \equiv \sin \zeta$. The algebra then proceeds in the same way as before, giving us

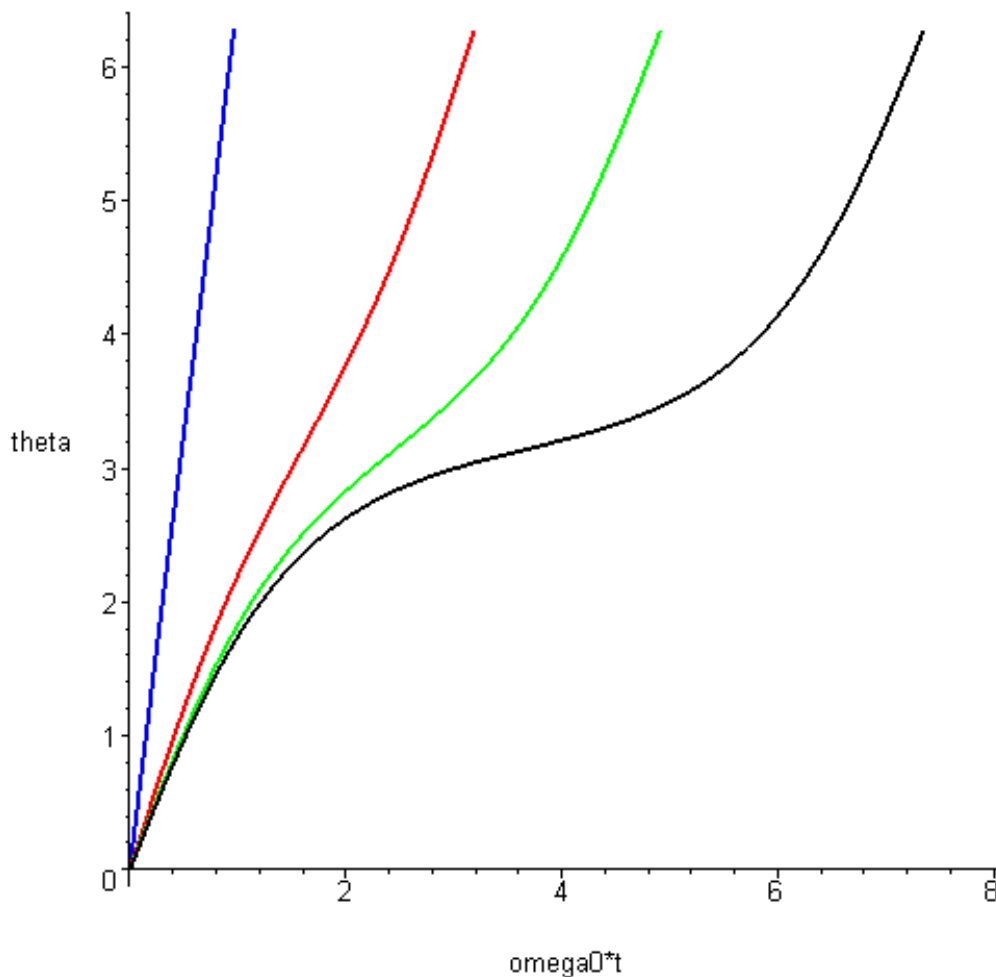
$$\pm \frac{\omega_0}{k} dt = \frac{dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}}, \quad v \equiv \sin \zeta = \sin(\theta/2). \quad (4.2)$$

However, here the plus or minus sign is not overly interesting. We know that the motion of the pendulum will simply be to swing round and round, with its speed varying as it goes but never changing sign. The plus or minus sign simply tells us that it can go around one way all the time,

or the other direction all the time. Therefore, we will simply choose the plus sign, saying that it will always go in the direction of increasing θ , i.e., counterclockwise. Choosing to have $t = t_0$ when the pendulum is at the bottom of the swing, and setting $\omega_0 t_0 = k\delta$, we have the following integrals:

$$\begin{aligned} \frac{\omega_0}{k} t - t_0 &= \frac{\omega_0}{k} \int_{t_0}^t dt = \int_0^v \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}} = \operatorname{sn}^{-1}(v; k), \\ \text{or } \sin(\theta/2) &= v = \operatorname{sn}(\omega_0 t/k - \delta; k), \quad k^2 \equiv 2mg\ell/E. \end{aligned} \quad (4.3)$$

Since $1/k$ is a measure of the total energy, we anticipate that for k rather near 1, i.e., the energy just barely larger than $2mg\ell$, the pendulum would slow down very much near the top of its swing; this slowing down would be much less as the energy becomes much larger, i.e., as k gets smaller. This effect can indeed be seen on the graphs below of the swing for one complete revolution, i.e., between 0 and 2π . The graph shows θ versus $\omega_0 t$ for $k = 0.3$, $k = 0.8$, $k = 0.95$, and $k = 0.995$, which are easily distinguished since the amount of time needed for one complete revolution increases (noticeably) as k increases. This is of course appropriate since a larger k means a smaller energy, and therefore an energy closer to the minimum required to be able to swing over the top. Therefore one can also use the fact that the slope of the curves is the angular speed, which decreases near $\theta = \pi$ more and more as k increases.



Top of the plot is $\theta = 2\pi$, for one full revolution.
 k increases from left to right.

Since in this case the angle continues to increase, we had to properly choose the value of the multi-valued inverse sine function, so that it was continuous in the desired region. For this particular graph, we used

$$\theta(t) = 2 \sin^{-1}[\operatorname{sn}(\omega_0 t/k; k)] = 2 \begin{cases} \arcsin[\operatorname{sn}(\omega_0 t/k; k)] , & 0 \leq \omega_0 t \leq kK(k) , \\ \pi - \arcsin[\operatorname{sn}(\omega_0 t/k; k)] , & kK(k) \leq \omega_0 t \leq 2kK(k) , \end{cases} \quad (4.4)$$

I have used the name “arcsin” to denote that particular branch of \sin^{-1} which always chooses a value between $-\pi/2$ and $+\pi/2$, which is the most common choice to make it single-valued, and the one chosen by Maple. However, for this problem we needed a choice that varied between 0 and π —since the angle θ was twice that value and we wanted the region where it varied over one complete revolution beginning at $\theta = 0$, so the equality used just above is an alternate way of choosing that branch that we needed.