Physics 406
Time Dependence of a Gaussian Wave Packet
Moving Through a Dispersive Medium With a Cutoff Frequency

An electromagnetic wave moves through a dispersive material for which $k = k(\omega)$ is given by

$$(ck)^2 = \omega^2 - \omega_m^2 \ ,$$

where ω_m is a constant value, determined by the properties of the material. One may see that waves cannot pass through the material if their original frequency is smaller than ω_m; therefore, ω_m is often referred to as a *cutoff frequency.*

We want to follow the motion of a wavepacket through this material, which begins, at time $t = 0$, in a Gaussian form; i.e., at time $t = 0$ the electric field is given by

$$\vec{E}(z, 0) = \vec{E}_0 \, \Re \left[e^{-z^2/2L^2} e^{ipz} \right] \ ,$$

where p, L, and also $\vec{E}_0$ are real constants. The problem, in principle, is to determine the form for the electric field at an arbitrary later time, $\vec{E}(z, t)$. In fact the integral that determines this behavior is quite difficult to calculate; therefore, we instead intend to acquire some approximations for the behavior, which will be based on the initial precept that the initial wavepacket is quite strongly peaked. It is worth noting that the wave packet that has been given has $< z >= 0$, $\Delta z = L/\sqrt{2}$, while we will also show that $< k >= p$ and $\Delta k = 1/(\sqrt{2}L)$.

The wavepacket will be approximately a “monochromatic” wave provided it is strongly peaked in wave number space, which will occur provided L is quite large. Therefore, we will begin by approximating the first-order behavior of the wave packet at later times, i.e., the behavior if it were almost exactly monochromatic, valid for times that are not too long after the initial time of $t = 0$. Then we can continue and find the explicit second-order behavior of the wave packet, presumably valid at yet later times.

In order to do all this, we need the Fourier transform of our given electric field at time $t = 0$. To simplify matters, I will first ignore the constant factor $\vec{E}_0$, and refer to the remainder of the expression as $\psi(z, 0)$. We then compute the Fourier transform

$$[\mathcal{F}(\psi)](k) = \int_{-\infty}^{+\infty} dz \, \psi(z, 0) e^{-ikz} = \int_{-\infty}^{+\infty} dz \, e^{-x^2/2L^2} e^{i(p-k)x} = \sqrt{2\pi} L \, e^{-L^2(k-p)^2/2} \ .$$

We may then write out the expression—to be integrated—for the (complex) time-dependent solution to the wave equation with this initial form:

$$\psi(z,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, [\mathcal{F}(\psi)](k)\, e^{i[kz-\omega(k)t]} \,, \qquad \omega(k) = \sqrt{(ck)^2 + \omega_m^2} \,.$$

Obviously, by Fourier's Theorem, this integral gives the correct value for $\psi(z,0)$. Our interest in it now is to find the values for later times. Using this Fourier transform, it is indeed straightforward to determine the averages in wave number space described above; i.e., that the initial wave function is peaked at $k = p$:

$$< k >= p \,, \qquad \Delta k = \frac{1}{\sqrt{2}\,L} \,.$$

Therefore, a legitimate first approximation is given by expanding our equation for the frequency in a Taylor series about the "most probable" wave number, namely $k = p$:

$$\omega(k) = \omega(p) + (k-p)\omega'(p) + \tfrac{1}{2}(k-p)^2\omega''(p) + \ \ldots = \omega_0 + (k-p)\frac{c^2 p}{\omega_0} + \tfrac{1}{2}(k-p)^2\frac{c^2\omega_m^2}{\omega_0^3} + \ldots \ ,$$

where we have referred to that most probable value of the frequency, $\omega(p)$, as just $\omega_0 \equiv \omega(p) = \sqrt{(cp)^2 + \omega_m^2}$.

Inserting this expression into the above integral, keeping only the first two terms, and defining a new variable of integration $s \equiv k - p$, we have

$$\psi(z,t) \approx \frac{L}{\sqrt{2\pi}} e^{i(pz-\omega_0 t)} \int_{-\infty}^{+\infty} ds\, e^{-L^2 s^2/2} e^{is(z-c^2pt/\omega_0)} = e^{i(pz-\omega_0 t)}\, e^{-(z-c^2pt/\omega_0)^2/2L^2} \,.$$

We see, as expected, that this has the value that it should at $t = 0$, namely $\psi(z,0)$, while for later times it has two distinct parts, moving with two different speeds:

a. It has an oscillatory part which moves at speed $v_{\rm ph} = \omega_0/p$, referred to as the phase velocity. Do note that this phase velocity is simply $\omega/k|_{k=p}$, i.e., it is the usual equation for the velocity of a monochromatic wave, ω/k, evaluated at the value of k for which our wave packet is peaked.

b. The packet also has an overall Gaussian shape, which we now see is moving at speed $v_g = c^2p/\omega_0$, usually referred to as the group velocity. In this case we recall that c^2p/ω_0 came from the second term in the Taylor expansion about the peak value of k; i.e., it is $d\omega/dk$, evaluated at $k = p$.

c. Therefore we may also provide a summary of these two statements about velocities by re-phrasing them in the following way, specialized only to the case where the packet in question has a single peak wave number around which it falls off:

$$\text{the phase velocity is } \; v_{\rm ph} = \left.\frac{\omega}{k}\right|_{k=p} ,$$
$$\text{the group velocity is } \; v_{\rm gr} = \left.\frac{d\omega}{dk}\right|_{k=p} .$$

Continuing on with this particular dispersive relationship—i.e., this particular functional form of $\omega = \omega(k)$, we now note that this approximation amounted to dropping, or ignoring, the term

$$e^{-\frac{i}{2}(k-p)^2 \frac{c^2\omega_m^2}{\omega_0^3} t}$$

in the integral. Since we anticipate that the majority of the packet has k-values near to p, then this is a legitimate thing to do, provided also that t is not too large; i.e., the behavior above is the anticipated "not-too-late" time behavior. Eventually that third term will become important; at even later times, even higher-order terms will of course become important. Therefore, let's also now go ahead and insert the third term in the Taylor series expansion for the frequency. This requires that we evaluate the integral:

$$\begin{aligned}\psi(z,t) &\approx \frac{L}{\sqrt{2\pi}} e^{i(pz-\omega_0 t)} \int_{-\infty}^{+\infty} ds\, e^{-L^2 s^2/2} e^{is(z-c^2pt/\omega_0)} e^{-(it/2)(c^2\omega_m^2/\omega_0^3)s^2} \\ &= e^{i(pz-\omega_0 t)} \frac{L}{\sqrt{L^2+iQt}}\, e^{-(z-c^2pt/\omega_0)^2/2(L^2+iQt)} , \qquad Q \equiv \frac{1}{2}\frac{c^2\omega_m^2}{\omega_0^3} .\end{aligned}$$

This is, however, a rather nasty expression because of all the complex quantities contained within it; therefore, we want to simplify its form considerably, remembering always that in the end we need to take the real part to acquire the physical electric field. We first write our complex number in terms of its magnitude and its phase, and also in terms of its real and imaginary parts:

$$\frac{L^2+iQt}{L^2} \equiv 1 + iHt = \sqrt{1+(Ht)^2}e^{i\phi} , \quad \tan\phi = Ht \equiv \frac{Qt}{L^2}$$
$$\Longrightarrow \quad 1/\sqrt{1+iHt} = e^{-i\phi/2}/\sqrt{\sqrt{1+(Ht)^2}}$$
$$1/(1+iHt) = (1-iHt)/[1+(Ht)^2] .$$

We may therefore re-write our form as follows:

$$\psi(z,t) \approx \frac{1}{\sqrt[4]{1+(Ht)^2}} e^{i\{pz-\omega_0 t+\phi/2+Ht(z-c^2pt/\omega_0)^2/[L^2+(Qt)^2]\}} e^{-\frac{1}{2}(z-c^2pt/\omega_0)^2/[L^2+(Qt)^2]} .$$

The actual electric field is then just the real part of this multiplied by our constant vector $\vec{E}_0$:

$$\vec{E}(z,t) \approx \vec{E}_0 \frac{e^{-\frac{1}{2}(z-c^2pt/\omega_0)^2/[L^2+(Qt)^2]}}{\sqrt[4]{1+(Ht)^2}} \cos\left\{pz - \omega_0 t + \phi/2 + \frac{Ht(z-c^2pt/\omega_0)^2}{[L^2+(Qt)^2]}\right\} .$$

The behavior of the part with the phase velocity is as before, but with a time-dependent phase shift; the part with the group velocity, namely $v_g = c^2p/\omega_0$, is much the same, except that now the wave packet is spreading out over time; i.e., the width of the Gaussian was once L but has now become $L\sqrt{1+(Ht)^2}$, so that it is becoming less and less peaked in location space. Additionally we also see that the overall normalization is multiplied by a time-dependent denominator, causing the amplitude to decrease as a function of time as well.