Introduction to the Fourier Transform

I. General Introduction

Looking at waves that move in one spatial dimension, or direction, we say that the solution to the wave equation has the form of some function, say $f(x,t)$. However, as it is a wave there is a relationship between its dependence on x and its dependence on t. Said in a more physical way, we could suppose that we examine the wave by sitting at one spot, say $x=0$, and watch it as it comes by, so that we are actually studying the function $f(0,t)$. On the other hand, we, with the help of a large body of collaborators, could study the wave at a certain fixed time, say $t=0$, at all the possible relevant locations in space, thereby actually studying the behavior of the function $f(x,0)$. Because it is a wave either of these two approaches should give us the same information. In these notes we will choose to use the second approach, studying, most of the time, just the behavior at a given fixed time, which we might as well choose to call $t=0$, having set our clocks that way.

II. General Superpositions of Monochromatic Waves

The simplest sort of (scalar) wave is a *monochromatic* one, which corresponds to a single, given, constant, angular frequency, ω, often also referred to as a *harmonic wave*:

$$f(x,t) = A\cos(kx-\omega t+\phi)\ , \qquad k\equiv \omega/v\ .$$

Here we refer to k as the *wave number*, and, remembering that it is describing a wave moving in the $\hat{x}$-direction, we say that $\vec{k}\equiv k\hat{x}$ is the *wave vector*. Alternative concepts are the slightly-more geometric notions of wavelength, $\lambda\equiv 2\pi/k$, measuring the repetition period in some units of length, and also the cyclic frequency, $\nu\equiv\omega/2\pi$, measured in units of cycles per second, often, in a rather fancy tone of voice, referred to as the unit of Hertz, so that one can resurrect the familiar formula from freshman physics, namely that $\lambda\nu = v$, the velocity at which the wave travels. Following the line of argument given in §I above, our function to be studied at time $t=0$ is then just

$$f(x)\equiv f(x,0) = A\cos(kx+\phi)\ , \qquad \omega = kv\ .$$

However, while the mathematical simplicity of a monochromatic wave is clear, it is also clear that they are, at most, only an approximation to any wave that we would actually want to study, since they describe a wave that has always exactly the same structure—amplitude, frequency, and shape—in all parts of space and for all time, both in the infinite past and in the infinite future. We would expect, instead, that any physically realistic wave would extend only over some finite region of space, and also would have both a beginning and a termination. Nonetheless, as we will see soon, we may create physical waves of that nature simply by adding together some large number of monochromatic waves, so that it is indeed important to understand well the behavior of monochromatic waves, even though they are unphysical. Such a wave is often referred to as a ***wave packet***, since these aspects of an extension only over finite regions of space and/or time do require that we add together many different waves of slightly differing frequencies.

To get a quick intuition about how this happens we will first think about adding together simply two different harmonic waves, with slightly different wave numbers. The behavior of this sum will have some destructive interference properties that were not immediately apparent in either of its parts. We begin by writing down the desired sum of two monochromatic waves, taking them to have the same phase angles and amplitudes, for mathematical simplicity:

$$f(x) = A\,\cos(k_1 x + \phi) + A\,\cos(k_2 x + \phi)\ .$$

What we desire to obtain from this is some reasonable understanding of how it acts as a wave, where it vanishes, where it has its maximum amplitude, etc., which will be most easily determined if we put this sum together in the form of a product of single terms. However, already at this point the difficulties with trigonometric identities begin; therefore, it is much simpler to immediately switch to a description that involves complex, oscillating exponentials, which will allow the (anticipated) algebra to have a much simpler description, re-writing k_1 and k_2 as sums of two different quantities:

$$\begin{aligned} f(x) &= \Re\left\{A\,e^{k_1 x+\phi} + A\,e^{k_2 x+\phi}\right\} \\ &= \Re\left\{Ae^{i[(k_1+k_2)x/2+\phi]}\left[e^{i[(k_1-k_2)x/2]} + e^{-i(k_1-k_2)x/2}\right]\right\} \\ &= 2\Re\left\{Ae^{i[(k_1+k_2)x/2+\phi]}\,\cos[(k_1-k_2)x/2]\right\} \\ &= 2\cos[(k_1+k_2)x/2+\phi]\cos[(k_1-k_2)x/2]\ . \end{aligned}$$

The argument of the first cosine is just the average of the two wave numbers, along with the common phase; however, when we look at the case where the two wave numbers are nearly the same, this means that the argument of the second cosine is rather small, so that it varies quite rapidly. The result of this addition of only two waves gives us a rapidly-varying harmonic wave—usually much more rapidly than either of the first two—oscillating under an **envelope** which is the average behavior of the original two waves. Therefore the rapid oscillations are suppressed at, and near, the zeroes of the envelope. It is this sort of behavior that we hope to use to arrange for an oscillation that is completely suppressed outside some finite region; however, to accomplish that we will have to superpose an infinite number of waves, with differing amplitudes.

The waves in which we have a major interest propagate through an infinite region of space. If, instead, we had desired to study only, say, waves propagating inside some finite cavity, such as the acoustic waves inside an organ pipe, we could say that all the desired waves would be periodic in space, with a wavelength that was some fraction of the size of the cavity. In such a case, we would have used Fourier series to create the appropriate (usually-infinite) sums of monochromatic waves. At the end of these notes I have appended some details of this use of Fourier series, which might be a useful review for you of material you have already studied, both last semester in electrostatics and in some mathematics class. However, in the main section of these notes, I want to discuss the correct approach to summing large numbers of waves, each with different periods, so that there is no overall period of the sum. This would, for instance, be the appropriate way to describe a single "pulse" of electromagnetic energy propagating through space. We refer to this approach as the use of ***Fourier integrals***. It is of course possible to approach Fourier integrals by taking some limit of the mathematics as the size of the cavity becomes infinite. Again, in the appendix I will give some fairly careful description of how to take such a limit. However, it is probably best to only keep that approach in the back of your mind as a physical justification, and begin directly with the appropriate integral presentation of the behavior of these integrals.

III. The Fourier Transform, in one spatial dimension

When taking a sum of distinct monochromatic waves, it is quite usual and normal to suppose that one might want differing amounts of the waves which have different frequencies; i.e., we might well want to have the amplitude of a given wave—which has been called just A in the preceding pages—to depend on the wave number k. As well, in that earlier discussion we found that the algebra was very much simpler to handle if we used the complex form of the exponentials, rather than the trigonometric functions, with the intent to take the real part of the result after all mathematical calculations were completed. Therefore, we can suppose that we should be interested in integrals of the following form:

$$\int dk A(k)\, e^{ikx} \ ,$$

where it might occasionally be true that only a particular sum is required instead of an entire integral, but those cases will in fact be quite rare. Since we know that for any value of k, and any function, $A(k)$, of that wave number, the quantity $A(k)\, e^{i(kx-\omega t)}$ is a solution of the wave equation, with the relation $\omega = kv$ telling us the velocity of the wave, then surely any sum, even an infinite one such as this, is also an acceptable solution of the wave equation, as it is an equation linear in the unknown function. This sort of summing up of all possibilities, with some system for determining the "weighting function," $A(k)$, is what we want to declare as a quite general solution of that equation.

As the tentative sum above is intended to create a wave with some definite sort of (desired) shape, my putting together many different waves, with different weighting functions, let us define clearly the ***Fourier transform***, which relates a pair of functions: $f(x)$ is the function such that its real part is the shape of the wave packet—at the time $t = 0$ as already stated; while we denote by $F(k)$ the (possibly complex) amplitude of the monochromatic wave with wave number k:

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, F(k)\, e^{ikx} \ . \tag{1}$$

You will recall that a general solution of the (1-dimensional) wave equation may propagate either in the direction of increasing x, so that it might be described as $h(x - vt)$, or in the direction of decreasing x, with a description as $g(x + vt)$. By arranging our integral to include both positive and negative values of k, we automatically include both possible directions of motion for our wave packet. Again the important physical statement here is simply that we are considering a physical system which is best described as an infinite linear superposition of distinct harmonic waves, and the role of $F(k)$ is to tell us just how much, and with what phase, each individual monochromatic wave contributes to this superposition. [I note that it does tell us about the phase because the usual case is that $F(k)$ is complex-valued, so that it may be thought of as its magnitude multiplied by a phase-factor, such as $F(k) = |F(k)|e^{i\phi(k)}$.]

You may also have thought it strange that I have written the weighting function in the form $F(k)/(2\pi)$. This is obviously just some sort of normalization convention, a reason for which we will see in just a moment. As it is a convention, in fact, there are many different choices that one can make there, and any selection of textbooks will have several of them. The reason for the need for such a factor, somewhere, however, is given by *Fourier's Theorem*, which tells us how to determine the weighting function for a particular desired shape:

$$F(k) = \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-ikx} \; . \tag{2}$$

The (related) pair of functions, $f(x)$ and $F(k)$, are referred to as Fourier transforms, one of the other. That this statement is true, for some appropriate restriction of the space of functions from which $f(x)$, and also $F(k)$, are chosen, is the content of Fourier's Theorem, the generalization of the similar theorem in the study of Fourier series, for periodic functions. Truthfully, the size of these spaces of functions has been the subject of much research work since Fourier's original studies, with many successful generalizations of his work having been achieved. We do now regularly take Fourier transforms of various sorts of distributions, such as Dirac's delta distribution, all quite legitimately. Lastly, looking at the fact that there is a denominator of 2π in one of the formulae and not the other displays a certain lack of symmetry between the two formulae, irrespective of the fact that the sign of the exponential is opposite in the two. Therefore, many different choices of normalization can be made to distribute that $1/2\pi$ wherever a particular author desires it. To my mind, the association of the 2π with the quantity that is measured with radians, i.e., with k, is the most reasonable way to keep track of where to put it. [I note that such a denominator should also appear with ω, were we to be taking Fourier transforms for a single, fixed location in space but considering functions of time.]

One should now list many important properties that the Fourier transform has, as well as to display some simple examples for future reference. To begin with, let us think of the Fourier transform as a mapping of functions into other functions. Actually, this is better described as a mapping of an entire space of functions into itself, where we will let $\mathcal{L}$ denote some appropriate space of functions, and let $\mathcal{F}$ denote this operation we are calling the Fourier transform:

$$\mathcal{F} : \mathcal{L} \to \mathcal{L}\,, \tag{3.1}$$

$$\text{i.e., for every } f \in \mathcal{L}\,,\ f = f(x), \quad \mathcal{F}(f) = F \in \mathcal{L}\,,\ F = F(k) \equiv \mathcal{F}(f)(k)\ .$$

Then this mapping has a number of useful properties, which we name below, and then give examples explaining each of them. In these examples, we will take g and h as functions of x, while α and β are just constant scalars:

1. Linearity: $f(x) = \alpha g(x) + \beta h(x) \qquad \Longleftrightarrow \qquad \mathcal{F}(f)(k) = \alpha\mathcal{F}(g)(k) + \beta\mathcal{F}(h)(k)$;
2. Conjugation: $f(x) = g^*(x) \qquad \Longleftrightarrow \qquad \mathcal{F}(f)(k) = \{\mathcal{F}(g)\}^*(-k)$;
3. the transform of the product of two functions is the convolution) of their transforms:

$$f(x) = g(x)h(x) \qquad \Longleftrightarrow \qquad \mathcal{F}(f)(k) = \int_{-\infty}^{+\infty} \frac{dk'}{2\pi}\, \mathcal{F}(g)(k')\mathcal{F}(h)(k-k')\ ;$$

4. the transform of a derivative:

$$f(x) = \frac{d^n}{dx^n} h(x) \qquad \Longleftrightarrow \qquad \mathcal{F}(f)(k) = (ik)^n \mathcal{F}(h)(k)\ ;$$

5. Parseval's relation for norms:

$$\int_{-\infty}^{+\infty} dx\, |f(x)|^2 = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, |\mathcal{F}(f)(k)|^2\ .$$

An especially important property of the Fourier transform is usually referred to as *the completeness property.* In principle, the idea of completeness is an important one when one has some infinite set of objects: How do you know you have "all of them," since if, for instance, one or two were missing, you would still have infinitely many? The answer is obtained by inserting the equation above for $F(k)$ back into the equation for $f(x)$, or vice versa. Both of these should be true, and each tells us, in a somewhat different way about the validity of Fourier's theorem. Let's begin with the one already described, namely inserting the value for $F(k)$ in terms of an integral over $f(y)$ in terms of the integral that should result in $f(x)$. [Do note that in our copy of the integral presentation for $F(k)$ we must denote the "dummy" integration variable in that formula by some other symbol than x, since we are already using that symbol for the variable on the left-hand-side of the equality.] We have

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, F(k)\, e^{ikx} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} dy\, f(y)\, e^{-iky} e^{+ikx} = \int_{-\infty}^{+\infty} dy\, f(y) \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ik(x-y)} \;.$$

As the final result on the right-hand side needs to be just $f(x)$, we need something inside the integral over dy which would arrange for that. Of course, we are "familiar" with what that should be, namely we understand the definition of the distribution referred to as the Dirac delta:

$$\int_a^b dy\, f(y)\, \delta(y-x) = \begin{cases} f(x), & a < x < b\;, \\ 0, & \text{otherwise.} \end{cases}$$

Since any value of x surely lies between $-\infty$ and $+\infty$, we see that we can indeed be successful with our statement of completeness of the transform, or our re-writing of the content of Fourier's Theorem, by saying that the following is the value of this somewhat unfamiliar integral:

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ik(x-y)} = \delta(x-y)\;. \tag{3}$$

By following the same argument in reverse, i.e., inserting the form for $f(x)$ into the equation for $F(k)$, we may also determine what is mathematically the same statement, namely

$$\in_{-\infty}^{+\infty} dx\, e^{ix(k-p)} = 2\pi\, \delta(k-p)\;,$$

where it is also useful to recall that the Dirac delta is an even function of its argument, and is zero at all points where its argument is not zero.

Now let me give a few useful examples of particular Fourier transforms, to see particular ways to construct interesting wave packets:

$$\begin{aligned}
\mathcal{F}(1) &= 2\pi\,\delta(k)\ , \\
\mathcal{F}(e^{isx}) &= 2\pi\,\delta(s-k)\ , \\
\mathcal{F}(e^{-\gamma x}e^{i\kappa x}) &= \frac{2\gamma}{(k-\kappa)^2+\gamma^2}\ , \\
\mathcal{F}(e^{-\gamma x}\cos(\kappa x)) &= \frac{\gamma}{(k-\kappa)^2+\gamma^2}+\frac{\gamma}{(k+\kappa)^2+\gamma^2} \\
\mathcal{F}(e^{-\gamma x^2}e^{i\kappa x}) &= \sqrt{\frac{\pi}{\gamma}}e^{-(k-\kappa)^2/4\gamma}\ ,
\end{aligned}$$

An interesting thing to recall at this point is the **uncertainty principle**. We may first look at this principle as simply a re-statement of things already discussed above, namely that a plane wave is defined over all space, rather than some finite region, so that it is impossible to answer reasonably the question as to where it is located. We suggested that this was related to the fact that we knew exactly what was the frequency of this wave. To now re-phrase that notion somewhat more precisely, we use the absolute square, $|f(x)|^2$, of our complex-valued wave as a measure of the probability that one finds the wave at location x, and then define the standard notion of the "width" of a wave packet, namely

$$\begin{aligned}
(\Delta x)^2 &\equiv < x^2 > - < x >^2\ , \\
<h(x)> &\equiv \frac{\int_{-\infty}^{+\infty} dx\, h(x)\,|f(x)|^2}{\int_{-\infty}^{+\infty} dx\,|f(x)|^2}\ ,
\end{aligned}$$

where $h(x)$ is any function of location. Defining the width of the Fourier transform in an equivalent way, we have already described the fact that these two widths may not both be completely zero, i.e., we may not know exactly both the location and the frequency. The uncertainty principle is a particular quantitative statement of this fact:

$$\Delta x\,\Delta k \geq \frac{1}{4\pi}\ .$$

IV. Fourier Transforms in three dimensions:

The same statements may be made in three dimensions, with the "normative factor" 2π appearing once for each dimension:

$$f(\vec{r}) = \int_{(\infty)} \frac{d\vec{k}}{(2\pi)^3} F(\vec{k})\, e^{i\vec{k}\cdot\vec{r}} \quad \Longleftrightarrow \quad F(\vec{k}) = \int_{(\infty)} d\vec{r}\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} \;, \tag{4.1}$$

where the symbol (∞) under the integral sign is meant to indicate that one should integrate over the entire 3-dimensional space of the corresponding (vector-valued) variable of integration. In particular, then the truth of this statement requires the three-dimensional integral

$$\int_{(\infty)} d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = (2\pi)^3\, \delta^{(3)}(\vec{r}-\vec{r}\,') \;,$$

or, equivalently,

$$\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z\, e^{i[k_x(x-x')+k_y(y-y')+k_z(z-z')]} = (2\pi)^3\, \delta(x-x')\delta(y-y')\delta(z-z') \;.$$

It is also worth recalling that the 3-dimensional Dirac delta, in spherical coordinates, has the following form:

$$\delta^{(3)}(\vec{r}-\vec{r}\,') = \frac{\delta(r-r')}{r^2}\,\frac{\delta(\theta-\theta')}{\sin\theta}\,\delta(\varphi-\varphi') \;.$$

Appendix

The purpose of this appendix is simply to begin with Fourier series presentations of (possibly-infinite) sums of harmonic waves, all with a common period, and to take the appropriate limits as that period increases without bound, thereby acquiring the theory of the Fourier integral. Such a construction is only of a somewhat pedantic interest, unless you are already relatively familiar with Fourier series themselves, but is included here for some reasons of completeness.

We begin by supposing that we have been given a function, $f = f(x)$, which is periodic, with period L:

$$\begin{gathered} f = f(x) \text{ is a periodic function, with period } L \\ \Longleftrightarrow \\ f(x) = f(x \pm L) = f(x + mL)\,, \quad \text{for any integer } m\,. \end{gathered} \tag{1.2}$$

The theory of Fourier series then tells us that we may find
sequences of numbers $\{a_n \mid n = 0, 1, 2, \ldots\}$ and $\{b_n \mid n = 0, 1, 2, \ldots\}$ such that

$$f(x) = \sum_{n=0}^{\infty} \{a_n \cos k_n x + b_n \sin k_n x\}\,, \qquad k_n \equiv 2n\pi/L\,, \tag{1.3}$$

where it is in fact true that the equality is not necessarily exactly so at the (two) ends of the period L, but only just at those two points, at least for continuous functions or interesting, not-too-radical generalizations of that. An important part of the underlying rationale for the existence, and use, of such an expression is that all the various wave numbers k_n are chosen so that all possible integer multiples of the lowest frequency, $2\pi/L$ are included, or, equivalently, so that all possible integer fractions of the original period, L, are included, i.e., the so-called "higher harmonics." However, it is also valuable to rewrite this expression using deMoivre's theorem about complex-valued exponentials, namely

$$e^{iz} = \cos z + i \sin z \quad \Longleftrightarrow \quad \begin{cases} 2\cos z = e^{+iz} + e^{-iz}\,, \\ 2i \sin z = e^{+iz} - e^{-iz}\,. \end{cases} \tag{1.4}$$

By setting $c_n = \frac{1}{2}(a_n - ib_n)$ for positive values of n and $c_n = \frac{1}{2}(a_{-n} + ib_{-n})$ for negative values of n, we may rewrite our expression using complex exponentials instead:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{ik_n x} \; . \tag{1.5}$$

The apparent discrepancy when $n = 0$, as to whether it is positive or negative, disappears when we note that $\sin k_0 x = \sin 0 = 0$, so that we ought to just go ahead and choose $b_0 = 0$, so that $c_0 = \frac{1}{2}a_0$ according to both the definition for positive values of n and that for negative values. The equation may be solved, to determine the various quantities c_n, which are referred to as the Fourier coefficients of the periodic function f:

$$c_n = \frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx \; f(x) \, e^{-ik_n x} \; . \tag{1.6}$$

To verify this we should show that the two equations, (1.5) and (1.6), are consistent, **in both directions.** We begin by inserting the claimed definition for $f(x)$ into Eq. (1.6), but noting that the index n in Eq. (1.5) for $f(x)$ is just a "dummy" variable, i.e., a variable indicating a summation, so that when we insert this value for $f(x)$ into Eq. (1.6) we must change the name of that index, from n to something else, so we do not confuse it with the actual index n that is on the left-hand side of the equation; we agree to call it m and then have the following, where we assume that the functions involved are sufficiently well-defined that it is allowable to interchange the order of the integral and the sum. We therefore begin with the right-hand side of Eq. (1.6), intending to show that it indeed equals the left-hand side:

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx \; f(x) \, e^{-ik_n x} = \sum_{m=-\infty}^{+\infty} c_m \, \frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx \; e^{ik_m x} e^{-ik_n x} \; . \tag{1.7}$$

The value of the integral is given by

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx \; e^{ik_m x} e^{-ik_n x} = \frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx \; e^{i(2\pi/L)(m-n)x} = \frac{\sin[\pi(m-n)]}{\pi(m-n)} \; . \tag{1.8}$$

However, as both m and n are integers, this quantity is of course exactly zero, **except** for the special case when $m - n = 0$. In that case we must take the limit as $m \to n$, and, for instance, the power-series expansion of the sine function tells us that the limit is just +1, so that the value of the integral may be stated as δ_{mn}, the Kronecker delta, which has value +1 when its indices are equal, and is zero otherwise. The result of our calculation is then

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx\, f(x)\, e^{-ik_n x} = \sum_{m=-\infty}^{+\infty} c_m \delta_{mn} = c_n\ . \tag{1.9}$$

This is in fact the value of the left-hand side, so that we have indeed proved consistency in this direction. This particular direction the result in question is often referred to by saying that the (basis) set $\{e^{(2\pi/L)nx}\}_{n=-\infty}^{+\infty}$ is *orthonormal,* as functions of x, over the interval of length L.

We now will show consistency in the other direction; as it turns out, this is somewhat more difficult, which is reasonable as, after all, it is the original proof of this direction that gave Fourier's name to the construction. We begin with the right-hand side of Eq. (1.5) and insert into it the value for c_n given by Eq. (1.6), but, again, noticing that the symbol x in Eq. (1.6) for c_n is just a "dummy" variable, i.e., a variable of integration, so that when we insert this value for c_n into Eq. (1.5) we must change the name of the variable of integration from x to something else, so we do not confuse it with the actual variable x that is on the left-hand side of the equation; we agree to call it y and then have the following, where, again, we assume that the functions involved are sufficiently well-defined that it is allowable to interchange the order of the integral and the sum:

$$\sum_{n=-\infty}^{+\infty} c_n\, e^{ik_n x} = \frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dy\ f(y) \sum_{n=-\infty}^{+\infty} e^{-ik_n y} e^{+ik_n x}\ . \tag{1.10}$$

Our problem then is to evaluate the sum above, which may also be written in a somewhat simpler form:

$$\sum_{n=-\infty}^{+\infty} e^{-ik_n y} e^{+ik_n x} = \sum_{n=-\infty}^{+\infty} e^{(2\pi i/L)n(x-y)} = 1 + 2\sum_{n=1}^{\infty} \cos[(2\pi/L)n(x-y)]\ . \tag{1.11}$$

It is clear that this sum is divergent when $x = y$. However, when this is not the case, it is not so clear what its value might be since the sign of its terms oscillate wildly as $|n|$ becomes larger and larger. To nonetheless determine a value we re-write it as follows:

$$\sum_{n=1}^{\infty} \cos[\alpha n(x-y)] = \frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty} \frac{\sin[\alpha n(x-y)]}{\alpha} = \frac{\partial}{\partial\alpha}\,\frac{\pi-\alpha}{2} = -\frac{1}{2}\ , \tag{1.12}$$

where we have looked up the value of the last sum in a table of series. For instance, it is equation (1.44.1.1) in the book by Ryzhik and Gradshteyn, one of the best published tables of integrals and series. It gives the constraint on $\alpha(x-y)$ in order for the value to be valid, which is that $\alpha(x-y)$ must be larger than 0 and less than 2π, i.e., away from the boundaries of the period cell for the sine function. This obviously says that $x \neq y$; as well it says that $L > x-y$, which should be alright because we really only want the values of our periodic functions when their arguments range between $-L/2$ and $+L/2$.

Inserting this value into our equation we see that whenever $0 < x-y < L$, the value of our sum is simply zero! These two properties suggest to us that it should be proportional to a Dirac delta. We notice that Fourier's theorem would be true, using the definition of a Dirac delta, were it so that the sum is given by

$$\sum_{n=-\infty}^{+\infty} c_n \, e^{2\pi i(x-y)(n/L)} = L\delta(x-y) \ , \tag{1.13}$$

where we recall for the record the definition of the distribution named the Dirac delta. Denoting it by $\delta(x-y)$, so that it resembles the symbols for a function, and inserting it under an integral sign along with some other function, a so-called "test function," which is required to be at least reasonably nice, the definition is the following rule for the evaluation of the integral:

$$\int_a^b dy\, f(y)\, \delta(x-y) = \begin{cases} f(x)\,, & a < x < b\,, \\ \quad 0\,, \text{ otherwise,} \end{cases} \tag{1.14}$$

where the "point" of the rule is that it vanishes everywhere **except** where its argument vanishes, at which point it evaluates the integral under which it has been placed. As an interesting, and probably useful, aside, we also note the following propositions about its behavior, which can be verified by changes of integration variables and/or integration by parts, where in all the cases below we would assume that the argument of the Dirac delta does indeed vanish during the region of integration, i.e., that both $+x$ and $-x$ lie within the interval between a and b:

$$\begin{aligned} \int_a^b dy\, f(y)\, \delta[\alpha(x-y)] &= \frac{1}{|\alpha|} f(x)\ , \\ \int_a^b dy\, f(y)\, \delta(x^2-y^2) &= \frac{f(x)+f(-x)}{2|x|}\ , \\ \int_a^b dy\, f(y)\, \frac{\partial}{\partial y}\delta(x-y) &= -\frac{\partial}{\partial x} f(x)\ . \end{aligned} \tag{1.15}$$

Returning to our Fourier series problem, and inserting this value for the sum into our integral, given above in Eq. (1.10), causes the desired result, namely that the integral has exactly the value $f(x)$, whenever $x-y$ lies between $-L/2$ and $+L/2$. As this is of course the content of Fourier's theorem, we presume that this evaluation is in fact correct—as indeed it is, provided both sides are interpreted as distributions, instead of just functions. In order to come up with a reasonable approach to a proof in terms of functions, we should consider not the sum from $n=-\infty$ to $+\infty$, but rather the sum from some $-N$ to $+N$, evaluate that sum, and then take the limit as $N\to\infty$. As an aside I note that the rigorous, mathematical reason this happens is that we really are not allowed to interchange the order of the integral and the sum, if we insist that all the quantities involved must be well-behaved functions. Lastly, one can note that there are a number of useful books outlining approaches to the theory of distributions, as "rules to evaluate integrals," and as limits of sequences of functions, which one might consult if desired:

To eliminate the need to have periodic functions, i.e., so that we may take the approach above for a much larger class of functions, that are no longer required to be periodic, we take the limit as the period, L, approaches infinity. We will surely omit all the careful details concerning the requirements on the functions, $f(x)$, such that the limits all exist, but only give here an overall description of the process. We first take the sum in Eq. (1.5) and multiply it by $+1$, written in the form of the ratio of the two equal quantities, $2\pi/L$ and $\Delta k\equiv k_{n+1}-k_n=2\pi/L$, which gives

$$f(x)=\sum_{n=-\infty}^{+\infty}\left(\frac{\Delta k}{2\pi/L}\right)c_n\,e^{ik_nx}=\sum_{n=-\infty}^{+\infty}\left(\frac{\Delta k}{2\pi}\right)[Lc_n]\,e^{ik_nx}\ . \tag{2.1}$$

In the limit as L becomes very large, heading toward infinity, clearly Δk becomes very small, heading toward the infinitesimal notion dk, while the sum over all values of n may be treated as an integral over dk, which still runs from $-\infty$ to $+\infty$, replacing n by its equivalent $kL/2\pi$, so that we may take k as the summation variable, that in the limit becomes an integration variable. This allows us to define the limit, as $L\to\infty$, of the (former) coefficients by a function of the (now continuous) variable k,namely $F(k)$, which is referred to as the *Fourier transform* of $f(x)$:

$$Lc_n=Lc_{{}_{kL/2\pi}}\quad\underset{L\to\infty}{\longrightarrow}\quad F(k)\ . \tag{2.2}$$

Using that limit, we may write the limit, as $L \to \infty$, of Eq. (1.5) for arbitrary, sufficiently-nice functions $f(x)$ in terms of their Fourier transforms:

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, F(k)\, e^{ikx} \ . \tag{2.3}$$

The inverse of this relationship will then allow for the determination of the Fourier transform when given the original function; this is of course the analog of Eq. (1.6) above, and should be taken easily in the limit of Eq. (1.6), after we multiply both sides by L, and take account of the limit of Lc_n as given in Eq. (2.2):

$$\lim_{L\to\infty} Lc_n = \lim_{L\to\infty} Lc_{kL/2\pi} = F(k) = \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-ikx} \ . \tag{2.4}$$

Again, surely, one should check the consistency of these two relationships. As it turns out insertion of one into the other and interchanging the orders of integration requires only one constraint, or definition, or rigorous mathematical proof, of the value of the following integral, which we have already discussed in class:

$$\int_{-\infty}^{+\infty} dk\, e^{ik(x-y)} = 2\pi\, \delta(x-y) \ . \tag{2.5}$$